# Cauchy completions and $\mathcal{V}$-fully faithful lax epimorphisms


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# Goal

Exhibit the relationship between

- $\mathcal{V}$-Cauchy completions,
- $\mathcal{V}$-fully faithful lax epimorphisms.

# Overview

- Metric spaces as illustration ($\mathcal{V} = [0, \infty]$).
- Results for general $\mathcal{V}$.
- Application to descent theory.
- The case of (monad, quantale)-enrichment.

## Generalized metric spaces

Set $M$, distance function $d\colon M \times M \to [0, \infty]$,

- $0 \geqslant d(x, x)$,
- $d(y, z) + d(x, y) \geqslant d(x, z)$,
- ~~$d(x, y) = d(y, x)$~~,
- ~~$d(x, y) = 0 \implies x = y$~~.

Function $f\colon M \to N$

$$d(x, y) \geqslant d(fx, fy)$$

# Isometries and lax epimorphisms

Let $f\colon M \to N$ be a metric map.

$f\colon M \to N$ is an *isometry* if

$$d(x, y) = d(fx, fy)$$

for all $x, y \in M$.

$f$ is a *lax epimorphism* if

$$h \circ f \geqslant g \circ f \implies h \geqslant g$$

for all $g, h\colon N \to [0, \infty]$.

# Lax epimorphisms and density

$f$ is *left/right dense* if

$$\inf_{x \in X} d(y, fx) = 0 \qquad \inf_{x \in X} d(fx, y) = 0$$

for all $y \in N$.

$f$ is *absolutely dense* if

$$\inf_{x \in X} d(fx, z) + d(y, fx) = d(y, z)$$

for all $y, z \in Y$.

### Lemma (Lucatelli Nunes, Sousa, 2022)

$$f \textit{ is absolutely dense} \iff f \textit{ is a lax epimorphism} \implies f \textit{ is left and right dense.}$$

## Cauchy completion

$M$ metric space.

### Theorem (Lawvere, 1973)

*There is a bijection between*

- *Equivalence classes of Cauchy sequences in* $M$.
- *Pairs of metric maps* $L\colon M^{\mathsf{op}} \to [0,\infty]$, $R\colon M \to [0,\infty]$ *such that*

$$0 \geqslant \inf_{y \in Y} Ry + Ly,$$

$$Ly + Rz \geqslant d(y,z).$$

# $\mathcal{V}$-categories

A (small) $\mathcal{V}$-category $\mathcal{C}$ consists of

- a (small) set $\mathsf{ob}\,\mathcal{C}$ of objects,
- a hom-object $\mathcal{C}(x, y) \in \mathcal{V}$ for each pair $x, y$,
- a unit morphism $\mathsf{u}_x \colon I \to \mathcal{C}(x, x)$ for each $x$,
- a composition morphism $\mathsf{c}_{x,y,z} \colon \mathcal{C}(y, z) \otimes \mathcal{C}(x, y) \to \mathcal{C}(x, z)$,

satisfying adequate identity and associativity laws.

# $\mathcal{V}$-functors

A $\mathcal{V}$-functor $F\colon \mathcal{C} \to \mathcal{D}$ consists of

- A function $F\colon \ \mathsf{ob}\,\mathcal{C} \to \mathsf{ob}\,\mathcal{D}$,
- A hom-morphism $F_{x,y}\colon \mathcal{C}(x,y) \to D(Fx,Fy)$

satisfying adequate unit and composition preservation properties.

$\mathcal{V}$-$\mathsf{CAT}$ and $\mathcal{V}$-$\mathsf{Cat}$ are the 2-categories of $\mathcal{V}$-categories and small $\mathcal{V}$-categories.

# Two notions of full faithfulness

Let $F\colon \mathcal{C} \to \mathcal{D}$ be a $\mathcal{V}$-functor.

$F$ is *$\mathcal{V}$-fully faithful* if

$$F_{x,y}\colon \mathcal{C}(x,y) \to \mathcal{D}(Fx,Fy)$$

is an isomorphism for all $x, y$.

$F$ is a *fully faithful morphism* if

$$\begin{aligned} F_!\colon \mathcal{V}\text{-}\mathsf{Cat}(\mathcal{B},\mathcal{C}) &\to \mathcal{V}\text{-}\mathsf{Cat}(\mathcal{B},\mathcal{D}) \\ G &\mapsto F \circ G \end{aligned}$$

is fully faithful for all $\mathcal{B}$.

# Two notions of full faithfulness

**Lemma (Lucatelli Nunes, Sousa, 2022)**

*If $F$ is $\mathcal{V}$-fully faithful, then $F$ is a fully faithful morphism.*

*The converse holds if $F$ has a (left or right) adjoint.*

# Lax epimorphisms

Let $F\colon \mathcal{C} \to \mathcal{D}$ be a $\mathcal{V}$-functor.

$F$ is a *lax epimorphism* if

$$\begin{aligned} F^*\colon \mathcal{V}\text{-}\mathsf{Cat}(\mathcal{D}, \mathcal{B}) &\to \mathcal{V}\text{-}\mathsf{Cat}(\mathcal{C}, \mathcal{B}) \\ G &\mapsto G \circ F \end{aligned}$$

is fully faithful for all $\mathcal{B}$.

# Lax epimorphisms

Let $F\colon \mathcal{C} \to \mathcal{D}$ be a $\mathcal{V}$-functor.

**Lemma (Lucatelli Nunes, Sousa, 2022)**

*$F$ is a lax epimorphism if and only if*

$$F^*\colon \mathcal{V}\text{-}\mathsf{CAT}(\mathcal{D},\mathcal{V}) \to \mathcal{V}\text{-}\mathsf{CAT}(\mathcal{C},\mathcal{V})$$

*is fully faithful.*

# Relationship with adjunctions

## Lemma

*If we have an adjunction*

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D} \quad (F \dashv G, \ \bot)$$

*then*

- $F$ *is a fully faithful morphism* $\iff$ $G$ *is a lax epimorphism,*
- $F$ *is a lax epimorphism* $\iff$ $G$ *is a fully faithful morphism.*

# Relationship with Cauchy completion

Let $F\colon \mathcal{C} \to \mathcal{D}$ be a $\mathcal{V}$-functor, let $\mathfrak{C}F\colon \mathfrak{C}X \to \mathfrak{C}Y$ be the induced $\mathcal{V}$-functor.

## Lemma (Lucatelli Nunes, P., Sousa, 2023)

*The following are equivalent:*

- $F$ *is* $\mathcal{V}$*-fully faithful.*
- $\mathfrak{C}F$ *is* $\mathcal{V}$*-fully faithful.*
- $\mathcal{V}\text{-}\mathsf{CAT}(F, \mathcal{V})$ *is a lax epimorphism.*

## Lemma (Lucatelli Nunes, P., Sousa, 2023)

*The following are equivalent:*

- $F$ *is a lax epimorphism.*
- $\mathfrak{C}F$ *is a lax epimorphism.*

## Relationship with Cauchy completion

Let $F\colon \mathcal{C} \to \mathcal{D}$ be a $\mathcal{V}$-functor, let $\mathfrak{C}F\colon \mathfrak{C}X \to \mathfrak{C}Y$ be the induced $\mathcal{V}$-functor.

### Theorem (Lucatelli Nunes, P., Sousa, 2023)

*The following are equivalent:*

- *$F$ is a $\mathcal{V}$-fully faithful lax epimorphism.*
- *$\mathfrak{C}F$ is an equivalence.*

# Application to descent theory

Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

## Lemma (Lucatelli Nunes, P., Sousa 2023)

*The following are equivalent:*

- $F$ *is a fully faithful lax epimorphism.*
- $\mathsf{CAT}(F, \mathsf{Set})$ *is an equivalence.*
- $\mathsf{CAT}(F, \mathsf{Cat})$ *is an equivalence.*

# Application to descent theory

Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

## Theorem (Lucatelli Nunes, P., Sousa, 2023)

*The following are equivalent:*

- *$F$ is an effective* $\mathsf{Cat}(-,\mathsf{Cat})$*-descent morphism.*
- *$F$ is an effective* $\mathsf{Cat}(-,\mathsf{Set})$*-descent morphism.*
- *$\mathcal{K}_F$ is a fully faithful lax epimorphism.*

# The case of (monad, quantale)-enrichment

## Open problem

Find conditions for a functor of $(T, \mathcal{V})$-categories to be of effective étale descent.

Thank you!