Minimality conditions on automata

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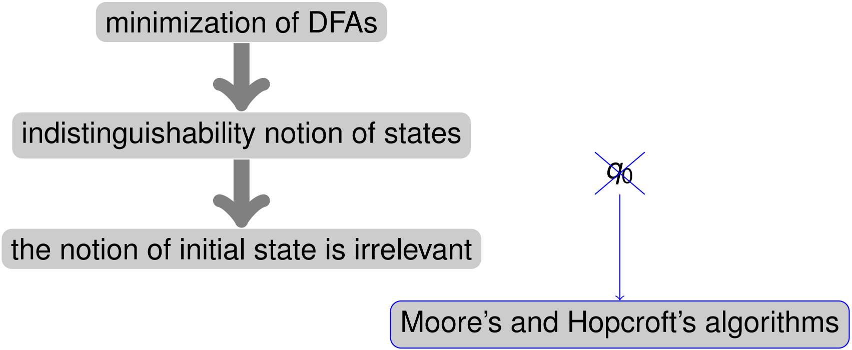
Abstract

- We investigate the “Dynamical Aspects of Automata minimality”. We are interested on how the choice of the final states can affect the minimality of the automata.

- A particular attention is devoted to the analysis of some extremal cases such as, for example, the automata that are minimal for any choice of final states (uniformly minimal automata) and the automata that are never minimal, under any assignment of final states (never-minimal automata).

Minimization of DFAs and role of q_0 (initial state)



minimization of DFAs

indistinguishability notion of states

the notion of initial state is irrelevant

$\cancel{q_0}$

Moore's and Hopcroft's algorithms

indistinguishable states

Let $\mathcal{A} = (Q, \Sigma, \delta)$ a DFA, $F \subseteq Q$ the set of final states and $\{p, q\} \subseteq Q$.

$$p \equiv q \Leftrightarrow \forall w \in \Sigma^* \ : \ \delta^*(p, w) \in F \text{ iff } \delta^*(q, w) \in F$$

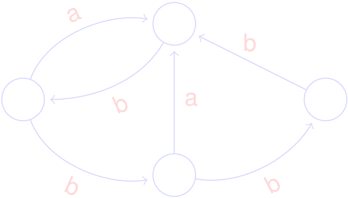
Objects of study

DFA:

- the initial state is not specified
- the set of final states is not specified
- strongly connected

↪ path from each vertex to every other vertex

$\mathcal{A} = (Q, \Sigma, \delta)$



synchronization problem and Černý's conjecture

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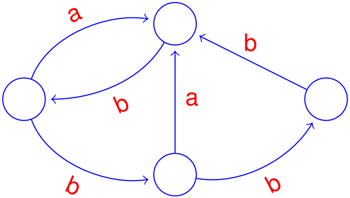


synchronization problem and Černý's conjecture

A useful tool for our investigation: the state-pair graph

$$A \longrightarrow \mathcal{G}(A)$$

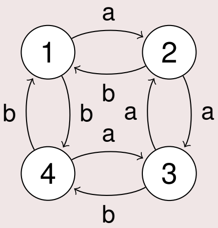
Definition

The state-pair graph of $\mathcal{A} = (Q, \Sigma, \delta)$ is the graph $G(\mathcal{A}) = (V_G, E_G)$ where:

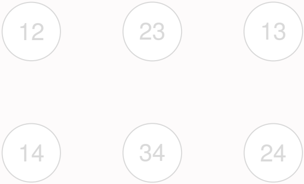
i. V_G consists of all not ordered pairs of distinct states of $\mathcal{A}$;

ii. $E_G = \{((p, q), (p', q')) \mid \delta(p, a) = p', \delta(q, a) = q' \text{ and } a \in \Sigma\}$.

Example

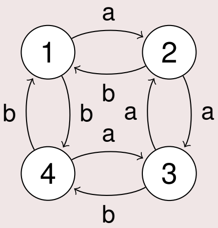


$\mathcal{A}$



$G(\mathcal{A})$

Example



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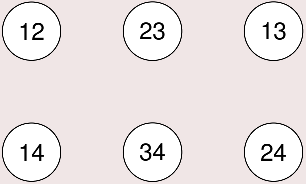

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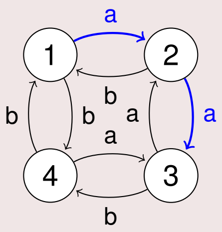

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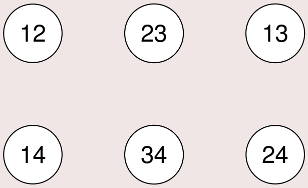

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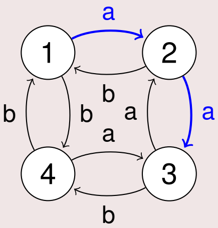

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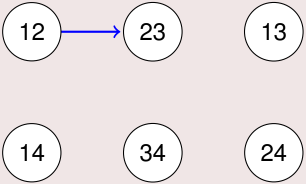

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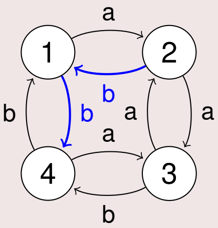

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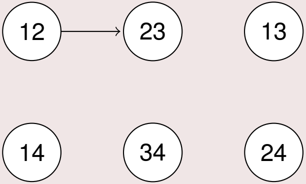

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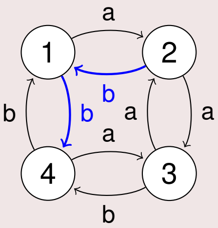

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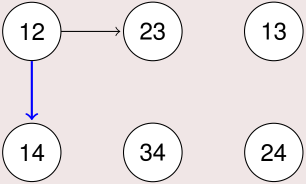

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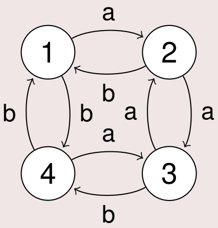

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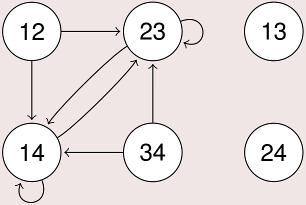


$G(\mathcal{A})$

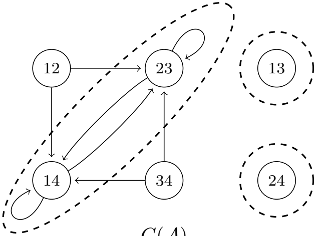
Notation and terminology

- $\mathcal{A} = (Q, \Sigma, \delta)$
- $\hat{\mathcal{A}}$: *completion* of $\mathcal{A}$
- $\mathcal{A}(i, F)$: DFA with initial state $i \in Q$ and $F \subseteq Q$ as set of final states
- $\mathcal{A}(i, F)$ is said to be *trim* if all its states are both *accessible* and *coaccessible*.

Closed components of a $G(\hat{\mathcal{A}})$

A *closed component* of a graph *G* is a subset *S* of the set of the vertices of *G* such that

- there exists a path from any element of *S* to any other element of *S* (i.e. *S* is a strongly connected component), and
- there is no outgoing edge from one element of *S* to a vertex of *G* which is not in *S*.



$G(\mathcal{A})$

To check the minimality of a DFA

$$\gamma_F : V_G \rightarrow \{B, W\}$$

$$\gamma_F(p, q) = \begin{cases} B & \text{if } p \in F \text{ and } q \notin F\text{, or vice versa;} \\ W & \text{otherwise.} \end{cases}$$

Theorem

Let $\mathcal{A} = (Q, \Sigma, \delta)$, $i \in Q$ and $F \subseteq Q$ such that $\mathcal{A}(i, F)$ is a trim DFA. Then $\mathcal{A}(i, F)$ is minimal iff in any closed component of $G(\hat{\mathcal{A}})$ there is at least an element v such that $\gamma_F(v) = B$.

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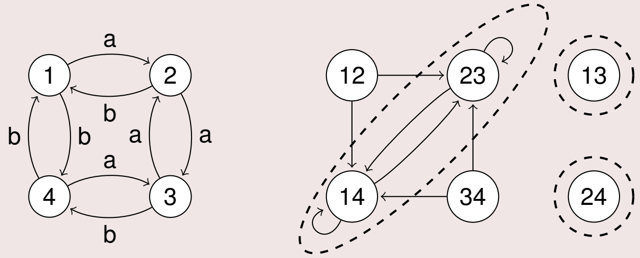

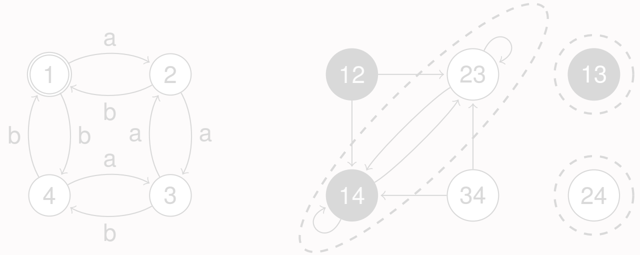

$F = \{1\}$

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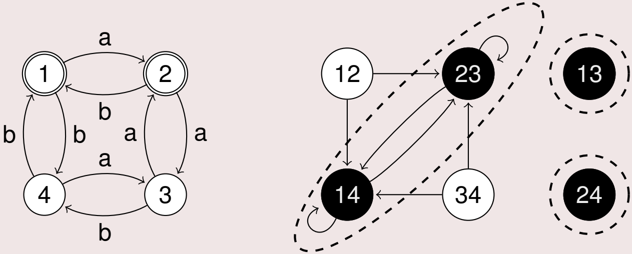
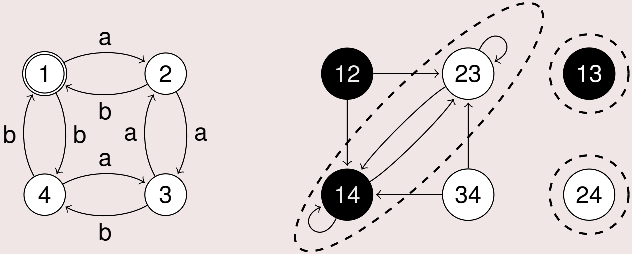


$F = \{1, 2\}$



$F = \{1\}$

A main question

Do there exist minimal automata whose minimality is not affected by the choice of the final states?

Remark

$\mathcal{A}(i, F)$ is trim for some $i \in Q$ and for all $F \subseteq Q$ if and only if $\mathcal{A}$ is strongly connected. Thus the above question makes sense only if we consider strongly connected automata.

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Uniformly minimal automata

Definition

A strongly connected automaton $\mathcal{A} = (Q, \Sigma, \delta)$ is called *uniformly minimal* if, for all $F \subseteq Q$, it is minimal.

Remark

If A is complete and $F = Q$, then A is minimal only if it corresponds to the trivial automaton with only one state. So a nontrivial uniformly minimal automaton is not complete.

Lemma

A strongly connected (incomplete) automaton $\mathcal{A}$ is uniformly minimal if and only if the only closed component of $G(\hat{\mathcal{A}})$ is $\{(q, s) \mid q \in Q$ and s is the sink state$\}$.

consequence

polynomial algorithm to test uniform minimality

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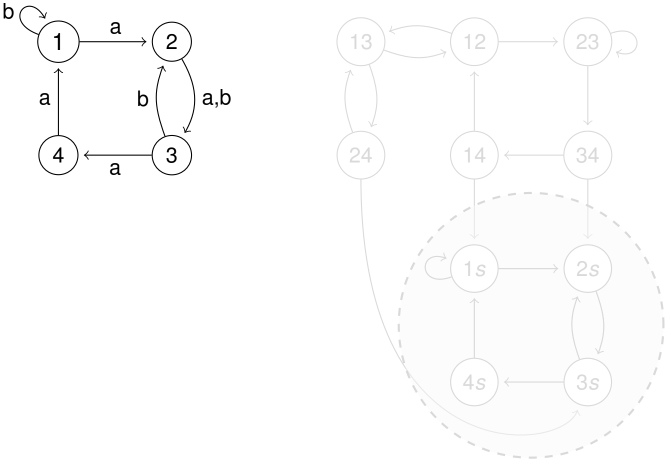

Figure: A uniformly minimal automaton $\mathcal{A}$ and the associated state-pair graph $G(\hat{\mathcal{A}})$.

Uniformly minimal automata

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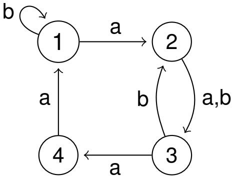


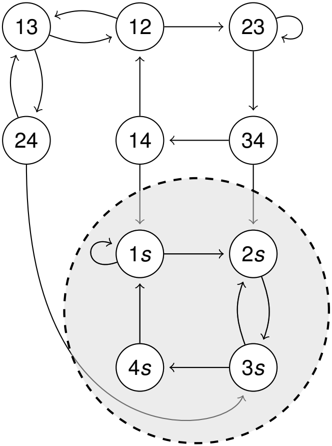


Figure: A uniformly minimal automaton $\mathcal{A}$ and the associated state-pair graph $G(\hat{\mathcal{A}})$.

Remark

Uniformly minimal automata are related to well-known objects in different contexts:

- *multiple*-entry DFAs
- Fisher covers of irreducible sofic shifts in Symbolic Dynamics

FA with a Limited Nondeterminism

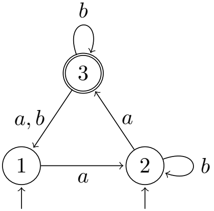
DFAs with multiple initial states (*multiple*-entry DFAs)

$\mathcal{A} = (Q, \Sigma, \delta)$
$I, F \subseteq Q$

$$\mathcal{A}(I, F) = (Q, \Sigma, \delta, I, F)$$

I set of initial states
F set of final states

If $\mid I \mid \leq k$, $\mathcal{A}(I, F)$ is called k-entry DFA.



Different notions of minimality

For an arbitrary regular language L, we have:

1. minimal DFA
2. minimal *multiple*-entry DFA
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More relevant,

- in general, minimal *multiple*-entry (resp. k-entry) DFAs are not unique, and
- the related minimization problems are computationally hard.

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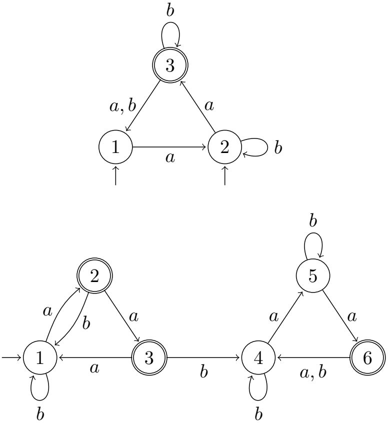
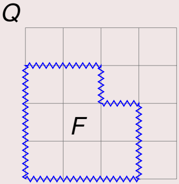


Figure: A 2-entry DFA and the corresponding minimal DFA.

Remark

DFA

The minimal DFA $\mathcal{A}$ recognizing a regular language L has a minimal number of final states.

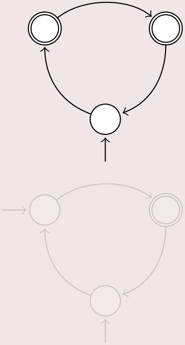


the Nerode equivalence $\sim_{\mathcal{A}}$ is the largest congruence saturating F

$$\forall \mathcal{A}' \ : \ L(\mathcal{A}') = L \ \rightarrow \ \sim_{\mathcal{A}} \leq \sim_{\mathcal{A}'}$$

k-entry DFA

$L \leftarrow$ unary string language whose length is not a multiple of 3



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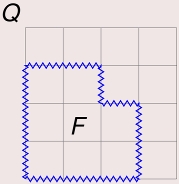

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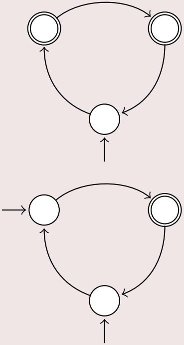

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Symbolic dynamics

Sofic shifts are recognized by finite automata where all states are both initial and final.
A sofic shift is irreducible if it is recognized by a strongly connected automaton.

In general, the minimal automaton for an arbitrary sofic shift is not unique. However, it is unique (up to the labeling of the states) in the case of an irreducible sofic shift L.

This minimal automaton (called Fisher cover) can be obtained from a strongly connected deterministic automaton recognizing L, by merging the indistinguishable states.

Theorem

Let $\mathcal{A} = (Q, \Sigma, \delta)$ a strongly connected DFA. The following conditions are equivalent:

1. *$\mathcal{A}(\{q\}, F)$ is minimal for some $q \in Q$ and for all $F \subseteq Q$, i.e. $\mathcal{A}$ is uniformly minimal.*
2. *$\mathcal{A}(\{q\}, F)$ is minimal for all $q \in Q$ and for all $F \subseteq Q$.*
3. *$\mathcal{A}(\{q\}, Q)$ is minimal for some $q \in Q$.*
4. *$\mathcal{A}(\{q\}, Q)$ is minimal for all $q \in Q$.*
5. *$\mathcal{A}(I, F)$ is $|I|$-entry minimal for all $I \subseteq Q$ and for all $F \subseteq Q$.*
6. *$\mathcal{A}(I, F)$ is multiple-entry minimal for all $I \subseteq Q$ and for all $F \subseteq Q$.*
7. *$\mathcal{A}(Q, Q)$ is the Fisher cover of some irreducible sofic shift.*
8. *$\mathcal{A}(Q, Q)$ is multiple-entry minimal.*

Scheme of the proof

① $\mathcal{A}(\{q\}, F)$ is minimal for some $q \in Q$ and for all $F \subseteq Q$.
② $\mathcal{A}(\{q\}, F)$ is minimal for all $q \in Q$ and for all $F \subseteq Q$.
③ $\mathcal{A}(\{q\}, Q)$ is minimal for some $q \in Q$.
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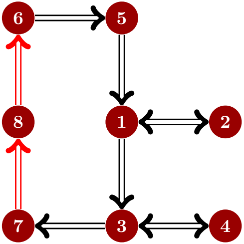

consequence

Uniformly minimal automata correspond to Fisher covers of irreducible sofic shifts in Symbolic Dynamics.

There are infinitely many uniformly minimal automata.

Almost uniformly minimal automata

A strongly connected DFA $\mathcal{A} = (Q, \Sigma, \delta)$ is *almost uniformly minimal* if, for all *proper* subsets $F \subset Q$, it is minimal.

Almost uniformly minimal automata

Theorem

For any integer $n \geq 2$ there exists a (complete) almost uniformly minimal DFA with n states.

$$\delta(i,a) = \begin{cases} \text{i+1}, & \text{if } 1 \leq i < n; \\ 1, & \text{if } i = n. \end{cases}$$

$$\delta(i,b) = \begin{cases} \text{i}, & \text{for } i \in \{1,n\}; \\ \text{i+1}, & \text{if } i = 2k \text{ for positive integers } k \leq \frac{n}{2} - 1; \\ \text{i-1}, & \text{if } i = 1 + 2k \text{ for positive integers } k \leq \frac{n}{2} - 1; \end{cases} \qquad n \text{ even}$$

$$\delta(i,b) = \begin{cases} \text{i}, & \text{for } i \in \{1,n\}; \\ \text{i}, & \text{if } i = 2k \text{ for integers } k \in [\frac{n+1}{4}, \frac{n+3}{4}]; \\ \text{i+1}, & \text{if } i = 2k \text{ for positive integers } k < \frac{n+1}{4}; \\ \text{i-1}, & \text{if } i = 1 + 2k \text{ for positive integers } k < \frac{n+1}{4}; \\ \text{i+1}, & \text{if } i = n - 2k \text{ for positive integers } k \leq \frac{n-3}{4}; \\ \text{i-1}, & \text{if } i = n + 1 - 2k \text{ for positive integers } k \leq \frac{n-3}{4}. \end{cases} \qquad n \text{ odd}$$

Almost uniformly minimal automata

Example

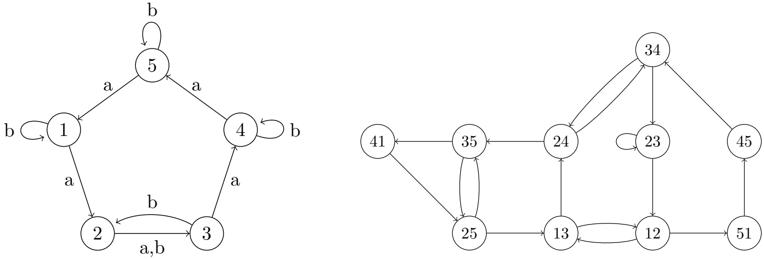

Figure: The automaton $\mathcal{M}_5$ and its state-pair graph (strongly connected).

Remark

If $G(\hat{\mathcal{A}})$ is strongly connected then, for all proper subsets $F \subset Q$, it has at least one vertex v such that $\gamma_F(v) = B$.

Almost uniformly minimal automata

On the complexity of the decisional problem

Remark

Almost uniformly minimal automata do not correspond to strongly connected DFAs which are minimal for all choices of the set of final states F with maximal cardinality.

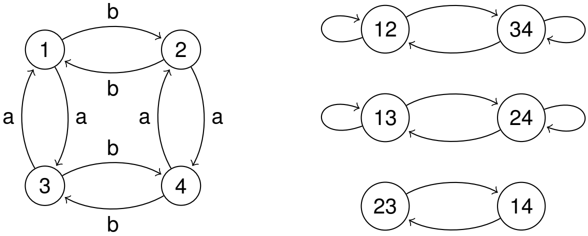

Figure: minimal for all F with $|F| = 3$, but not almost uniformly minimal

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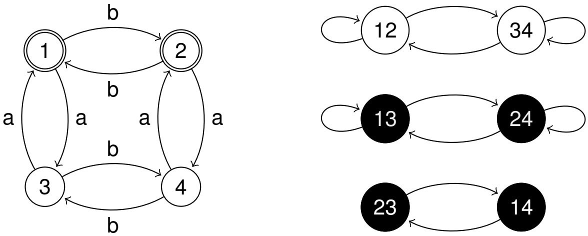

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Theorem

Let $\mathcal{A} = (Q, \Sigma, \delta)$ be a strongly connected DFA which is not uniformly minimal. $\mathcal{A}$ is almost uniformly minimal if and only if for any closed component S of $G(\hat{\mathcal{A}})$ and any pair of states $q, q' \in Q$ there exists a sequence $q_1, ..., q_t \in \hat{Q}$, with $t \geq 1$, such that $q = q_1, q_t = q'$ and $(q_i, q_{i+1}) \in S$, for $1 \leq i < t$.

consequence

polynomial algorithm to decide whether an automaton is almost uniformly minimal

Almost uniformly minimal automata

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consequence

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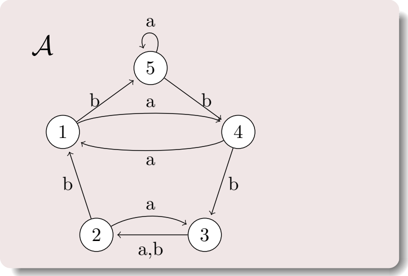
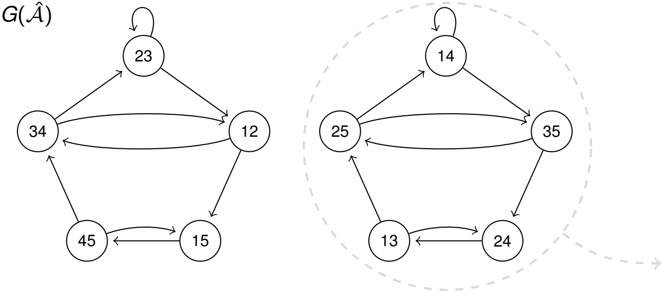
Almost uniformly minimal automata

Example





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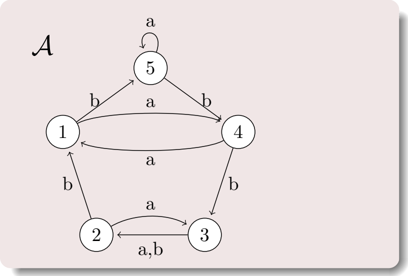
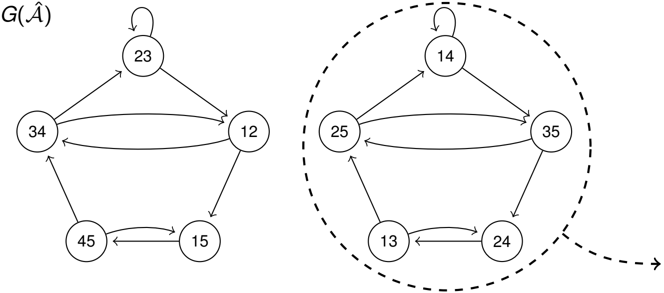


$G(\hat{\mathcal{A}})$



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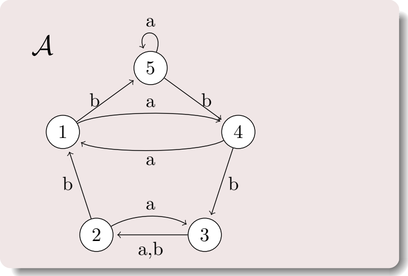

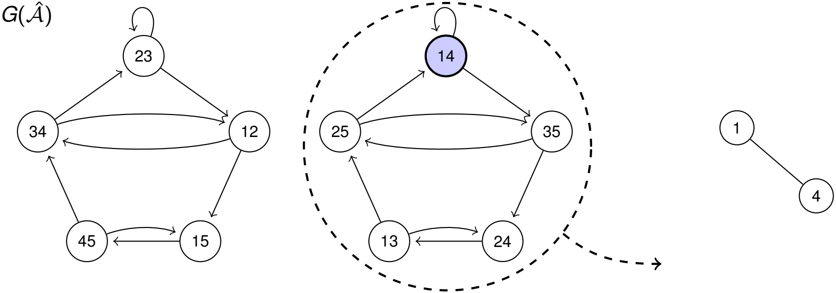

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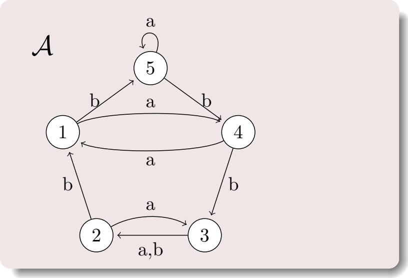

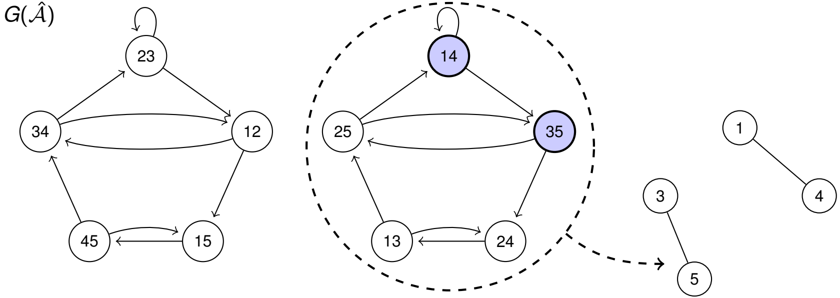

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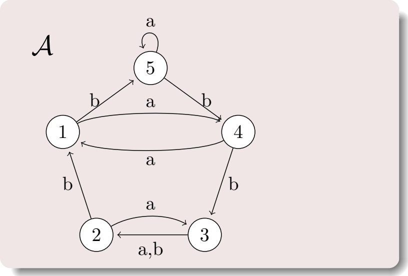

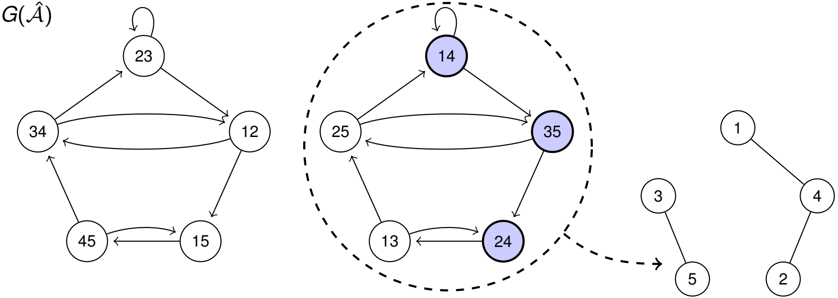

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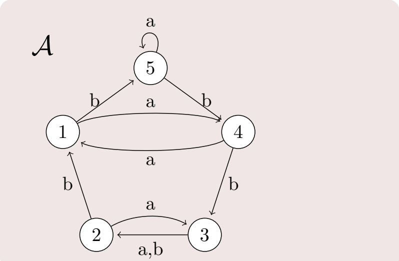

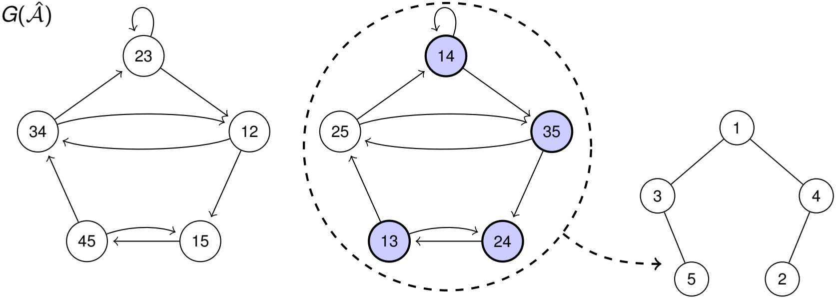

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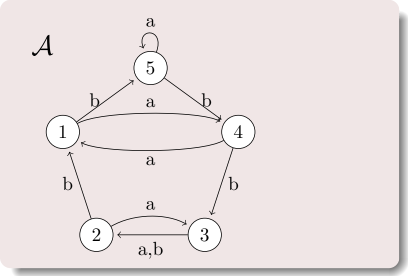

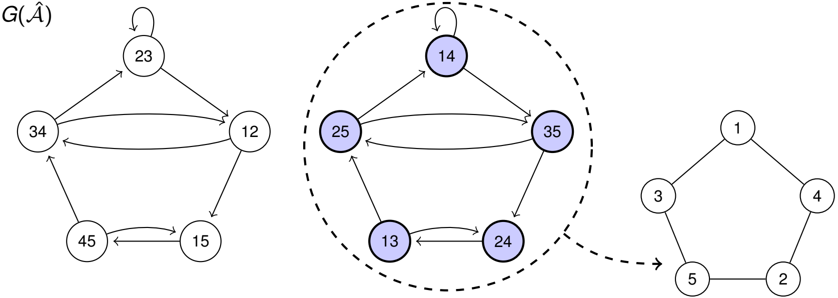

Never-minimal automata

Question

Do there exist strongly connected automata which aren't minimal for any choice of their final states?

We call *never-minimal* a DFA which isn't minimal for any choice of their final states.

Theorem

For any integer $n \geq 4$ there exists a never-minimal strongly connected DFA with n states.

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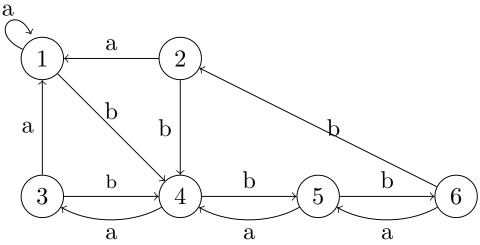
Proof:

$Q = \{1, 2, ..., n\}, \ \Sigma = \{a, b\}$

$$\delta(i,a) = \begin{cases} 1, & \text{if } i \leq 3 \\ \text{i-1}, & \text{if } 4 \leq i \leq n \end{cases} \qquad \delta(i,b) = \begin{cases} 4, & \text{if } i \leq 3 \\ \text{i+1}, & \text{if } 3 < i \leq n-1 \\ 2, & \text{if } i = n \end{cases}$$

12 23 13 have no outgoing edge

$1 \in F \Rightarrow 2 \notin F \Rightarrow 3 \in F \Rightarrow \gamma_F(1,3) = W.$



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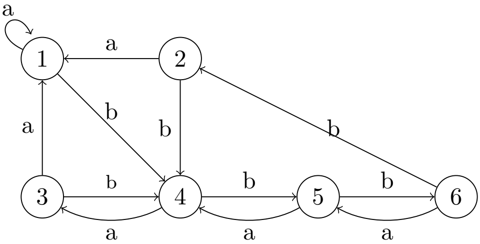

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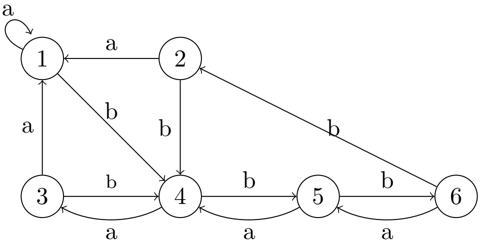

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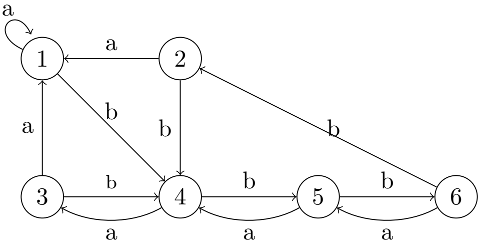

Never-minimal automata

A sufficient condition

Let $A = (Q, \Sigma, \delta)$ a *DFA* and $a \in \Sigma$:

$$\delta_a : Q \to Q$$

$$q \mapsto \delta(q, a)$$

Definition

We say that a DFA $\mathcal{A} = (Q, \Sigma, \delta)$ satisfies condition C_h if there is $Q_h \subseteq Q$, with $| Q_h | = h$, such that, for all $a \in \Sigma$, the restriction of δ_a to Q_h is a constant or an identity function.

Theorem

Let $\mathcal{A} = (Q, \Sigma, \delta)$ a DFA. If $\mathcal{A}$ satisfies C_3 then it is never-minimal.

Never-minimal automata

C_3 is not a necessary condition

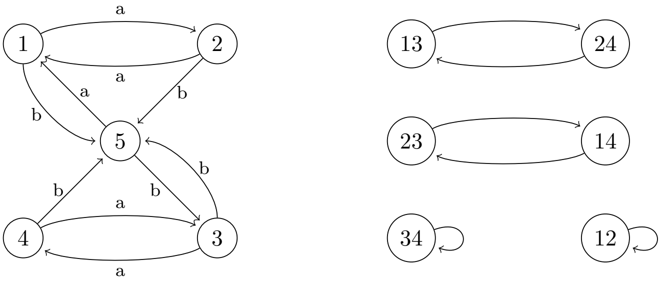

Figure: A never-minimal automaton $\mathcal{A}$ that doesn't satisfy condition C_3 and the closed components of $G(\hat{\mathcal{A}})$.

polynomial time algorithm for never-minimal DFA?

Never-minimal automata

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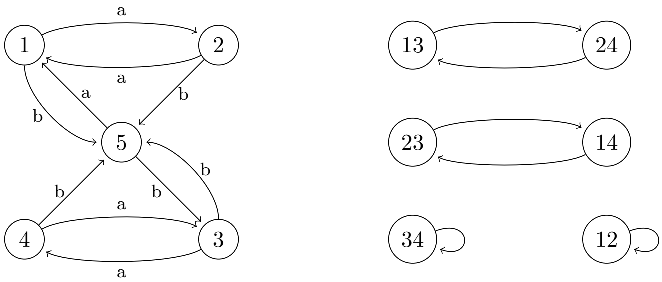

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Never-minimal automata

Relationships to the “syntactic monoid problem”

If M is a finite monoid and P a subset of M, there is a largest congruence σ_P saturating P defined by:

$$x \sigma_P y \;\Leftrightarrow\; \forall s, t \in M \; (sxt \in P \;\Leftrightarrow\; syt \in P).$$

The set P is called *disjunctive* if σ_P is the equality in M.
A monoid M is *syntactic* if it has a disjunctive subset.

Syntactic monoid problem

Instance: A finite monoid M
Question: is M syntactic?

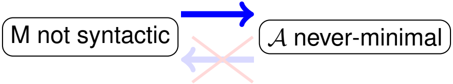
P. Goralcik, V. Koubek (98)

- Polynomial-time algorithm ($O(|M|^3)$) for the syntactic monoid problem for a large class of finite monoids.
- A slide generalization of syntactic monoid problem makes it *NP-complete*.
- ○ Is there any chance to have a polynomial-time algorithm for the “syntactic monoid problem” ?

Never-minimal automata

Relationships to the "syntactic monoid problem"

Let M be the transition monoid of a DFA $\mathcal{A}$.



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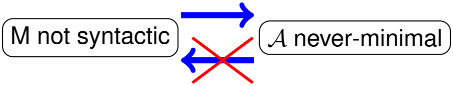

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Complete characterization for the automata over a unary alphabet

Strongly connected DFAs are *cyclic*.

Uniformly minimal automata

There do not exist nontrivial uniformly minimal automata.

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All vertices of the associated state-pair graphs are covered by disjoint cycles. Moreover, for each $q \in Q$ there is at least one vertex in any cyclic component of $G(\mathcal{A})$ that contains q. It follows that $\mathcal{A}$ is minimal for every choice of the set of final states F with $|F| = 1$.
$\Rightarrow$ There do not exist never-minimal automata.

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Almost uniformly minimal automata

Theorem

Let $\mathcal{A} = (Q, \{\sigma\}, \delta)$ be a cyclic DFA with $\mid Q \mid = n$. $\mathcal{A}$ is almost uniformly minimal if and only if n is a prime number.

Proof:

($\Leftarrow$) $n = hk, \quad F = \{q_1, ..., q_h\}$

$$\delta^*(q_i, \sigma^k) = \begin{cases} q_{i+1}, & \text{if } i \in \{1, ..., h-1\}; \\ q_1, & \text{if } i = h. \end{cases}$$

If $i \in F \Rightarrow L(\mathcal{A}) = \{w \mid |w| = k \cdot c, c \geq 0\} \Rightarrow \mathcal{A}(i, F)$ isn't minimal.

($\Rightarrow$) n prime, $|F| = m < n$. $L(\mathcal{A}(i, F))$, $\forall i$, is given by all words over $\{\sigma\}$ whose length belongs to the union of exactly m equivalence classes modulo n. Since n is prime, this set of integer numbers cannot be equal to the union of classes modulo different integers. Therefore $\mathcal{A}(i, F)$ is minimal.

Thank you for your attention!