L-algebras, normality and exact completions

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Outline

Normal categories

The quasi-variety of L-algebras

Exact completion

Commutator of ideals

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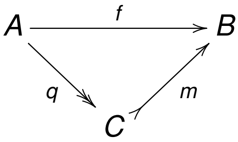
Definition

Let $\mathbb{C}$ be a finitely complete category with a zero object 0. Then $\mathbb{C}$ is a normal category if

- any morphism $f \colon A \to B$ in $\mathbb{C}$ admits a factorization $f = m \cdot q$, where q is a normal epimorphism and m is a monomorphism :



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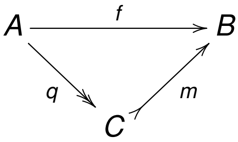

- normal epimorphisms are stable under pullbacks.

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In all these categories the Split Short Five Lemma holds.

A preordered group $(G, \leq, +)$ is a group G endowed with a preorder relation $\leq$ on G that is "compatible" with the group operation $+$:

$$[a \leq c, \text{ and } b \leq d] \Rightarrow [a + b \leq c + d].$$

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PreOrdGrp is a normal category that is not homological, as shown by M.M. Clementino, N. Martins-Ferreira and A. Montoli, 2019.

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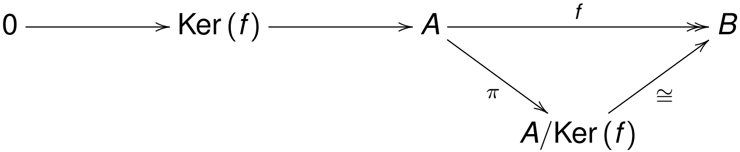
Noether's Isomorphism Theorems

The first isomorphism theorem of group theory holds in any normal category $\mathbb{C}$:
for any regular epimorphism $f \colon A \to B$,



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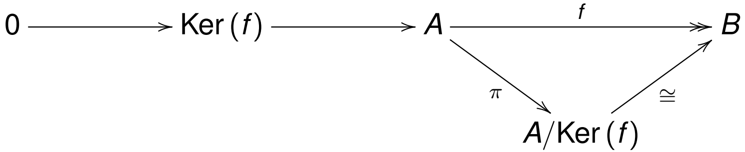

there is an isomorphism $B \cong \frac{A}{\mathrm{Ker}\,(f)}$.

This uses the property that $f \colon A \to B$ is monomorphism if and only if $\mathrm{Ker}\,(f) = 0$.

The double quotient isomorphism theorem also holds in any normal category $\mathbb{C}$ (T. Everaert and M. Gran, 2013) : given normal subobjects $K \subset L \subset A$ of an $A \in \mathbb{C}$, there is an isomorphism

$$A/L \cong \frac{A/K}{L/K}.$$

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The Zassenhaus Lemma, used in the proof of the Jordan-Hölder theorem, also holds in any normal category (O. Ngaha and F. Sterck, 2019).

Outline

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Definition (W. Rump, 2008)

An L-algebra is a set X with a binary operation $\cdot$ and a 0-ary operation 1 such that

$$x \cdot x = x \cdot 1 = 1, \; 1 \cdot x = x, \tag{1}$$

$$(x \cdot y) \cdot (x \cdot z) = (y \cdot x) \cdot (y \cdot z), \tag{2}$$

$$x \cdot y = y \cdot x = 1 \implies x = y \tag{3}$$

for every $x, y, z \in X$.

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for every $x, y, z \in X$.

Remark

The identity (2) holds in most generalizations of classical logic, including intuitionistic and many valued logic ("$\cdot$" can represent an implication "$\to$").

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These include Heyting algebras, Boolean algebras, MV-algebras and other algebraic structures in logic.

Boolean algebras

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MV-algebras

In 2005 Gispert and Mundici characterised MV-algebras as commutative monoids $(M, \cdot, 1)$ with an involution $(\,)' \colon M \to M$ (the "negation") such that $0 = 1'$ satisfies $x \cdot 0 = 0$ and

$$x \cdot (x \cdot y')' = y \cdot (y \cdot x')'.$$

Rump proved that the operation $x \to y = (x \cdot y')'$ defines an L-algebra.

The constant 1, called the logical unit, is the unique element with the property that

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If we think of $\cdot$ as an "implication" these identities become

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where 1 can be interpreted as "truth".

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The trivial algebra $\{1\}$ is the zero object of LAlg.

Let us show that any surjective homomorphism is the cokernel of its kernel.

First observe that the terms $t_1(x, y) = x \cdot y$ and $t_2(x, y) = y \cdot x$ satisfy

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Let $A \xrightarrow{f} B$ be a surjective homomorphism, $K \xrightarrow{k} A$ its kernel, and $A \xrightarrow{g} C$ any morphism such that $g \circ k = 1$

$$\begin{array}{ccccc} K & \xrightarrow{k} & A & \xrightarrow{f} & B \\ \vdots & & \downarrow{\scriptstyle g} & & \\ \{1\} & \cdots\cdots\to & C & & \end{array}$$

For any $b \in B$ there is an $a \in A$ such that $f(a) = b$. Let us show that by setting

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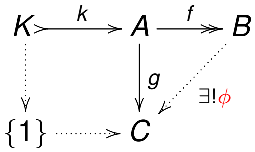
Indeed, let a and a' be such that $f(a) = f(a')$. Then, for any $i \in \{1, 2\}$,

$$f(t_i(a, a')) = t_i(f(a), f(a')) = t_i(f(a), f(a)) = 1,$$

hence $t_i(a, a') \in K$.

This implies that $t_i(g(a), g(a')) = g(t_i(a, a')) = 1$, so that $g(a) = g(a')$ by (3).

In the commutative diagram



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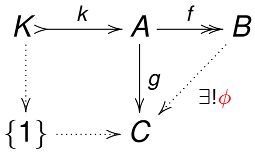

the homomorphism ϕ such that $\phi \circ f = g$ is unique by the surjectivity of f.

It follows that $f = \text{coker}(k)$, and f is then a normal epimorphism.
Accordingly, LAlg is a normal category. $\square$

The quasivariety LAlg is a full subcategory of the variety PreLAlg of pre-*L*-algebras (also called *unital cycloids* in the literature), determined by

$$x \cdot x = x \cdot 1 = 1, \; 1 \cdot x = x, \tag{1}$$

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There is then an adjunction

$$\mathrm{LAlg} \underset{U}{\overset{F}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathrm{PreLAlg},$$

where the reflection $A \xrightarrow{\eta_A} UF(A) = \frac{A}{\sim}$ of a pre-L-algebra A is a quotient, with

$$(x, y) \in \sim \Leftrightarrow x \cdot y = 1 = y \cdot x.$$

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Exact completion of a regular category

What is the relationship between the regular category LAlg and the exact category PreLAlg in terms of the exact completion $\mathbb{C}_{\text{ex/reg}}$ of a regular $\mathbb{C}$?

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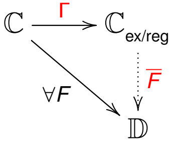
A functor $F\colon \mathbb{C} \to \mathbb{D}$ between regular categories is regular if it preserves finite limits and regular epimorphisms.

This functor $\Gamma \colon \mathbb{C} \to \mathbb{C}_{\text{ex/reg}}$ satisfies the following universal property : for any regular functor $F \colon \mathbb{C} \to \mathbb{D}$



there is an essentially unique regular functor $\overline{F} \colon \mathbb{C}_{\text{ex/reg}} \to \mathbb{D}$ with $\overline{F} \circ \Gamma \cong F$.

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Note that the exact completion of a normal category is not normal, in general (M. Gran and Z. Janelidze, 2014). We observe that a new example is given here by the variety $\mathsf{PreLAlg} = \mathsf{LAlg}_{ex/reg}$.

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The quotient $\frac{X}{\sim}$ of a pre-L-algebra X by the congruence $\sim$ defined by $(x, y) \in \sim$ if and only if $x \cdot y = 1 = y \cdot x$ "forces" the quasivariety LAlg to be normal.

A property that is “stable” under the exact completion is the Mal’tsev property.

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Indeed, one can show that a regular $\mathbb{C}$ is a Mal’tsev category if and only if $\mathbb{C}_{\text{ex/reg}}$ is a Mal’tsev category.

Given a regular Mal'tsev category $\mathbb{C}$, consider a reflexive relation

$$R \underset{c}{\overset{d}{\rightrightarrows}} A \quad (e \colon A \to R)$$

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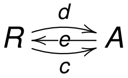
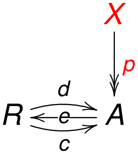
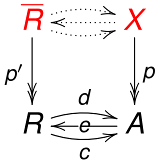


in $\mathbb{C}_{\text{ex/reg}}$,

there is a regular epimorphism $p\colon X \to A$ with $X \in \mathbb{C}$:

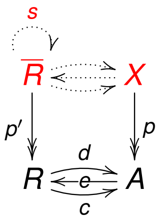


It is then easy to complete the diagram



with $\overline{R}$ a reflexive relation on X in $\mathbb{C}$, p and p' regular epis.

Since $\mathbb{C}$ is a Mal'tsev category, $\overline{R}$ is a symmetric relation in $\mathbb{C}$:



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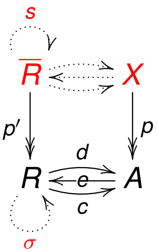

This implies that R is a symmetric relation in $\mathbb{C}_{ex/reg}$ as well.

The fact that

$$[\ \mathbb{C} \text{ is a Mal'tsev category }] \text{ if and only if } [\mathbb{C}_{\text{ex/reg}} \text{ is a Mal'tsev category}]$$

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Similar results can be proved for other exactness properties, such as :
- protomodularity (M. Gran and S. Lack, 2014)
- subtractivity and unitality (M. Gran and D. Rodelo, 2012).

Both LAlg and PreLAlg are clearly subtractive (A. Ursini, 1994), with subtractive term $s(x, y) = y \cdot x$:

$$s(x, 1) = 1 \cdot x = x, \qquad s(x, x) = x \cdot x = 1.$$

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However, the quasivariety of LAlg is not a Mal’tsev category, as we now explain.

Consider the two element L-algebra X whose multiplication is defined by

$\cdot$	0	1
0	1	1
1	0	1

The relation $R = \{(0,1),(1,0),(1,1)\}$ on X is a subalgebra of the L-algebra $X \times X$.

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The kernel pairs $Eq(p_1)$ and $Eq(p_2)$ of the projections

$$X \xleftarrow{p_1} R \xrightarrow{p_2} X$$

do not permute : $Eq(p_1) \circ Eq(p_2) \neq Eq(p_2) \circ Eq(p_1)$.

Indeed,

$$Eq(p_1) = \{((0,1),(0,1)),((1,0),(1,0)),((1,1),(1,1)),((1,0),(1,1)),((1,1),(1,0))\}$$

and

$$Eq(p_2) = \{((0,1),(0,1)),((1,0),(1,0)),((1,1),(1,1)),((0,1),(1,1)),((1,1),(0,1))\}.$$

Indeed,

$$Eq(p_1) = \{((0,1),(0,1)),((1,0),(1,0)),((1,1),(1,1)),((1,0),(1,1)),((1,1),(1,0))\}$$

and

$$Eq(p_2) = \{((0,1),(0,1)),((1,0),(1,0)),((1,1),(1,1)),((0,1),(1,1)),((1,1),(0,1))\}.$$

Accordingly,

$$(1,0)Eq(p_1)(1,1)Eq(p_2)(0,1)$$

showing that

$$((1,0),(0,1)) \in Eq(p_2) \circ Eq(p_1).$$

However,

$$((1,0),(0,1)) \notin Eq(p_1) \circ Eq(p_2), \quad \textit{since} \quad (0,0) \notin R$$

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Remark

It is interesting to note that both LAlg and PreLAlg are "permutable at 1", this meaning that

$$(x, 1) \in S \circ R \Leftrightarrow (x, 1) \in R \circ S,$$

for any pair of congruences R and S on the same algebra.

Indeed, consider the "subtractive" term $s(x, y) = y \cdot x$.
If there is y such that $xRyS1$, then

$$x = 1 \cdot x = s(x, 1)\, S\, s(x, y)\, R\, s(y, y) = y \cdot y = 1, \quad \textit{and} \quad (x, 1) \in R \circ S.$$

Outline

Commutator of ideals

A subset I of a pre-L-algebra X is an ideal of X if

$$\begin{aligned} &1 \in I, \\ &x \in I \text{ and } x \cdot y \in I \implies y \in I, \\ &x \in I \implies (x \cdot y) \cdot y \in I, \\ &x \in I \implies y \cdot x \in I, \\ &x \in I \implies y \cdot (x \cdot y) \in I \end{aligned}$$

for every $x, y \in X$.

Ideals of a pre-L-algebra X correspond to equivalence relations R on X such that the quotient X/R is in LAlg.

The correspondence between equivalence relations in PreLAlg and ideals is the following :

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given an ideal I of X, the associated equivalence relation $\sim$ is defined by

$$(x, y) \in \sim \quad \Leftrightarrow \quad (x \cdot y \in I) \wedge (y \cdot x \in I).$$

Commutator of ideals

Let X be an L-algebra and I, J two ideals of X. Define their commutator $[I, J]$ as the smallest ideal of X for which the multiplication $\cdot$ in X, i.e., the mapping

$$\mu \colon I \times J \to X/[I, J]$$

$$\mu(i, j) = [i \cdot j]^{[I,J]}$$

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For every pair I, J of ideals of an L-algebra X, one has

$$[I, J] = I \cap J.$$

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is an L-algebra morphism.

Proposition

For every pair I, J of ideals of an L-algebra X, one has

$$[I, J] = I \cap J.$$

Proof

This will follow from the fact that, for any $x \in I \cap J$, the equivalence class $[x]^{[I,J]} = [x]$ is the neutral element in the quotient $X/[I, J] : [x] = [1]$.

Indeed, for any $i \in I$, $j \in J$, $x \in I \cap J$ one has the equality

$$([x] \cdot [x]) \cdot ([i] \cdot [j]) = ([x] \cdot [i]) \cdot ([x] \cdot [j]).$$

Indeed, for any $i \in I$, $j \in J$, $x \in I \cap J$ one has the equality

$$([x] \cdot [x]) \cdot ([i] \cdot [j]) = ([x] \cdot [i]) \cdot ([x] \cdot [j]).$$

By choosing $i = 1$ and $j = x$ we get

$$([x] \cdot [x]) \cdot ([1] \cdot [x]) = ([x] \cdot [1]) \cdot ([x] \cdot [x]),$$

from which it follows that $[x] = [1]$, and $I \cap J = [I, J]$ as desired. $\square$

The result that $I \cap J = [I, J]$ implies that the only abelian object is 0, since $[A, A] = 0$ implies that $A \cap A = A = 0$.

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It would be interesting to investigate - from a categorical perspective - the commutator theory of congruences in *relatively modular quasivarieties* and in *relatively distributive quasivarieties*, such as LAlg.

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