# Building pretorsion theories from torsion theories

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**Definition:** Let $\mathbb{C}$ be an abelian category.
A pair $(\mathcal{T}, \mathcal{F})$ of full replete subcategories of $\mathbb{C}$ is a torsion theory if

- $\mathrm{Hom}(T, F) = 0$ for all $T \in \mathcal{T},\ F \in \mathcal{F}$;
- for every $X \in \mathbb{C}$ there exists a short exact sequence

$$0 \to T_X \to X \to F_X \to 0 \qquad \text{with } T \in \mathcal{T}, F \in \mathcal{F}.$$

**Example:**
$(\mathcal{T}, \mathcal{F})$ in the category Ab of abelian groups, where

- $\mathcal{T}$ = torsion groups;
- $\mathcal{F}$ = torsionfree groups

$$0 \longrightarrow t(G) \longrightarrow G \longrightarrow G/t(G) \longrightarrow 0 \qquad s.e.s$$

with $t(G)$ = torsion subgroup of $G$.

**Definition:** Let $\mathbb{C}$ be any pointed category.
A pair $(\mathcal{T}, \mathcal{F})$ of full replete subcategories of $\mathbb{C}$ is a torsion theory if

- $\mathrm{Hom}(T, F) = 0$ for all $T \in \mathcal{T},\ F \in \mathcal{F}$;
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**Two examples of non-abelian torsion theories**

- (NilCRng, RedCRng) in the category CRngs of commutative rings, where
  - NilCRng = nilpotent rings
  - RedCRng = reduced rings

$$0 \longrightarrow Nil(R) \longrightarrow R \longrightarrow R/Nil(R) \longrightarrow 0 \qquad s.e.s.$$

- (GrpInd, GrpHaus) in the category GrpTop of topological groups, where
  - GrpInd = groups with the indiscrete topology
  - GrpHauss = Hausdorff groups

$$0 \longrightarrow \overline{\{1\}} \longrightarrow G \longrightarrow G \,/\, \overline{\{1\}} \longrightarrow 0 \qquad s.e.s.$$

# Torsion theories in pointed categories

[Many authors in several contexts]

**Definition:** Let $\mathbb{C}$ be any pointed category.
A pair $(\mathcal{T}, \mathcal{F})$ of full replete subcategories of $\mathbb{C}$ is a torsion theory if

- $\mathrm{Hom}(T, F) = 0$ for all $T \in \mathcal{T}, \ F \in \mathcal{F}$;
- for every $X \in \mathbb{C}$ there exists a short exact sequence

$$0 \to T_X \to X \to F_X \to 0 \qquad \text{with } T \in \mathcal{T}, F \in \mathcal{F}.$$

**Definition:** Let $\mathbb{C}$ be any category.
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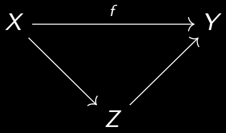
- $\mathrm{Hom}(T, F) =$ ?...? for all $T \in \mathcal{T}, \ F \in \mathcal{F}$;
- for every $X \in \mathbb{C}$ there exists ?...?...?...?...?

$$T_X \to X \to F_X \qquad \text{with } T \in \mathcal{T}, F \in \mathcal{F}.$$

Let $\mathbb{C}$ be any category.

- Consider two full replete subcategories $\mathcal{T}$ and $\mathcal{F}$ of $\mathbb{C}$.
- Define $\mathcal{Z} := \mathcal{T} \cap \mathcal{F}$, the class of trivial objects.
- We say that a morphism $X \xrightarrow{f} Y$ in $\mathbb{C}$ is $\mathcal{Z}$-trivial if it factors through an object in $\mathcal{Z}$:




$Z \in \mathcal{Z}$

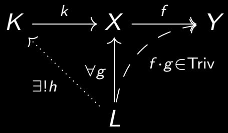
- The class of trivial morphisms forms an ideal of morphisms (denoted by Triv) in $\mathbb{C}$: if $f \in \mathrm{Triv}(A, B)$ or $g \in \mathrm{Triv}(B, C)$, then $g \cdot f \in \mathrm{Triv}(A, C)$.

**Remark:** if $\mathbb{C}$ is pointed and $\mathcal{Z} = \mathcal{T} \cap \mathcal{F} = 0$, then the ideal of trivial morphisms is the ideal of zero morphisms of $\mathbb{C}$.

A morphism $k\colon K \to X$ is a $\mathbb{Z}$-kernel of $f\colon X \to Y$ if

(i) $K \xrightarrow{k} X \xrightarrow{f} Y$ is $\mathbb{Z}$-trivial;

(ii) for any $g\colon L \to X$ such that $f \cdot g$ is trivial, there is a unique $h\colon L \to K$ such that $k \cdot h = g$




The notion of $\mathbb{Z}$-cokernel is defined dually. A sequence

$$W \xrightarrow{f} X \xrightarrow{g} Y$$

is a short $\mathbb{Z}$-exact sequence if $f$ is the $\mathbb{Z}$-kernel of $g$ and $g$ is the $\mathbb{Z}$-cokernel of $f$.

**Remark:** Any $\mathbb{Z}$-kernel is a monomorphisms and any $\mathbb{Z}$-cokernel is an epimorphisms.

**Definition:** Let $\mathbb{C}$ be any pointed category.
A pair $(\mathcal{T}, \mathcal{F})$ of full replete subcategories of $\mathbb{C}$ is a torsion theory if

- $\mathrm{Hom}(T, F) = 0$ for all $T \in \mathcal{T},\ F \in \mathcal{F}$;
- for every $X \in \mathbb{C}$ there exists a short exact sequence

$$0 \to T_X \to X \to F_X \to 0 \qquad \text{with } T \in \mathcal{T}, F \in \mathcal{F}.$$

**Definition:** Let $\mathbb{C}$ be any category.
A pair $(\mathcal{T}, \mathcal{F})$ of full replete subcategories of $\mathbb{C}$ is a pretorsion theory if

- $\mathrm{Hom}(T, F) = \mathrm{Triv}(T, F)$ for all $T \in \mathcal{T},\ F \in \mathcal{F}$;
- for every $X \in \mathbb{C}$ there exists a short $\mathcal{Z}$-exact sequence

$$T_X \to X \to F_X \qquad \text{with } T \in \mathcal{T}, F \in \mathcal{F}.$$

Given a pretorsion theory $(\mathcal{T}, \mathcal{F})$ in a category $\mathbb{C}$, there are two functors:

- a "torsion functor" $T \colon \mathbb{C} \to \mathcal{T}$ which is a left-inverse right-adjoint of the full embedding $E_T \colon \mathcal{T} \hookrightarrow \mathbb{C}$;
- a "torsion-free functor" $F \colon \mathbb{C} \to \mathcal{F}$ which is a left-inverse left-adjoint of the full embedding $E_F \colon \mathcal{F} \hookrightarrow \mathbb{C}$.

For every object $X \in \mathbb{C}$ there is a short $\mathcal{Z}$-exact sequence

$$T(X) \xrightarrow{\varepsilon_X} X \xrightarrow{\eta_X} F(X)$$

where the monomorphism $\varepsilon_X$ is the $X$-component of the counit $\varepsilon$ of the adjunction

$$\mathcal{T} \underset{T}{\overset{E_T}{\rightleftarrows}} \mathbb{C} \quad (E_T \dashv T)$$

while the epimorphism $\eta_X$ is the $X$-component of the unit $\eta$ of the adjunction

$$\mathbb{C} \underset{E_F}{\overset{F}{\rightleftarrows}} \mathcal{F} \quad (F \dashv E_F)$$

# Basic properties of pretorsion theories

- $X \in \mathcal{T} \iff T(X) \cong X$ and $Y \in \mathcal{F} \iff F(Y) \cong Y$.
- Two classes determine the third one, in the sense that:
  - if $\mathrm{Hom}(X, \mathcal{F}) = \mathrm{Triv}(X, \mathcal{F})$ then $X \in \mathcal{T}$ and
  - if $\mathrm{Hom}(\mathcal{T}, Y) = \mathrm{Triv}(\mathcal{T}, Y)$ then $Y \in \mathcal{F}$.
- $\mathcal{T}$ is closed under extremal quotients and $\mathcal{F}$ is closed under extremal monomorphisms.
- The three classes $\mathcal{T}, \mathcal{F}$ and $\mathcal{Z}$ are all closed under retracts.
- The initial object 0 is in $\mathcal{T}$, while the terminal object 1 is in $\mathcal{F}$ (if they exist).
  - In particular, if $\mathbb{C}$ is pointed, the zero object is in $\mathcal{Z}$.

## Some examples

**Objects:** sets endowed with a preorder $(A, \rho)$ (reflexive + transitive relation)

**Morphisms:** monotone maps between (preordered) sets.

There is a pretorsion theory $(\mathcal{T}, \mathcal{F})$ in PreOrd given by:

- $\mathcal{T}$ = equivalence relations (symmetric preorders)
- $\mathcal{F}$ = partial orders (antysimmetric preorders)
- $\mathcal{Z}$ = discrete relations (the "equality" relations).

The short $\mathcal{Z}$-exact sequence of an object $(A, \rho)$ is of the form

$$(A, \equiv) \xrightarrow{Id_A} (A, \rho) \xrightarrow{\pi} (A/\equiv, \leq)$$

where
· $a \equiv b$ if and only if $a\rho b$ and $b\rho a$; · $[a] \leq [b]$ if and only if $a\rho b$.

There is an isomorphism of categories:

$$\mathsf{PreOrd} \xrightarrow{\sim} \mathsf{AlexTop}$$

$$(A, \rho) \longmapsto (A, \tau_\rho)$$

where

- AlexTop is the category of Alexandrov-discrete spaces (arbitrary intersections of open sets is open).
- $\mathcal{O} \in \tau_\rho$ if and only if [ $x \in \mathcal{O}$ and $a \rho x \Rightarrow a \in \mathcal{O}$ ].

The corresponging pretorsion theory in AlexTop is (PartAlex, $T_0$)

- $\mathcal{T}$ = PartAlex = partition spaces (there exists a partition of the set which is a basis)
- $\mathcal{F} = T_0$ - spaces
- $\mathcal{Z}$ = discrete topological spaces

# Two generalizations

1. A pretorsion theory in the category Cat of all small categories [Xarez]:
   - $\mathcal{T}$ = "symmetric categories" $\quad \mathrm{Hom}(X, Y) \neq \emptyset \Rightarrow \mathrm{Hom}(Y, X) \neq \emptyset$
   - $\mathcal{F}$ = "antisymmetric categories" $\quad \mathrm{Hom}(X, Y) \neq \emptyset,\ \mathrm{Hom}(Y, X) \neq \emptyset \Rightarrow X = Y$
   - $\mathcal{Z}$ = classes of monoids $\quad$ (no morphisms between distinct objects)

2. A pretorsion theory in the category $\mathrm{PreOrd}(\mathbb{C})$ of internal preorders in an exact category [Facchini, Finocchiaro, Gran]:
   - $\mathcal{T} = \mathrm{Eq}(\mathbb{C})$ = equivalence relations in $\mathbb{C}$
   - $\mathcal{F} = \mathrm{ParOrd}(\mathbb{C})$ = partial orders in $\mathbb{C}$
   - $\mathcal{Z} = \mathrm{Dis}(\mathbb{C})$ = discrete relations in $\mathbb{C}$.

# Cat = category of small categories    [Borceux, —, Gran, Tholen]

There is another pretorsion theory in Cat:

- $\mathcal{T}$ = groupoids (every morphism is an isomorphism)
- $\mathcal{F}$ = skeletal categories (every isomorphism is an automorphism)
- $\mathcal{Z}$ = classes of groups (every morphism is an automorphism)

The short $\mathcal{Z}$-exact sequence of a category $\mathbb{C}$ is of the form

$$\mathsf{Iso}(\mathbb{C}) \longrightarrow \mathbb{C} \xrightarrow{Q} \mathbb{Q}$$

where the second functor is the following coequalizer in Cat

$$\coprod_{iso} 1 \underset{c}{\overset{d}{\rightrightarrows}} \mathbb{C} \xrightarrow{Q} \mathbb{Q}$$

# Stable category $\mathsf{Stab}(\mathbb{L})$ associated with a pretorsion theory

**Question:** is it possible to associate a torsion theory (in an universal way) to a given pretorsion theory $(\mathcal{T}, \mathcal{F})$ in a category $\mathbb{C}$?

The idea is to consider a congruence $\mathcal{R}$ on $\mathbb{C}$ and a quotient <u>pointed</u> category

$$\Sigma \colon \mathbb{C} \longrightarrow \mathbb{C}/\mathcal{R} =: \mathsf{Stab}(\mathbb{C})$$

"trivials" $\rightsquigarrow$ "zeros"

Pretorsion theory $\rightsquigarrow$ Torsion theory

# Torsion theory functor

Let $(\mathbb{A}, \mathcal{T}, \mathcal{F})_{pret}$ be a category $\mathbb{A}$ with a given pretorsion theory $(\mathcal{T}, \mathcal{F})$ in $\mathbb{A}$. If $(\mathbb{B}, \mathcal{T}', \mathcal{F}')_t$ is a pointed category $\mathbb{B}$ with a given torsion theory $(\mathcal{T}', \mathcal{F}')$ in it, we say that a torsion theory functor is a functor $G \colon \mathbb{A} \to \mathbb{B}$ satisfying the following two properties:

- $G(\mathcal{T}) \subseteq \mathcal{T}'$ and $G(\mathcal{F}) \subseteq \mathcal{F}'$;
- if $T_A \to A \to F_A$ is the canonical short $\mathcal{Z}$-exact sequence associated with $A \in \mathbb{A}$ in the pretorsion theory $(\mathcal{T}, \mathcal{F})$, then

$$0 \to G(T_A) \to G(A) \to G(F_A) \to 0$$

  is a short exact sequence in $\mathbb{B}$.

# Theorem [F. Borceux, —, M. Gran]

Let $(\mathcal{T}, \mathcal{F})$ be a pretorsion theory in a lextensive category $\mathbb{L}$ and assume that $\mathcal{T}$ is closed under complemented subobjects. Then, there exists a "**stable category**" $\mathsf{Stab}(\mathbb{L})$ and a **torsion theory functor** $\Sigma \colon \mathbb{L} \to \mathsf{Stab}(\mathbb{L})$ which is **universal** among all finite coproduct preserving *torsion theory functors* $G \colon \mathbb{C} \to \mathbb{X}$.

$$\begin{array}{ccc} \mathbb{L} & \xrightarrow{\ \Sigma\ } & \mathsf{Stab}(\mathbb{L}) \\ & {\scriptstyle \forall G} \searrow \quad \swarrow {\scriptstyle \exists ! H} & \\ & \mathbb{X}. & \end{array}$$

# Building pretorsion theories from torsion theories

## Comparable torsion theories [ —, Fedele ]:

Let $\mathbb{C}$ be a pointed category and consider two torsion theories $(\mathcal{T}_1, \mathcal{F}_1)$ and $(\mathcal{T}_2, \mathcal{F}_2)$ in it.

For $i = 1, 2$, let $T_i \colon \mathbb{C} \to \mathcal{T}_i$ and $F_i \colon \mathbb{C} \to \mathcal{F}_i$ denote respectively the torsion and torsion-free functors induced by the torsion theory $(\mathcal{T}_i, \mathcal{F}_i)$.

The following conditions are equivalent:

(i) $\mathcal{T}_2 \subseteq \mathcal{T}_1$ $(\mathcal{F}_1 \subseteq \mathcal{F}_2)$

(ii) $(\mathcal{T}_1, \mathcal{F}_2)$ is a pretorsion theory.

Moreover, if these conditions hold, then $\mathcal{T}_1 = \mathcal{T}_2 * \mathcal{Z}$ and $\mathcal{F}_2 = \mathcal{Z} * \mathcal{F}_1$, where $\mathcal{Z} := \mathcal{T}_1 \cap \mathcal{F}_2$, and the $\mathcal{Z}$-short exact sequence of an object $X \in \mathbb{C}$ is given by

$$T_1 X \longrightarrow X \longrightarrow F_2 X$$

*Notice: no hypothesis are required for* $\mathbb{C}$ *or the torsion theories.*

# Comparable torsion theories: example 1

Let $R$ be a unital commutative ring and $S \subseteq R$ a multiplicatively closed subset ($1 \in S$ and $r, s \in S \Rightarrow r \cdot s \in S$).

There is a torsion theory $(\mathcal{T}_S, \mathcal{F}_S)$ in $\mathrm{Mod}(R)$ where $M \in \mathcal{T}_S$ iff $M \otimes_R S^{-1}R = 0$.

Explicitly, $M \in \mathcal{T}_S$ if, for every $m \in M$, there exists $s \in S$ such that $sm = 0$, while $M \in \mathcal{F}_S$ if there are no non-zero elements of $M$ annihilated by elements of $S$.

Any inclusion $S \subseteq T$ of multiplicatively closed subsets of $R$ induces a pretorsion theory $(\mathcal{T}_T, \mathcal{F}_S)$ where the class $\mathcal{Z}$ of trivial objects consists of those modules $M$ with the following property: for every non-zero $m \in M$ we have $\mathrm{Ann}_R(m) \cap T \neq \emptyset$ and $\mathrm{Ann}_R(m) \cap S = \emptyset$.

As a particular case of what we have just seen, any inclusion of prime ideals induces a pretorsion theory, since the complement of a prime ideal is a multiplicatively closed set.

# Comparable torsion theories: example 1

Let $R$ be a domain of infinite Krull dimension and consider an infinite chain of prime ideals

$$0 = P_0 \subsetneq P_1 \subsetneq P_2 \subsetneq \dots$$

This chain induces a chain of torsion theories $(\mathcal{T}_i, \mathcal{F}_i)$, with $\mathcal{T}_0 \supsetneq \mathcal{T}_1 \supsetneq \mathcal{T}_2 \supsetneq \dots$

Thus we have pretorsion theories $(\mathcal{T}_0, \mathcal{F}_i)$, where $\mathcal{T}_0$ is the subcategory of "classical" torsion modules, while $N \in \mathcal{F}_i$ iff for every $n \in N$, $\mathrm{Ann}_R(n) \subseteq P_i$.

**Conclusion:**

A subcategory $\mathcal{T}$ of a given category $\mathbb{C}$ can be the torsion class of (possibly infinitely) many different pretorsion theories.

## Comparable torsion theories: example 2 (suggested by Marino Gran)

Let $\mathbb{C}$ be an homological category and consider Grpd($\mathbb{C}$). There are two (comparable) torsion theories:

$$( \text{Ab}(\mathbb{C}) , \text{Eq}(\mathbb{C}) ) \qquad \text{and} \qquad ( \text{connected groupoids} , \mathbb{C} )$$

which then gives us a pretorsion theory

$$( \text{connected groupoids} , \text{Eq}(\mathbb{C}) )$$

**One last remark:**

Not all pretorsion theories arise in this way.

## Extension with a Serre subcategory [ — , Fedele ]:

- Let $\mathbb{C}$ be a pointed category where every morphism admits an (epi, mono)-factorization, and assume that $\mathbb{C}$ has pullbacks and pushouts which preserve normal epimorphisms and normal monomorphisms respectively.
- Let $\mathcal{S}$ be a Serre epireflective and monocoreflective subcategory of $\mathbb{C}$.
- Let $(\mathcal{U}, \mathcal{V})$ be a torsion theory in $\mathbb{C}$.

**Then**

the pair $(\mathcal{T}, \mathcal{F}) = (\mathcal{U} * \mathcal{S}, \mathcal{S} * \mathcal{V})$ is a pretorsion theory with class of trivial objects $\mathcal{S}$.

The short $\mathcal{S}$-exact sequence is given by

$$\begin{array}{ccccccccc}
0 & \longrightarrow & U_X & \longrightarrow & T_X & \longrightarrow & S_X & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \varepsilon_X} & & \downarrow & & \\
0 & \longrightarrow & U_X & \longrightarrow & X & \longrightarrow & V_X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \eta_X} & & \| & & \\
0 & \longrightarrow & S'_X & \longrightarrow & F_X & \longrightarrow & V_X & \longrightarrow & 0
\end{array}$$

Thank you