On P_F-frames

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Outline of the Talk

Notation

- L denotes a frame.
- $\mathcal{R}L$ denotes the ring of real-valued continuous function on a frame L.
- Coz$L$ denotes the cozero part of a frame L.
- Cov$L$ denotes the set of all covers of L.

Preliminaries

Definition

A frame is a complete lattice L in which the infinite distributive law

$$a \wedge \bigvee S = \bigvee \{a \wedge s \mid s \in S\}$$

holds for all $a \in L$ and $S \subseteq L$.

Definition

A frame homomorphism is a map between frames that preserves finite meets, including the bottom element, and arbitrary joins, including the top element.

Definition

An element a *is said to be rather below an element* b *written* $a \prec b$ *if there is an element* $c \in L$ *such that* $a \wedge c = 0$ *and* $b \vee c = 1$. *We call an element* c *a separating element.*

Definition

An element a *is said to be completely below an element* b *written* $a \prec\prec b$ *if there is a sequence* (c_q) *indexed by the rationals* $\mathbb{Q} \cap [0, 1]$ *such that* $c_r \prec c_s$ *whenever* $r \leq s$.

Definition

A frame L is said to be regular if for all $a \in L$,

$$a = \bigvee \{x \in L \mid x \prec a\}.$$

Definition

A frame L is said to be completely regular if for all $a \in L$,

$$a = \bigvee \{x \in L \mid x \prec\prec a\}.$$

Definition

A frame L is said to be a P_F-frame if whenever $a, b \in CozL$ such that $a \wedge b = 0$, then at least one of them is complemented.

Proposition

If L is a P_F-frame, then $\downarrow a$ is a P_F-frame for each $a \in CozL$.

Examples

- Every P-frame is a P_F-frame.
- An almost P-frame with *ccc* is a P_F-frame.
- A basically disconnected almost P-frame is a P_F-frame.
- A weakly cozero complemented almost P-frame is a P_F-frame.
- An O_z-frame which is also an almost P-frame is a P_F-frame.

Recall that a frame L is said to be an F-frame if $\varphi : L \to \downarrow a$ is a C^*-quotient map for every $a \in \mathrm{Coz}L$.

Proposition

Every P_F-frame is an F-frame.

Proposition

Let $h : L \to M$ be coz-onto, dense frame homomorphism. If L is a P_F-frame, then so is M.

Proposition

Let $h : L \to M$ be naerly open and coz-codense frame homomorphism. If M is a P_F-frame, then so is L.

Proposition

A frame L is a P_F-frame if and only if βL is a P_F-frame.

Definition

An element φ is said to be a von Neumann inverse of α if $\varphi = \varphi^2\alpha$. A ring R is said to be von Neumann regular ring if every $\alpha \in R$ is von Neumann inverse.

Proposition

The following are equivalent for a frame L

- L *is a* P_F*-frame.*
- *For* $a, b \in CozL$ *such that* $a \wedge b$ *is complemented, then at least one is complemented.*
- *Of any two ideals of* $\mathcal{R}L$ *whose product is a* P*-ideal, then at least one is a* P*-ideal.*
- *Of any two principal ideals of* $\mathcal{R}L$ *whose product (intersection) is zero, then at least one is semiprime.*
- *Of any two principal ideals of* $\mathcal{R}L$ *whose product (intersection) is semiprime, then at least one is semiprime.*
- L *is an essential* P*-frame which is also an* F*-frame.*

Proposition

Every weakly cozero complemented P_F-frame is basically disconnected.

Definition

A frame L is said to be an essential almost P-frame if there is at most one cozero element which is not regular.

Example

An essential P-frame.

Proposition

Every basically disconnected essential almost P-frame is a P_F-frame.

Corollary

A frame L is weakly cozero complemented P_F-frame if and only if it is a basically disconnected essential almost P-frame.

Corollary

A frame L is a P_F-frame if and only if for any two comaximal principal ideals of $\mathcal{R}L$, one is semiprime and the other is convex.

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