CAUCHY COMPLETENESS FOR NORMED CATEGORIES

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Based on joint work with Maria Manuel Clementino and Walter Tholen.

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1. Background

ONCE AGAIN LAWVERE (1973)

Reference

LAWVERE, F. WILLIAM (1973). **“Metric spaces, generalized logic, and closed categories”.** In: *Rendiconti del Seminario Matemàtico e Fisico di Milano* **43**.(1), pp. 135–166. Republished in: Reprints in Theory and Applications of Categories, No. 1 (2002), 1–37.

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Neeman (2020)

> *“... To the best of my knowledge the myriad applications have essentially all gone in directions totally different from the one we will be pursuing in this note. There is only a handful of exceptions ...”*

Neeman, Amnon (2020). **“Metrics on triangulated categories”.** In: *Journal of Pure and Applied Algebra* **224**.(4), p. 106206.

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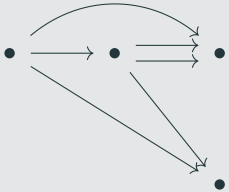
LAWVERE, F. WILLIAM (1973). **"Metric spaces, generalized logic, and closed categories".** In: *Rendiconti del Seminario Matemàtico e Fisico di Milano* **43**.(1), pp. 135–166. Republished in: Reprints in Theory and Applications of Categories, No. 1 (2002), 1–37.

The notion of «normed category» ...

Arrows have "lengths"



subject to

$$0 \geqslant |1|, \quad |f| + |g| \geqslant |gf|.$$

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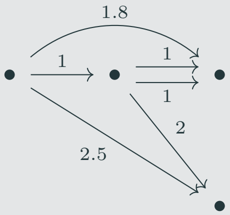

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An example

Lawvere (1973)

The notion of «normed category» can also be related to the (non-symmetric) Hausdorff metric

$$2^X(A, B) = \sup_{a \in A} \inf_{b \in B} X(a, b) \ \ldots$$

Example

Let X be a metric space. Define the normed category $H(X)$ as follows.

- Objects: subsets of X.
- An arrow $f : A \longrightarrow B$ in $H(X)$ is a map from A to B, with “norm”

$$|f| = \sup_{a \in A} X(a, fa).$$

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Some references

- Kubiś, Wiesław (2017). ***Categories with norms.*** Tech. rep. arXiv: `1705.10189 [math.CT]`.
- Lawvere, F. William (1973). **“Metric spaces, generalized logic, and closed categories”.** In: *Rendiconti del Seminario Matemàtico e Fisico di Milano* **43**.(1), pp. 135–166. Republished in: Reprints in Theory and Applications of Categories, No. 1 (2002), 1–37.
- Luckhardt, Daniel and Insall, Matt (2021). ***Norms on Categories and Analogs of the Schröder-Bernstein Theorem.*** Tech. rep. math.CT: `2105.06832` (arXiv).
- Neeman, Amnon (2020). **“Metrics on triangulated categories”.** In: *Journal of Pure and Applied Algebra* **224**.(4), p. 106206. arXiv: `1901.01453 [math.CT]`.

Definition

A **normed category** $\mathbb{X}$ is an ordinary category with (small) normed hom-sets

$$|-|\colon \mathbb{X}(x,y) \longrightarrow [0,\infty]$$

so that

$$0 \geqslant |1_x|, \quad |f| + |g| \geqslant |gf|.$$

A functor $F\colon \mathbb{X} \longrightarrow \mathbb{Y}$ is **normed** whenever

$$|f| \geqslant |Ff|.$$

Remark (Lawvere (1973))

We will leave as an exercise for the reader to define a closed category $\mathcal{S}(\mathbf{R})$ such that «normed categories» are just $\mathcal{S}(\mathbf{R})$-valued categories and a «closed functor» $\inf\colon \mathcal{S}(\mathbf{R}) \longrightarrow \mathbf{R}$ which induces the passage from any «normed category» to a metric space with the same objects.

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Definition

A **$\mathcal{V}$-normed category** $\mathbb{X}$ is an ordinary category with (small) normed hom-sets

$$|-| \colon \mathbb{X}(x,y) \longrightarrow \mathcal{V}$$

so that

$$k \le |1_x|, \quad |f| \otimes |g| \le |gf|.$$

A functor $F \colon \mathbb{X} \longrightarrow \mathbb{Y}$ is **$\mathcal{V}$-normed** whenever

$$|f| \le |Ff|.$$

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NORMED SETS

Definition

A **$\mathcal{V}$-normed** set is a set A that comes with a function $|-|\colon A \longrightarrow \mathcal{V}$, and a **$\mathcal{V}$-normed map** $(A, |-|) \longrightarrow (B, |-|)$ is a mapping $f\colon A \longrightarrow B$ satisfying $|a| \leq |fa|$ for all $a \in A$.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "|-|"', "\leq"] & B \arrow[d, "|-|"] \\
 & \mathcal{V}
\end{tikzcd}$$

This defines the category $\mathbf{Set}/\!\!/\mathcal{V}$.

Betti, Renato and Galuzzi, Massimo (1975). **“Categorie normate”.** In: *Bollettino dell’Unione Matematica Italiana* **4**.(11), pp. 66–75.

Remark

$\mathcal{V}$-normed Category $=$ Category enriched in $\mathbf{Set}/\!\!/\mathcal{V}$.

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THE MONOIDAL CLOSED CATEGORY $\mathrm{Set}/\!\!/\mathcal{V}$

Theorem

The category $\mathbf{Set}/\!\!/\mathcal{V}$ is symmetric monoidal-closed.

Proof.

- For $\mathcal{V}$-normed sets A and B, their tensor product $A \otimes B$ is carried by the cartesian product $A \times B$, normed by $|(a, b)| = |a| \otimes |b|$ in $\mathcal{V}$.
- The tensor-neutral set E is the set $\{*\}$ normed by $|*| = k$.
- $[A, B]$ has carrier set $\mathbf{Set}(A, B)$ (all mappings $\varphi \colon A \longrightarrow B$), with their norm defined by

$$|\varphi| = \bigwedge_{a \in A} [|a|, |\varphi a|]. \qquad \square$$

Notation

We simpy write

$$\mathbf{Cat}/\!\!/\mathcal{V} \quad \text{and} \quad \mathbf{CAT}/\!\!/\mathcal{V}$$

instead of $(\mathbf{Set}/\!\!/\mathcal{V})$-$\mathbf{Cat}$ respectively $(\mathbf{Set}/\!\!/\mathcal{V})$-$\mathbf{CAT}$.

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EXAMPLES OF $\mathcal{V}$-NORMED CATEGORIES

Examples

- For $\mathcal{V} = \mathbf{1}$ the terminal quantale, a $\mathbf{1}$-normed category is just an ordinary category.
- For the Boolean quantale $\mathcal{V} = \mathbf{2}$, a $\mathbf{2}$-normed category $\mathbb{X}$ can be described as a category $\mathbb{X}$ that comes with a distinguished class $\mathcal{M}$ of morphisms which is closed under composition and contains all identity morphisms.

  The $\mathbf{2}$-normed functors preserve the distinguished morphisms.
- We may consider the category $\mathbf{Set}$ as $[0, \infty]$-normed:

  $$|f : X \longrightarrow Y| = \text{“size of } Y \setminus f(X)\text{”} \in \mathbb{N} \cup \{\infty\}.$$

  Hence $|f|$ may be seen as a (predominantly finitary) measure of the degree to which f fails to be surjective.

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Examples

- The monoidal functor $\mathbf{Set}/\!\!/\mathcal{V} \longrightarrow \mathbf{Set}$ (“forget norm”) induces the forgetful functor

  $$\mathbf{Cat}/\!\!/\mathcal{V} \longrightarrow \mathbf{Cat}.$$

  Note. This functor is topological.

- The lax mondoidal functor

  $$s\colon \mathbf{Set}/\!\!/\mathcal{V} \longrightarrow \mathcal{V}, \quad A \longmapsto \bigvee_{a\in A} |a|$$

  induces the functor

  $$s\colon \mathbf{Cat}/\!\!/\mathcal{V} \longrightarrow \mathcal{V}\text{-}\mathbf{Cat}, \quad \mathbb{X} \longmapsto (\text{objects of } \mathbb{X}, s\mathbb{X}(x,y) = \bigvee_{f\colon x\to y} |f|).$$

  Note. For every metric space X,

  $$s(H(X)) = \text{the usual (non-symmetric) Hausdorff metric space.}$$

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- The functor $s\colon \mathbf{Set}/\!\!/\mathcal{V} \longrightarrow \mathcal{V}$ has a right adjoint

  $$i\colon \mathcal{V} \longrightarrow \mathbf{Set}/\!\!/\mathcal{V}, \quad v \longmapsto (\{\star\}, |\star| = v)$$

  which induces the functor

  $$i\colon \mathcal{V}\text{-}\mathbf{Cat} \longrightarrow \mathbf{Cat}/\!\!/\mathcal{V}, \quad i(X) = \mathbb{X} \text{ indiscrete with } |(x,y)| = X(x,y)$$

  which is right adjoint to $s\colon \mathbf{Cat}/\!\!/\mathcal{V} \longrightarrow \mathcal{V}\text{-}\mathbf{Cat}$.

The underlying category à la Kelly

Remark

There is also the “forgetful functor”

$$(-)_\circ : \mathbf{Cat}/\!\!/\mathcal{V} \to \mathbf{Cat}$$

represented by the tensor-neutral element E.

That is, $(-)_\circ$ sends a small $\mathcal{V}$-normed category $\mathbb{X}$ to the category $\mathbb{X}_\circ$ with the same objects as $\mathbb{X}$, but with only those morphisms $f : x \longrightarrow y$ in $\mathbb{X}$ with $k \leq |f|$.

THE NORMED CATEGORY $\mathbf{Set}/\!/\mathcal{V}$

Recall

For every closed symmetric monoidal category $\mathcal{W}$,

$$[-,-]\colon \mathcal{W}\times\mathcal{W}\longrightarrow\mathcal{W}$$

makes $\mathcal{W}$ a $\mathcal{W}$-category.

Example

$\mathbf{Set}/\!/\mathcal{V}$ becomes a $\mathcal{V}$-normed category whose objects are $\mathcal{V}$-normed sets, but whose hom-sets of morphisms $A\to B$ are given by the internal hom $[A,B]$ of $\mathbf{Set}/\!/\mathcal{V}$, that is, by all $\mathbf{Set}$-maps $A\to B$.

To avoid (or increase) confusion, we write $\mathbf{Set}\|\mathcal{V}$ to denote this normed category.

Remark

$(\mathbf{Set}\|\mathcal{V})_\circ = \mathbf{Set}/\!/\mathcal{V}$

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2. Convergence for normed categories

FORWARD-CAUCHY SEQUENCES

Definition (Bonsangue, Breugel, and Rutten (1998))

A sequence $s = (x_n)$ in a metric space X is **forward Cauchy** whenever

$$\inf_{N \in \mathbb{N}} \sup_{n \geq m \geq N} X(x_m, x_n) = 0.$$

An element $x \in X$ is a **forward limit** of s whenever

$$X(x, y) = \inf_{N \in \mathbb{N}} \sup_{n \geq N} X(x_n, y),$$

for all $y \in X$.

Reference

Bonsangue, Marcello M., Breugel, Franck van, and Rutten, Jan (1998). **"Generalized metric spaces: completion, topology, and powerdomains via the Yoneda embedding".** In: *Theoretical Computer Science* **193**.(1-2), pp. 1–51.

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Theorem

A metric space X is net-wise forward Cauchy complete if and only if X has all weighted colimits of flat weights $\psi\colon X \mathrel{\circ\!\!\!\rightarrow} 1$.

Vickers, Steven (2005). **"Localic completion of generalized metric spaces. I".** In: *Theory and Applications of Categories* **14**.(15), pp. 328–356.

CAUCHY COMPLETENESS À LA KUBIŚ

Definition (Kubiś (2017))

Let $s = (x_m \xrightarrow{s_{m,n}} x_n)_{m \leq n \in \mathbb{N}}$ be a sequence in the normed category $\mathbb{X}$.

- Then s is **Cauchy** whenever $0 \geqslant \inf_{N \in \mathbb{N}} \sup_{n \geq m \geq N} |s_{m,n}|$.
- A **limit** of the diagram s is given by a colimit $(x_n \xrightarrow{\gamma_n} x)$ of s in the ordinary category $\mathbb{X}$ so that $0 \geqslant \inf_{N \in \mathbb{N}} \sup_{n \geq N} |\gamma_n|$.

Remark

- Colimits are not unique up to 0-isomorphisms.
- Kubiś constructs a Cauchy completion with a “kind of” universal property,
- proves a fixpoint theorem, and
- has a further condition in the definition of normed category which, for a metric space, is equivalent to symmetry.

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- Colimits are not unique up to 0-isomorphisms.
- Kubiś constructs a Cauchy completion with a “kind of” universal property,
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CAUCHY COMPLETENESS À LA KUBIŚ

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Definition

Let $s = (x_m \xrightarrow{s_{m,n}} x_n)_{m \leq n \in \mathbb{N}}$ be a sequence in the $\mathcal{V}$-normed category $\mathbb{X}$. An object x is a **normed colimit** of s in $\mathbb{X}$ if

- x is a colimit of s in the ordinary category $\mathbb{X}$, with a colimit cocone $(x_n \xrightarrow{\gamma_n} x)$ so that
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For any cocone $\alpha\colon s \longrightarrow \Delta x$ over a sequence $s = (x_n)_{n\in\mathbb{N}}$ in $\mathbb{X}$, tfae:

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Corollary

A normed colimit of a sequence in a $\mathcal{V}$-normed category $\mathbb{X}$ is uniquely determined up to an isomorphism in $\mathbb{X}_\circ$.

Corollary

Let $\mathbb{X}$ be a $\mathcal{V}$-normed category satisfying the condition

$$|f| \geq |f\cdot h| \otimes |h| \quad \text{for all composable morphisms } h \text{ and } f.$$

Then an object x is a normed colimit of a sequence s in $\mathbb{X}$ if, and only if, x is an ordinary colimit of s with a colimit cocone that is a ***k-cocone****.*

ANALYSING NORMED COLIMITS

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ANALYSING FURTHER

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if $\mathbb{X} = i(X)$ is given by a $\mathcal{V}$-category X, then the condition of the page before means equivalently that X is symmetric.

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The dual condition reads as

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as above, for $\mathbb{X} = i(X)$ this means that X is symmetric.

For $\mathcal{V} = [0, \infty]$, Kubiś included this condition in the definition of normed category; however, this does not make colimits unique up to a 0-isomorphism.

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Definition

For a $\mathcal{V}$-normed category $\mathbb{X}$, we say that

- a sequence $s = (x_m \xrightarrow{s_{m,n}} x_n)_{m \leq n \in \mathbb{N}}$ in $\mathbb{X}$ is **Cauchy** if

$$k \leq \bigvee_{N \in \mathbb{N}} \bigwedge_{n \geq m \geq N} |s_{m,n}|,$$

- and $\mathbb{X}$ is **Cauchy cocomplete** if every Cauchy sequence in $\mathbb{X}$ has a normed colimit in $\mathbb{X}$.

Example

We consider the case $\mathbb{X} = iX$ for a metric space X.

The sequence $s = (x_n)$ is Cauchy in $\mathbb{X}$ if, and only if,

$$\inf_{N\in\mathbb{N}} \sup_{n\geq m\geq N} X(x_m, x_n) = 0,$$

if and only if s is forward Cauchy in X.

Since the category $\mathbb{X}$ is indiscrete, any cocone $(x_n \longrightarrow x)_{n\in\mathbb{N}}$ is a colimit cocone.

Finally, x is a normed colimit of s if, and only if,

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Assumption

In the sequel we consider a quantale $\mathcal{V}$ satisfying one of the following conditions.

(A) k **is approximated from totally below**, that is:

$$k = \bigvee \{u \in \mathcal{V} \mid u \ll k\},$$

where $u \ll k$ means that, whenever $k \leq \bigvee_{i \in I} v_i$, then $u \leq v_i$ for some $i \in I$.

(B) k **$\wedge$-distributes over arbitrary joins**, that is:

$$k \wedge \bigvee_{i \in I} v_i = \bigvee_{i \in I} k \wedge v_i.$$

PRESHEAF CATEGORIES ARE CAUCHY COCOMPLETE

Proposition

The $\mathcal{V}$-normed category $[\mathbb{X}, \mathbf{Set}\|\mathcal{V}]$ is Cauchy cocomplete, for every small $\mathcal{V}$-normed category $\mathbb{X}$.

Proof.

For a Cauchy sequence $\sigma = (P_m \xrightarrow{\sigma_{m,n}} P_n)_{m \leq n \in \mathbb{N}}$ in $[\mathbb{X}, \mathbf{Set}\|\mathcal{V}]$.

1. Take its colimit $(\gamma_n : P_n \longrightarrow P)_{n \in \mathbb{N}}$.
2. For every object x of $\mathbb{X}$, $n \in \mathbb{N}$ and $c \in Px$, put

$$|c| = \bigwedge_{N \in \mathbb{N}} \bigvee_{n \geq N} \bigvee_{a \in (\gamma_n)_x^{-1} c} |a|.$$

3. $P : \mathbb{X} \to \mathbf{Set}\|\mathcal{V}$ is indeed a normed functor.
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4. The difficult part is to get that $(\gamma_n : P_n \longrightarrow P)_{n \in \mathbb{N}}$ is a k-cocone. $\square$

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Remark

Normed functors preserve Cauchy sequences.

Proposition

Every left adjoint normed functor $F\colon \mathbb{X} \longrightarrow \mathbb{Y}$ preserves normed colimits of Cauchy sequences.

Proof.

The diagram

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\mathrm{Nat}(Fs|_N, \Delta u) \arrow[d] \\
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ADJOINTS VS. NORMED COLIMITS

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Let s be a Cauchy sequence in the normed category $\mathbb{X}$. Consider the Yoneda embedding

$$y_{\mathbb{X}} \colon \mathbb{X} \longrightarrow P\mathbb{X} := [\mathbb{X}^{\mathrm{op}}, \mathbf{Set} \| \mathcal{V}].$$

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Hence, $\mathbb{X}$ is Cauchy cocomplete if, and only if, $\mathbb{X}$ has all colimits weighted by normed colimits of representables.

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\mathrm{Cocone}(s|_N, y) \arrow[r, dashed] \arrow[d, "\wr"'] & \mathbb{X}(x, y) \arrow[d, dotted, "\wr"] & \text{colimit in } \mathbf{Set} /\!\!/ \mathcal{V} \\
\mathrm{Cocone}(y_{\mathbb{X}} \cdot s|_N, y_{\mathbb{X}}(y)) \arrow[r] & \mathrm{Nat}(\Phi, y_{\mathbb{X}}(y)) & \text{colimit in } \mathbf{Set} /\!\!/ \mathcal{V}
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For every object x in $\mathbb{X}$,

$$x = \mathrm{ncolim}(s) \implies x = \mathrm{colim}\, \Phi$$

Hence, $\mathbb{X}$ is Cauchy cocomplete if, and only if, $\mathbb{X}$ has all colimits weighted by normed colimits of representables.

NORMED COLIMITS VS. WEIGHTED COLIMITS

Remark

Let s be a Cauchy sequence in the normed category $\mathbb{X}$. Consider the Yoneda embedding

$$y_{\mathbb{X}} \colon \mathbb{X} \longrightarrow P\mathbb{X} := [\mathbb{X}^{\mathrm{op}}, \mathbf{Set}\|\mathcal{V}].$$

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