Semidirect products and split extensions of categories

Ülo Reimaa
University of Tartu

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Collaborators

- Nelson Martins-Ferreira – Polytechnic Institute of Leiria
- Fosco Loregian – Tallinn University of Technology
- Graham Manuell – University of Coimbra
- Greta Coraglia – University of Milan
- Davide Castelnovo – University of Padova

Prelude

Starting point

Marino Gran, Gabriel Kadjo, Joost Vercruysse - **A torsion theory in the category of cocommutative Hopf algebras**. *Appl. Categ. Structures* 24 (2016), no.3, 269–282.

$$
\begin{tikzcd}
\mathsf{Lie}_{\mathbb{K}} \arrow[r, hook, shift left=1.5, "U"] \arrow[r, phantom, "\perp"] & \mathsf{ccHopf}_{\mathbb{K}} \arrow[l, dotted, shift left=1.5, "P"] \arrow[r, "\mathcal{G}" description] & \mathsf{Gp} \arrow[l, hook', shift right=3, "\mathbb{K}[-]"'] \arrow[l, dotted, hook, shift left=3, "\mathbb{K}[-]"]
\end{tikzcd}
$$

Renaming things and forgetting the dotted arrows, we get the diagram

$$
\begin{tikzcd}
\mathsf{Lie}_{\mathbb{K}} \arrow[r, hook, "i"] & \mathsf{ccHopf}_{\mathbb{K}} \arrow[r, shift right=1.5, "p"'] \arrow[r, phantom, "\perp"] & \mathsf{Gp} \arrow[l, hook', shift right=1.5, "s"']
\end{tikzcd}
$$

where $sp \cong 1$ and i is the kernel of p, so this is a split extension of categories.

Question: How similar is this in behaviour to split extensions of groups?

$$\mathsf{Lie}_{\mathbb{K}} \overset{i}{\hookrightarrow} \mathsf{ccHopf}_{\mathbb{K}} \underset{p}{\overset{s}{\rightleftarrows}} \mathsf{Gp} \qquad (s \dashv p)$$

Problem

Can we write $\mathsf{ccHopf}_{\mathbb{K}} \simeq \mathsf{Lie}_{\mathbb{K}} \rtimes \mathsf{Gp}$, *expressing the category* $\mathsf{ccHopf}_{\mathbb{K}}$ *as some sort of a semidirect product of* $\mathsf{Lie}_{\mathbb{K}}$ *and* Gp*?*

An action of a group G on a Lie algebra L is a monoid homomorphism

$$G \to \mathsf{Lie}_{\mathbb{K}}(L, L)$$

from G to the monoid of endomorphisms of L.

Given such an action, we can form the semidirect product $L \rtimes G \in \mathsf{ccHopf}_{\mathbb{K}}$.

This gives us a functor

$$\{\text{actions of groups on Lie algebras}\} \to \mathsf{ccHopf}_{\mathbb{K}}\,.$$

The functor

$$\{\text{actions of groups on Lie algebras}\} \to \mathsf{ccHopf}_{\mathbb{K}}\,,$$
$$(\text{action of } G \text{ on } L) \mapsto L \rtimes G$$

is (under assumptions on the field $\mathbb{K}$) an equivalence of categories, giving us:

A solution to our problem

If we define

$$\mathsf{Lie}_{\mathbb{K}} \rtimes \mathsf{Gp} := \text{actions of groups on Lie algebras}$$

then we have achieved the goal $\mathsf{ccHopf}_{\mathbb{K}} \simeq \mathsf{Lie}_{\mathbb{K}} \rtimes \mathsf{Gp}$.

We can think of actions as being *formal* semidirect products, so the functor

$$\mathsf{Lie}_{\mathbb{K}} \rtimes \mathsf{Gp} \xrightarrow{\rtimes} \mathsf{ccHopf}_{\mathbb{K}}$$

realizes formal semidirect products as concrete objects in the category $\mathsf{ccHopf}_{\mathbb{K}}$.

Since semidirect products are constructed from actions and

$$\mathrm{Lie}_{\mathbb{K}} \rtimes \mathrm{Gp} := \text{actions of groups on Lie algebras}$$

is a semidirect product, then what is the corresponding action of Gp on $\mathrm{Lie}_{\mathbb{K}}$?

An action of a group G on a Lie algebra L can be given as an algebra

$$G \cdot L \to L$$

over the monad $G \cdot -$ on the category $\mathrm{Lie}_{\mathbb{K}}$, acting by copowering L with G:

$$G \cdot L := \coprod_{g \in G} L\,.$$

We get such a monad for each group G, collectively giving us a functor

$$\mathrm{Gp} \to \mathrm{Mnd}(\mathrm{Lie}_{\mathbb{K}})$$

from Gp to the category of monads on $\mathrm{Lie}_{\mathbb{K}}$. This functor is our action of Gp on $\mathrm{Lie}_{\mathbb{K}}$.

Actions via monads in general

Suppose we are given a functor

$$\mathbb{B} \to \mathsf{Mnd}(\mathbb{X})\,, \quad B \mapsto (X \mapsto B\curvearrowright X)\,,$$

meaning each object B of $\mathbb{B}$ has an associated monad

$$B\curvearrowright \_ : \mathbb{X} \to \mathbb{X}\,, \quad X \mapsto B\curvearrowright X\,.$$

This functor into monads gives meaning to the otherwise meaningless phrase:

- **an action of an object B of $\mathbb{B}$ on an object X of $\mathbb{X}$.**

Such an action is now just an algebra

$$\xi \colon B\curvearrowright X \to X$$

over the monad $B\curvearrowright \_$.

The category $\mathbb{X} \rtimes \mathbb{B}$ of algebras of a parameterized monad $\mathbb{B} \to \mathrm{Mnd}(\mathbb{X})$

Given the functor

$$\mathbb{B} \to \mathrm{Mnd}(\mathbb{X})\,, \quad B \mapsto (X \mapsto B \curvearrowright X)\,,$$

we can form the category $\mathbb{X} \rtimes \mathbb{B}$ of all these actions, which explicitly means that

- an **object** in $\mathbb{X} \rtimes \mathbb{B}$ is a triple $(X, B, B \curvearrowright X \xrightarrow{\xi} X)$, where $X \in \mathrm{Ob}(\mathbb{X})$, $B \in \mathrm{Ob}(\mathbb{B})$ and ξ is an algebra over the monad $B \curvearrowright \_$ on the category $\mathbb{X}$ and
- **a morphism** $(X, B, \xi) \to (X', B', \xi')$ is a pair $(X \xrightarrow{g} X', B \xrightarrow{f} B')$ of morphisms that make the diagram

$$\begin{tikzcd}
B \curvearrowright X \arrow[r, "\xi"] \arrow[d, "f \curvearrowright g"'] & X \arrow[d, "g"] \\
B' \curvearrowright X' \arrow[r, "\xi'"'] & X'
\end{tikzcd}$$

commute.

The fibration of algebras

Associated to the category $\mathbb{X} \rtimes \mathbb{B}$ is the fibration

$$\mathbb{X} \rtimes \mathbb{B} \to \mathbb{B}\,, \quad (X, B, \xi) \mapsto B$$

that we call the **fibration of algebras** of the parameterized monad $\mathbb{B} \to \mathsf{Mnd}(\mathbb{X})$.

Example

The category $\mathsf{Mon}(\mathbb{C})$ of monoids in a monoidal category $\mathbb{C}$ acts on the category $\mathbb{C}$ via monads by mapping a monoid M in $\mathbb{C}$ to the monad

$$M \otimes \_ : \mathbb{C} \to \mathbb{C}\,.$$

The category $\mathsf{Mon}(\mathbb{C}) \rtimes \mathbb{C}$ is the category of left modules and the fiber over a monoid M is the category of left M-modules.

The split extension associated to the fibration of algebras

Around the fibration of algebras, we can build the diagram

$$\begin{tikzcd}
\mathbb{X} \rtimes \{\varnothing\} \arrow[r, hook, shift left=2ex, "i"] \arrow[r, phantom, "\perp"] & \mathbb{X} \rtimes \mathbb{B} \arrow[l, shift left=2ex, "i^R"] \arrow[r, "p" description] & \mathbb{B} \arrow[l, hook', shift right=3ex, "s"'] \arrow[l, hook, shift left=3ex, "p^R"] \arrow[l, phantom, shift right=1.5ex, "\perp"] \arrow[l, phantom, shift left=1.5ex, "\perp"]
\end{tikzcd}$$

of adjunctions, assuming the categories $\mathbb{X}$ and $\mathbb{B}$ have initial and terminal objects. The category $\mathbb{X} \rtimes \{\varnothing\}$ is the fiber over the initial object $\varnothing$ of $\mathbb{B}$. Note that $\mathbb{X} \rtimes \{\varnothing\} \simeq \mathbb{X}$ if $\varnothing \curvearrowright \_$ is the identity monad.

- This diagram can be reasonably called an (adjoint) split exact sequence (of adjunctions).
- In the nicely pointed setting, this exactness implies that the full and faithful functors i and p^R exhibit a torsion theory inside of $\mathbb{X} \rtimes \mathbb{B}$.
- More generally, we can think of this exactness as giving us a generalized notion of torsion theory.

Split extensions from fibrations

More generally, given any fibration $p\colon \mathbb{A} \to \mathbb{B}$ whose fibers have initial and terminal objects, we can build the (not necessarily as exact) split extension diagram

$$\begin{tikzcd} \mathbb{X} \arrow[r, hook, shift left=2, "i"] \arrow[r, phantom, "\perp"] & \mathbb{A} \arrow[l, shift left=2, "i^R"] \arrow[r, "p" description] & \mathbb{B} \arrow[l, hook', shift right=4, "s"'] \arrow[l, hook', shift left=4, "p^R"] \end{tikzcd}$$

by taking the fiber $\mathbb{X} := \mathbb{A}_\varnothing$ over the initial object of $\mathbb{B}$.

Problem

How far is a split extension diagram being equivalent to the one we had for $\mathbb{X} \rtimes \mathbb{B}$?

The span $\mathbb{X} \xleftarrow{i^R} \mathbb{A} \xrightarrow{p} \mathbb{B}$ induces a functor $(i^R, p)\colon \mathbb{A} \to \mathbb{X} \times \mathbb{B}$.

- If this functor is monadic, we have $\mathbb{A} \simeq \mathbb{X} \rtimes \mathbb{B}$.
- The closer the functor (i^R, p) is to being monadic, the nicer our extension is.

Sometimes, all nice extensions are monadic

If we apply our construction to a pointed fibration (fibers are pointed categories), the left adjoint of p and right adjoint of p coincide, meaning we get the diagram

$$\begin{tikzcd} \mathbb{X} \arrow[r, hook, shift left=2, "i"] \arrow[r, phantom, "\perp"] & \mathbb{A} \arrow[l, shift left=2, "i^R"] \arrow[r, "p" description] & \mathbb{B} . \arrow[l, hook', shift right=4, "s"'] \arrow[l, hook, shift left=4, "s"] \end{tikzcd}$$

In the semi-abelian setting, such extensions are monadic, meaning $\mathbb{A} \simeq \mathbb{X} \rtimes \mathbb{B}$.

This nicely follows the microcosm principle and further justifies the notation $\mathbb{X} \rtimes \mathbb{B}$.

Example

Applying this to the fibration $\mathrm{Beck}(\mathbb{B}) \to \mathbb{B}$ of Beck modules in a semi-abelian category $\mathbb{B}$, we get $\mathrm{Beck}(\mathbb{B}) \simeq \mathrm{Ab}(\mathbb{B}) \rtimes \mathbb{B}$, assuming $\mathrm{Beck}(\mathbb{B})$ is semi-abelian.

We can dualize everything

After dualizing

- Left adjoints become right adjoints and right adjoints become left adjoints.
- Monads become comonads.
- We instead take the fiber over the terminal object as the kernel of $p\colon \mathbb{A} \to \mathbb{B}$.

It was shown in

- P. Faul, G. Manuell, J. Siqueira, **Artin glueings of toposes as adjoint split extensions**. *J. Pure Appl. Algebra* 227 (2023), no. 5

that adjoint split extensions

$$
\begin{tikzcd}
\mathcal{E}' \arrow[r, hook, "i"] & \mathcal{E} \rtimes \mathcal{E}' \arrow[r, shift right, "p"'] \arrow[r, phantom, "\top"] & \mathcal{E} \arrow[l, shift right, "s"']
\end{tikzcd}
$$

of toposes are *comonadic*, and $\mathcal{E} \rtimes \mathcal{E}'$ is just the Artin glueing of toposes.

Monads on the product $\mathbb{X} \times \mathbb{B}$

The monads and adjunctions we get on the product are quite particular, in that they are fibered over $\mathbb{B}$ in the sense that they commute with the projection $\mathbb{X} \times \mathbb{B} \to \mathbb{B}$:

$$
\begin{tikzcd}
\mathbb{A} \arrow[r, "p"] \arrow[d, shift left=2] \arrow[d, phantom, "\dashv"] & \mathbb{B} \\
\mathbb{X} \times \mathbb{B} \arrow[u, shift left=2] \arrow[ur, "\pi_B"'] &
\end{tikzcd}
\,.
$$

A monad on $\mathbb{X} \times \mathbb{B}$ can be decomposed into two interacting parameterized monads

$$\mathbb{B} \to \mathsf{Mnd}(\mathbb{X}), \quad \mathbb{X} \to \mathsf{Mnd}(\mathbb{B}).$$

If $\mathbb{X} \to \mathsf{Mnd}(\mathbb{B})$ maps everything to the identity monad, we get our previous situation.

The more general situation is interesting as well. For example, internal crossed modules arise from the pair of parameterized monads

$$B \mapsto B \flat \_\,, \qquad X \mapsto \_ + X\,.$$

Thank you for listening!