Around Cobham's theorem and some of its extensions

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November 25, 2010

Dynamical Aspects of Automata and Semigroup Theories
Satellite Workshop of Highlights of AutoMathA

Two influential papers

Cobham wrote two very influential papers

- On the Base-Dependence of Sets of Numbers Recognizable by Finite Automata, *Mathematical Systems Theory* **3** (1969) 186-192
- Uniform Tag Sequences, *Mathematical Systems Theory* **6** (1972) 164-192

First Cobham's Theorem

Definition

Given a base $r \geq 2$, a set $X \subseteq \mathbb{N}$ is called *r-recognizable* if X written in base r is accepted by a finite automaton
(All possible leading 0 are considered)

Example

$X = \{2^n \mid n \geq 0\}$ is 2-recognizable and 4-recognizable

- base 2 : 0^*10^*
- base 4 : $0^*(1+2)0^*$
- 3-recognizable ? $0^*(1+2+11+22+121+1012+2101+\cdots)$

Theorem (Cobham 1969)

A set $X \subseteq \mathbb{N}$ is r-recognizable for every base $r \geq 2$ iff X is a finite union of constants and arithmetic progressions.

More precisely ...

Definition

Two bases $r, s \geq 2$ are multiplicatively dependent if $r^k = s^l$ for some $k, l \in \mathbb{N} \setminus \{0\}$.

Example

Bases $2, 4$ are multiplicatively dependent. Bases $2, 3$ are not.

Theorem (Cobham 1969)

Let $r, s \geq 2$ be two multiplicatively independent bases.
A set $X \subseteq \mathbb{N}$ is r- and s-recognizable iff X is a finite union of constants and arithmetic progressions.

Example

$X = \{2^n \mid n \geq 0\}$ is not 3-recognizable.
(It is exactly 2^k-recognizable for every $k \geq 1$).

Second Cobham's Theorem

Theorem (Cobham 1972)

A set $X \subseteq \mathbb{N}$ is r-recognizable iff its characteristic sequence is generated by the iteration of a r-uniform morphism, followed by a coding.

Example

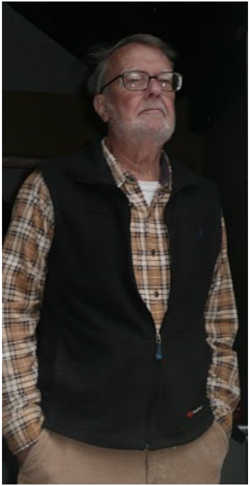
$X = \{2^n \mid n \geq 0\}$

g :	$a \to ab$,	$b \to bc$,	$c \to cc$	2-uniform morphism
f :	$a \to 0$,	$b \to 1$,	$c \to 0$	coding

a
ab
$abbc$
$abbcbccc$
$abbcbcccbccccccc$
$abbcbcccbcccccccbccccccccccccccc$

$01101000100000001000000000000000 \cdots$

Picture



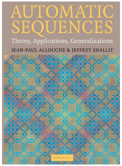
Alan Belmont Cobham
Born November 4, 1927, San Francisco
He lives in Middletown, Connecticut

Picture from Jeffrey O. Shallit 's blog
http ://www.cs.uwaterloo.ca/∼shallit/

Great impact of the two Cobham's theorems

Basis of hundreds of papers exploring the theory of automatic sequences and generalizing them.

- J.-P. Allouche, J. Shallit
  Automatic Sequences :
  Theory, Applications, Generalizations
  Cambridge University Press (2003)



First Cobham's theorem

- Simpler proofs and generalizations to various contexts : multidimensional setting, logical framework, non standard bases, substitutive systems, fractals and tilings, . . .
- B. Adamczewski, J. Bell, A. Bès, B. Boigelot, J. Brusten, V. Bruyère, F. Durand, S. Fabre, I. Fagnot, G. Hansel, C. Michaux, A. Muchnik, D. Perrin, F. Point, M. Rigo, A. Semenov, R. Villemaire, . . .

Great impact of the two Cobham's theorems

First Cobham's theorem - surveys

- D. Perrin, Finite automata, In *Handbook of TCS*, Vol B, Elsevier - MIT Press (1990) 1-57
- V. Bruyère, G. Hansel, C. Michaux and R. Villemaire, Logic and p-recognizable sets of integers, *Bull. Belg. Math. Soc.* **1** (1994) 191-238
- M. Rigo, Numeration systems : a link between number theory and formal language theory, *Proc. DLT'10*, LNCS **6224** Springer (2010) 33-53
- F. Durand, M. Rigo, On Cobham's theorem, In Handbook of Automata Theory (AutoMathA project), in preparation, 39 p

Outline of this talk

First Cobham's theorem

- Some known extensions
  - Logical characterizations
  - Extension to $\mathbb{Z}$
  - Extension to $\mathbb{N}^m$
- Recent extensions to $\mathbb{R}$ and $\mathbb{R}^m$
  - Recognizability
  - Logical characterizations
  - Cobham's theorem extended to $\mathbb{R}$
  - Weak automata
  - Main steps of the proof
  - Cobham's theorem extended to $\mathbb{R}^m$
- Conclusion and other related works

Some known extensions

Logical characterizations

Theorem (Büchi 1960)

X is r-recognizable iff X is first-order definable in $\langle \mathbb{N}, +, V_r \rangle$.

- $V_r(x) = y$ means that y is the largest power of r dividing x.
  $V_r(0) = 1$
- Formulae - (first-order) variables $x, y, z, \ldots$ over $\mathbb{N}$
  equality $=$, addition $+$, function V_r
  connectives $\wedge, \vee, \neg, \rightarrow, \leftrightarrow$
  quantifiers $\exists, \forall$ over variables

Example

- $V_2(20) = 4,\ V_3(20) = 1$
- $\leq$ is first-order definable ; any constant is first-order definable
- $X = 2\mathbb{N} + 1$ is definable by the formula $\varphi(x)$:
  $(\exists y)\ (x = y + y + 1)$
- $X = \{2^n \mid n \geq 0\}$ is definable by the formula $\varphi(x)$: $V_2(x) = x$

Logical characterizations

Theorem

X is a finite union of constants and arithmetic progressions
iff X is ultimately periodic
iff X is first-order definable in Presburger arithmetic $\langle \mathbb{N}, + \rangle$.

- X is ultimately periodic iff
  $(\exists l \geq 0)(\exists p \geq 1)(\forall n \geq l)\ (n \in X \Leftrightarrow n + p \in X)$

Theorem (Cobham's theorem restated)

Let $r, s \geq 2$ be two multiplicatively independent bases.

A set $X \subseteq \mathbb{N}$ is r- and s-recognizable,
(X is first-order definable in $\langle \mathbb{N}, +, V_r \rangle$ and in $\langle \mathbb{N}, +, V_s \rangle$)

iff X is a finite union of constants and arithmetic progressions
(X is first-order definable in $\langle \mathbb{N}, + \rangle$)

Extension to $\mathbb{Z}$

Automata

- In base r, a positive (resp. negative) number always begins with 0 (resp. $r-1$).

Example

In base 2, $-6 = -8 + 2$ is written as 1010 (2's complement), and 10 as 01010

$X = \{2^n \mid n \geq 0\} \cup \{-2^n \mid n \geq 0\}$ is 2-recognizable.
Base 2 : $0^+10^* + 1^+10^*$

Logical structures

- Structures $\langle \mathbb{Z}, +, \leq \rangle$ and $\langle \mathbb{Z}, +, \leq, V_r \rangle$
- X is first-order definable in $\langle \mathbb{Z}, +, \leq \rangle$ iff X is a finite union of constants, arithmetic progressions, and opposite of arithmetic progressions

Extension to $\mathbb{N}^m$

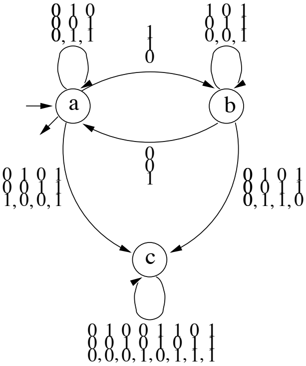
Automata

$\binom{3}{9}$ is written as $\binom{0011}{1001} = \binom{0}{1}\binom{0}{0}\binom{1}{0}\binom{1}{1}$ in base 2.

Example

$X = \{(x, y, z) \mid x + y = z\}$ is 2-recognizable

state a : no carry
state b : carry
state c : error



Extension to $\mathbb{N}^m$

Theorem (Büchi 1960)

Let $m \geq 1$. A set $X \subseteq \mathbb{N}^m$ is r-recognizable iff it is first-order definable in $\langle \mathbb{N}, +, V_r \rangle$.

Theorem (Semenov 1977)

Let $m \geq 1$. Let $r, s \geq 2$ be two multiplicatively independent bases. A set $X \subseteq \mathbb{N}^m$ is r- and s-recognizable iff X is first-order definable in $\langle \mathbb{N}, + \rangle$.

- Elegant proof by (Muchnik 1991), by induction on m

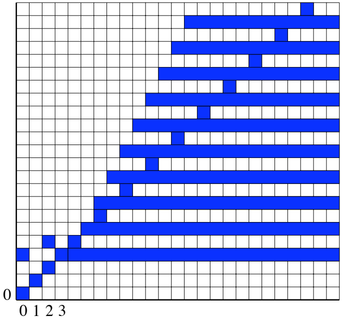
Extension of ultimate periodicity

- definability in $\langle \mathbb{N}, + \rangle$
- finite union of points and cones (semi-linear sets)
- Muchnik's definability criterion

Extension to $\mathbb{N}^m$

$\varphi(x, y)$

$$
\begin{array}{ll}
 & (x = 0 \land y = 3) \\
\lor & (x = 2 \land y = 4) \\
\lor & (x = y) \\
\lor & (\exists z)(\exists t)(x = z + t + 4) \land (y = t + t + 3)
\end{array}
$$



Two points and two cones :

- cone $\{(x, y) \mid (\exists z)(\exists t)\ (x, y) = z(1, 0) + t(1, 2) + (4, 3)\}$
- diagonal

Recent extensions to $\mathbb{R}$ and $\mathbb{R}^m$

Recognizability in $\mathbb{R}^m$

Definition

Given a base r, real numbers are positionally encoded as infinite words over $\{0, 1, \ldots, r-1, \star\}$

- a positive (resp. negative) number begins with 0 (resp. $r-1$)
- all possible encodings ; tuples
- integer numbers : infinite words $u \star 0^\omega$ and $u \star (r-1)^\omega$
- rational numbers : infinite words $u \star vw^\omega$

Example

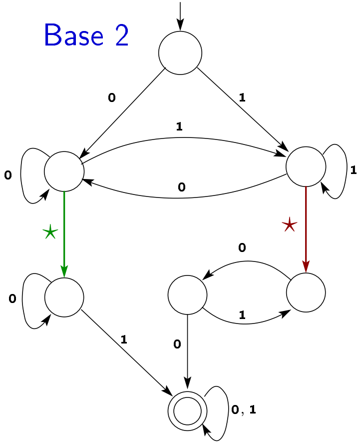
3.5 in base 10 : $0^+3 \star 50^\omega \ \cup \ 0^+3 \star 4\ 9^\omega$.

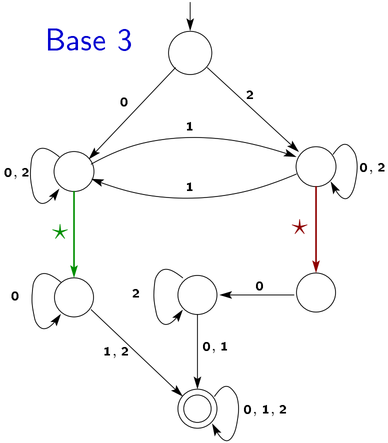
Definition

Let $m \geq 1$. Let $r \geq 2$ be a base.
A set $X \subseteq \mathbb{R}^m$ is r-recognizable if X written in base r is accepted by a finite (non deterministic) Büchi automaton

Example

$$\left\{2n + \left]0, \tfrac{4}{3}\right[\;\middle|\; n \in \mathbb{Z}\right\}$$





Logical characterization

Theorem (Boigelot-Rassart-Wolper 1998)

A set $X \subseteq \mathbb{R}^m$ is r-recognizable iff X is first-order definable in $\langle \mathbb{R}, +, \leq, \mathbb{Z}, V_r \rangle$.

- Variables $x, y, z, \ldots$ over $\mathbb{R}$
- Predicate $\mathbb{Z}(x)$ means that x is an integer variable
- $V_r(x) = y$ means y is the largest power of r dividing x as follows : $x = ky$ with $k \in \mathbb{Z}$ (if such a power exists)

Example

- $V_{10}(3.5) = \frac{1}{10}$, $V_{10}(3.55) = \frac{1}{10^2}$
- $X = \{2^n \mid n \in \mathbb{Z}\}$ is definable by : $V_2(x) = x$
- any rational constant is first-order definable
- $X = \left\{2n +]0, \frac{4}{3}[\;\mid\; n \in \mathbb{Z}\right\}$ is definable by : $(\exists y)(\exists z) \;\; \mathbb{Z}(y) \wedge (x = y + y + z) \wedge (0 < z < \frac{4}{3})$

Ultimately periodically simple sets

Theorem (Weispfenning 1999)

$X \subseteq \mathbb{R}$ is first-order definable in $\langle \mathbb{R}, +, \leq, \mathbb{Z} \rangle$ iff X is ultimately periodically simple.

- Characterization for dimension 1. Higher dimensions : see later.

Definition

X is ultimately periodically simple iff

- X is a finite union of sets of the form $Y_i + Z_i$ where
- each $Y_i \subseteq \mathbb{Z}$ is either an integer constant, either an arithmetic progression, or its opposite
- each $Z_i \subseteq [0,1]$ is an interval with rational endpoints

Example

$X = \left\{ 2n + \left]0, \frac{4}{3}\right[\;\middle|\; n \in \mathbb{Z} \right\}$ is ultimately periodically simple

Cobham's theorem extended to $\mathbb{R}$

Theorem (Boigelot-Brusten-Bruyère 2008)

Let r, s be two bases that do not have the same set of prime factors. A set $X \subseteq \mathbb{R}$ is r- and s-recognizable iff X is ultimately periodically simple

- If r, s do not have the same set of prime factors, then they are multiplicatively independent
- The converse is false (ex. $r = 6$ and $s = 12$)
- This theorem is false for two multiplicatively independent bases (see next slides)

Weak automata

Definition

A deterministic Büchi automaton is weak if each of its strongly connected components has either only accepting or only non accepting states.

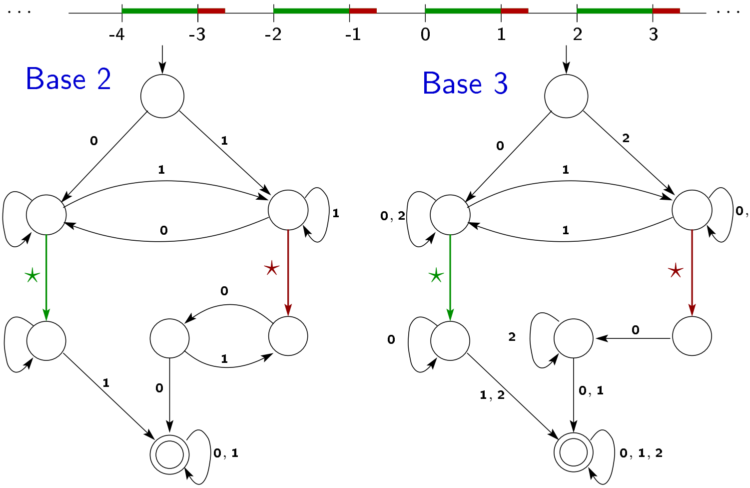
- Practically as easy to handle as finite-word automata
- Canonical minimal form (Löding 2001)

Theorem (Boigelot-Brusten 2007)

Let r, s be two independent bases. A set $X \subseteq \mathbb{R}$ is r- and s-recognizable by weak deterministic Büchi automata iff X is ultimately periodically simple

Weak automata

Example



Weak automata

The expressiveness of weak deterministic Büchi automata is limited

- level $\Sigma_2^0 \cap \Pi_2^0$ in Borel hierarchy
- instead of level $\Sigma_3^0 \cap \Pi_3^0$ for Büchi automata

Theorem (Boigelot-Jodogne-Wolper 2001)

Let $m \geq 1$. Let $r \geq 2$ be a base.
If a set $X \subseteq \mathbb{R}^m$ is definable in $\langle \mathbb{R}, +, \leq, \mathbb{Z} \rangle$, then X written in base r is recognizable by a weak determinstic Büchi automaton.

- Same result with $\langle \mathbb{R}, +, \leq, \mathbb{Z}, P_r \rangle$, where predicate $P_r(x)$ means that x is a power of r (Brusten 2006)
- False for $\langle \mathbb{R}, +, \leq, \mathbb{Z}, V_r \rangle$

Cobham's theorem extended to $\mathbb{R}$

Theorem (Boigelot-Brusten-Bruyère 2008)

Let r, s be two bases that do not have the same set of prime factors. A set $X \subseteq \mathbb{R}$ is r- and s-recognizable iff X is ultimately periodically simple

- In other words, if X is definable in $\langle \mathbb{R}, +, \leq, \mathbb{Z}, V_r \rangle$ and $\langle \mathbb{R}, +, \leq, \mathbb{Z}, V_s \rangle$, then it is definable in $\langle \mathbb{R}, +, \leq, \mathbb{Z} \rangle$
- In other words, if a set X is recognizable by a Büchi automaton independently of the base, then it is recognizable by a weak deterministic Büchi automaton
- Theoretical justification to the use of weak deterministic automata as an effective symbolic representation of sets in the context of computer-aided verification (LASH tool)

Cobham's theorem extended to $\mathbb{R}$

Counterexample

$r = 6$ and $s = 12$

$X = \{x \in \mathbb{R} \mid \text{x can be encoded in base 6 as } u \star v0^\omega\}$

X is both 6- and 12-recognizable

X is not ultimately periodically simple

Intuition

- To have an infinite queue of zeros in base 6 is equivalent to have an infinite queue of zeros in base 12.
- X written in base 6 cannot be accepted by a weak Büchi automaton.

Theorem (Boigelot-Brusten-Bruyère 2009)

For any pair of bases r, s that have the same set of prime factors, the set $X = \{x \in \mathbb{R} \mid x \text{ can be encoded in base } r \text{ as } u \star v0^\omega\}$

- *is both r- and s-recognizable,*
- *but is not ultimately periodically simple.*

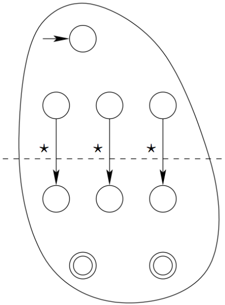
Main steps of the proof

Let r, s that do not have the same set of prime factors
Let $X \subseteq \mathbb{R}$ be a r- and s-recognizable set

1. Separate integer parts and fractional parts

We have $X = \bigcup_{i=1}^{n} Y_i + Z_i$ with

- $Y_i \subseteq \mathbb{Z}$ and $Z_i \subseteq [0, 1]$
- Y_i is r- and s-recognizable (finite-word automata)
- Z_i is r- and s-recognizable (Büchi automata)



By Cobham's theorem, each Y_i is first-order definable in $\langle \mathbb{Z}, +, \leq \rangle$
Thus, it is sufficient to prove each Z_i is a finite union of intervals with rational endpoints

Main steps of the proof

Let $X \subseteq [0, 1]$ be a r- and s-recognizable set.

2. Product stability

Definition
X is f-product-stable if for all x : $x \in X \Leftrightarrow f \cdot x \in X$

- r^j-product stability and s^k-product stability, for some $j, k \geq 1$
- In relation with some cycles in the automata

3. Sum stability

Definition
X is d-sum-stable if for all x : $x \in X \Leftrightarrow x + d \in X$

- Second use of Cobham's Theorem
- Very technical proof. Simpler proof?

Cobham's theorem extended to $\mathbb{R}^m$

Theorem (Boigelot-Brusten-Leroux 2009)

Let $m \geq 1$. Let r, s that do not have the same set of prime factors. A set $X \subseteq \mathbb{R}^m$ is r- and s-recognizable iff X is first-order definable in $\langle \mathbb{R}, +, \leq, \mathbb{Z} \rangle$.

$X \subseteq \mathbb{R}^m$ is first-order definable in $\langle \mathbb{R}, +, \leq, \mathbb{Z} \rangle$ iff

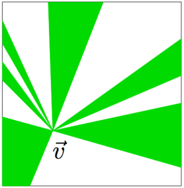
- X is a finite union of sets of the form $Y_i + Z_i$ where
- each $Y_i \subseteq \mathbb{Z}^m$ is first-order definable in $\langle \mathbb{Z}, +, \leq \rangle$
- each $Z_i \subseteq [0, 1]^m$ is first-order definable in $\langle \mathbb{R}, +, \leq, 1 \rangle$, i.e., is a boolean combination of linear constraints with rational coefficients

Boigelot-Brusten-Leroux's definability criterion like in (Muchnik 1991)

Cobham's theorem extended to $\mathbb{R}^m$

Main steps of the proof

- Separation of integer and fractional parts : $X = \bigcup_{i=1}^{n} Y_i + Z_i$
- By Semenov's theorem, each Y_i is first-order definable in $\langle \mathbb{Z}, +, \leq \rangle$
- For each Z_i,
  - r^j-product stability and s^k-product stability, for some $j, k \geq 1$
  - Conical structure of Z_i
  - Each face of $Z_i \cap [0,1]^m$ has dimension $m-1$ and is r-, s-recognizable



- By induction on the dimension, each Z_i is first-order definable in $\langle \mathbb{R}, +, \leq, 1 \rangle$

Conclusion and other related works

Conclusion

In this talk

- Extension of first Cobham's theorem to $\mathbb{Z}$, $\mathbb{N}^m$, $\mathbb{R}$ and $\mathbb{R}^m$
- Logical approach to find the right statements
- Precise description of the structure of automata, when the recognizability is independent of the base

Morphic approach

- Another extension of first's Cobham theorem for sets $X \subseteq \mathbb{N}$
- Orthogonal and beautiful extension
- See next slides

Morphic approach

Theorem (Cobham 1972)

A set $X \subseteq \mathbb{N}$ is r-recognizable iff its characteristic sequence is generated by the iteration of a r-uniform morphism, followed by a coding.

Example

$X = \{2^n \mid n \geq 0\}$

g :	$a \rightarrow ab$,	$b \rightarrow bc$,	$c \rightarrow cc$	2-uniform morphism
f :	$a \rightarrow 0$,	$b \rightarrow 1$,	$c \rightarrow 0$	coding

a
ab
$abbc$
$abbcbccc$
$abbcbcccbccccccc$
$abbcbcccbcccccccbccccccccccccccc$

$0110100010000000100000000000000 \cdots$

Morphic approach

Definition

A set $X \subseteq \mathbb{N}$ is α-recognizable if its characteristic sequence is generated by the iteration of a morphism g, followed by a coding f, such that $\alpha > 1$ is the dominating eigenvalue of the incidence matrix of g.

- Example Fibonacci morphism $g: \quad a \to ab, \quad b \to a$ with incidence matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and dominating eigenvalue $\frac{1+\sqrt{5}}{2}$

Theorem (Durand 2010)

Let α and β two mutiplicatively independent Perron numbers. A set $X \subseteq \mathbb{N}$ is α- and β-recognizable iff X is a finite union of constants and arithmetic progressions.

- See reference "F. Durand, M. Rigo, On Cobham's theorem, In Handbook of Automata Theory, in preparation, 39 pages"

Thank you . . .