LAX FRACTIONS IN 2-CATEGORIES

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ongoing work with Graham Manuell

R ring, $S \subseteq R$ multiplicative

Ring of fractions: $\frac{a}{s}$, $a \in R$, $s \in S$, $\frac{a}{s} \equiv \frac{ac}{sc}$

$$\begin{tikzcd} R \arrow[r, "h"] \arrow[d] & R[S^{-1}] \arrow[dl, dashed] \\ A & \end{tikzcd}$$

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$\mathcal{X}$ category, $\Sigma \subseteq \mathrm{Mor}(\mathcal{X})$

Category of fractions:

$$\begin{tikzcd} \mathcal{X} \arrow[r, "P_\Sigma"] \arrow[d] & \mathcal{X}[\Sigma^{-1}] \arrow[dl, dashed] \\ \mathcal{A} & \end{tikzcd}$$

$P_\Sigma s$ iso, $s \in \Sigma$

universal

A special case:

$$
\begin{tikzcd}
\mathcal{X} \arrow[r, shift left=2, "F"] \arrow[r, phantom, "\perp"] & \mathcal{A}_\ell \arrow[l, hook', shift left=2]
\end{tikzcd}
= \mathcal{X}[\Sigma^{-1}] \quad \text{for } \Sigma = \mathcal{A}_\ell^{\perp}
$$

reflective full subcategory

$$
\mathcal{A}^{\perp} = \left\{ h : \forall
\begin{tikzcd}
\cdot \arrow[r, "h"] \arrow[d] & \cdot \arrow[dl, dashed, "\exists!"] \\
A &
\end{tikzcd}
, \; A \in \mathcal{A} \right\}
$$

$\mathcal{X}$ 2-category

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, phantom, "\perp(\mathrm{id},\varepsilon)"] & B \arrow[l, bend left, "g"] \end{tikzcd}$$

in particular, $gf = 1_A$

f is a left adjoint right inverse (lari)

$f_* := g$ (unique up to an invertible 2-cell)

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$$\begin{tikzcd} A \arrow[r, "a"] & B \arrow[r, bend right, "s_*"'] & C \arrow[l, "s"'] \end{tikzcd}$$

$s_* a$ is a (left) lax fraction

Next:

- The Kleisli category of a lax-idempotent monad as a category of lax fractions

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(For order-enriched categories, [S., JPAA, 2016])

A is **left Kan injective** to h if

$$\begin{tikzcd} \cdot \arrow[r, "h"] \arrow[d, "f"'] & \cdot \arrow[dl, "\mathrm{Lan}_h f"] \\ A & \end{tikzcd}$$

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A &
\end{tikzcd}$$

$A \xrightarrow{t} B$ is left Kan injective to h if

$$\begin{tikzcd}
{} \arrow[r, "h"] \arrow[d, "f"'] & {} \arrow[dl, "\mathrm{Lan}_h f"] \arrow[ddl, bend left=60, "\mathrm{Lan}_h (tf)"] \\
A \arrow[d, "t"'] & {} \arrow[l, phantom, "="] \\
B &
\end{tikzcd}$$

Idempotent monads	Lax-idempotent (pseudo)monads (= KZ-monads) [Kock]

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Category of algebras: $\mathcal{A} = \{\eta_x : x \in \mathcal{X}\}_{\perp} \hookrightarrow \mathcal{X}$	2-Category of algebras: $\mathcal{A}_{\ell} = \mathrm{LInj}\{\eta_x : x \in \mathcal{X}\} \xhookrightarrow[\text{locally full}]{} \mathcal{X}$ [DiLiberti, Lobbia, S., 2023]

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$\mathcal{A} \perp h \iff Th$ iso	$\mathcal{A}$ LInj $h \iff Th$ lari

$\mathcal{X}^{\to}$:= category of arrows

obj: morphisms of $\mathcal{X}$; mor: $(f,g): u \to v$

$$\begin{tikzcd} \cdot \arrow[r, "u"] \arrow[d, "f"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "v"'] & \cdot \end{tikzcd}$$

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For Σ a subcategory of $\mathcal{X}^{\to}$,

we write just

$$\begin{tikzcd} \cdot \arrow[r, "u"] \arrow[d, "f"'] \arrow[dr, phantom, "\Sigma"] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "v"'] & \cdot \end{tikzcd}$$

to indicate that $(f,g): u \to v$ is a morphism in Σ.

A commutative square $\begin{tikzcd} \cdot \arrow[r, "u"] \arrow[d, "f"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "v"'] & \cdot \end{tikzcd}$

with u and v lari 1-cells

satisfies the Beck-Chevalley condition if

its mate is an invertible 2-cell:

$$\begin{tikzcd} \cdot \arrow[d, "f"'] & \cdot \arrow[l, "u_*"'] \arrow[d, "g"] \arrow[dl, phantom, "\Swarrow \cong"] \\ \cdot & \cdot \arrow[l, "v_*"] \end{tikzcd}$$

$\mathcal{X}$ 2-category, Σ subcategory of $\mathcal{X}^{\to}$.

A (bi)category of lax fractions w.r.t. Σ

is given by

$$\mathcal{X} \xrightarrow{P_\Sigma} \mathcal{X}[\Sigma_*] \qquad \text{s.t.}$$

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(1) P_Σ sends objs. of Σ to lari 1-cells and Σ-squares (= morphisms of Σ) to Beck-Chevalley squares.

(2) For every 2-functor $\mathcal{X} \xrightarrow{G} \mathcal{C}$ with property (1), there is a pseudofunctor H with

$$\begin{tikzcd} \mathcal{X} \arrow[rr, "P_\Sigma"] \arrow[dr, "G"'] & & \mathcal{X}[\Sigma_*] \arrow[dl, "H"] \\ & \mathcal{C} & \end{tikzcd}$$

, unique up to a pseudonatural iso.

Let $\mathcal{A}$ be a locally full 2-subcat. of $\mathcal{K}$.

$$\mathcal{A}^{\mathrm{LInj}}$$

denotes the subcategory of $\mathcal{K}^{\to}$ given by:

Obj: all u s.t. $\mathcal{A}$ is left Kan injective to u.

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$\mathcal{A}^{\mathrm{LInj}}$ denotes the subcategory of $\mathcal{X}^{\to}$ given by:

obj: all u s.t. $\mathcal{A}$ is left Kan injective to u;

mor: all
$$\begin{tikzcd} \cdot \arrow[r, "u"] \arrow[d, "f"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "v"'] & \cdot \end{tikzcd}$$
s.t.
$$\begin{tikzcd} \cdot \arrow[r, "u"] \arrow[d, "f"'] & \cdot \arrow[d, "g"] \arrow[ddl, bend left=80, "\mathrm{Lan}_u(tf)"] \arrow[r, phantom, "="] & {} \\ \cdot \arrow[r, "v"] \arrow[d, "t"'] & \cdot \arrow[dl, "\mathrm{Lan}_v t"] & \\ \mathcal{A} \ni A & & \end{tikzcd}$$

Given a lax-idempotent monad $\mathbb{T}$ over $\mathcal{X}$, let $\mathcal{A} \hookrightarrow \mathcal{X}$ be the inclusion of the category of algebras (as described before).

Given a lax-idempotent monad T over $\mathcal{X}$, let $\mathcal{A} \hookrightarrow \mathcal{X}$ be the inclusion of the category of algebras (as described before).

Let $\mathcal{K}$ be the full and locally full 2-subcategory of $\mathcal{A}$ of all TX, $X \in \mathcal{X}$.

Then the corestriction of T to $\mathcal{K}$

$$\mathcal{X} \xrightarrow{T_{\mathcal{K}}} \mathcal{K}$$

gives a 2-category of lax fractions w.r.t. $\mathcal{A}^{\mathrm{LInj}}$.

Σ subcat. of $\mathcal{K}^{\to}$ admits a left calculus of lax fractions:

Σ subcat. of $\mathcal{K}^{\to}$ admits a left calculus of lax fractions:

(1) IDENTITY. $\forall x$, $1_x \in \Sigma$

$\forall s \in \Sigma$,

$$\begin{tikzcd} \cdot \arrow[r, "1_x"] \arrow[d, "1_x"'] \arrow[dr, phantom, "\Sigma"] & \cdot \arrow[d, "s"] \\ \cdot \arrow[r, "s"'] & \cdot \end{tikzcd}$$

Σ subcat. of $\mathcal{K}^{\to}$ admits a left calculus of lax fractions:

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$\forall s \in \Sigma$,
$$\begin{tikzcd} \cdot \arrow[r, "1_x"] \arrow[d, "1_x"'] \arrow[dr, phantom, "\Sigma"] & \cdot \arrow[d, "s"] \\ \cdot \arrow[r, "s"'] & \cdot \end{tikzcd}$$

(2) COMPOSITION. In
$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "1"] & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "2"] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \arrow[r] & \cdot \end{tikzcd}$$

1, 2 Σ-sqs. $\Rightarrow$ 1 + 2 Σ-sq.

Σ subcat. of $\mathcal{X}^{\to}$ admits a left calculus of lax fractions:

(1) IDENTITY. $\forall X,\ 1_X \in \Sigma$

$\forall s \in \Sigma,$
$$\begin{tikzcd} \cdot \arrow[r, "1_X"] \arrow[d, "1_X"'] \arrow[dr, phantom, "\Sigma"] & \cdot \arrow[d, "s"] \\ \cdot \arrow[r, "s"'] & \cdot \end{tikzcd}$$

(2) COMPOSITION. In
$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "1"] & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "2"] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \arrow[r] & \cdot \end{tikzcd}$$

1, 2 Σ-sqs. $\Rightarrow$ 1 + 2 Σ-sq.

(3) SQUARE. $\forall$ $\cdot \xrightarrow{s \in \Sigma} \cdot$, $f \downarrow$, $\exists s', f'$
$$\begin{tikzcd} \cdot \arrow[r, "s \in \Sigma"] \arrow[d, "f"'] \arrow[dr, phantom, "\Sigma"] & \cdot \arrow[d, "f'"] \\ \cdot \arrow[r, "s' \in \Sigma"'] & \cdot \end{tikzcd}$$

Σ subcat. of $\mathcal{X}^{\to}$ admits a left calculus of lax fractions:

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$\forall s \in \Sigma,$
$$\begin{tikzcd} \cdot \arrow[r, "1_x"] \arrow[d, "1_x"'] \arrow[dr, phantom, "\Sigma"] & \cdot \arrow[d, "s"] \\ \cdot \arrow[r, "s"'] & \cdot \end{tikzcd}$$

(2) COMPOSITION. In
$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "1"] & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "2"] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \arrow[r] & \cdot \end{tikzcd}$$

1, 2 Σ-sqs. $\Rightarrow$ 1 + 2 Σ-sq.

(3) SQUARE.
$$\forall \quad \begin{tikzcd} \cdot \arrow[r, "s \in \Sigma"] \arrow[d, "f"'] \arrow[dr, phantom, "\Sigma"] & \cdot \arrow[d, "f'"] \\ \cdot \arrow[r, "s' \in \Sigma"'] & \cdot \end{tikzcd}$$

$\exists s', f'$

(4) EQUALISATION.
$$\forall \quad \begin{tikzcd} \cdot \arrow[r, "\Omega"] \arrow[d, "f"'] \arrow[dr, phantom, "\Sigma"] & \cdot \arrow[d, "a"] \\ \cdot \arrow[r, "s"'] & \cdot \end{tikzcd}, \quad \begin{tikzcd} \cdot \arrow[r, "\Omega"] \arrow[d, "f"'] \arrow[dr, phantom, "\Sigma"] & \cdot \arrow[d, "b"] \\ \cdot \arrow[r, "s"'] & \cdot \end{tikzcd}, \quad \exists \quad \begin{tikzcd} \cdot \arrow[r, "s"] \arrow[d, equal] \arrow[dr, phantom, "\Sigma"] & \cdot \arrow[d, "d"] \\ \cdot \arrow[r, "ds"'] & \cdot \end{tikzcd}$$

with $da = db$

Σ subcat. of $\mathcal{H}^{\to}$ admits a left calculus of lax fractions:

(1) IDENTITY. $\forall x,\ 1_x \in \Sigma$

$\forall s \in \Sigma,$
$$\begin{tikzcd} \cdot \arrow[r, "1_x"] \arrow[d, "1_x"'] \arrow[dr, phantom, "\Sigma"] & \cdot \arrow[d, "s"] \\ \cdot \arrow[r, "s"'] & \cdot \end{tikzcd}$$

(2) COMPOSITION. In
$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "1"] & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "2"] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \arrow[r] & \cdot \end{tikzcd}$$

1, 2 Σ-sqs. $\Rightarrow$ 1 + 2 Σ-sq.

(3) SQUARE.
$$\forall \quad \begin{tikzcd} \cdot \arrow[r, "s \in \Sigma"] \arrow[d, "f"'] \arrow[dr, phantom, "\Sigma"] & \cdot \arrow[d, "f'"] \\ \cdot \arrow[r, "s' \in \Sigma"'] & \cdot \end{tikzcd}$$

$\exists s', f'$

(4) EQUALISATION.
$$\forall \quad \begin{tikzcd} \cdot \arrow[r, "r"] \arrow[d, "f"'] \arrow[dr, phantom, "\Sigma"] & \cdot \arrow[d, "a"] \\ \cdot \arrow[r, "s"'] & \cdot \end{tikzcd}, \quad \begin{tikzcd} \cdot \arrow[r, "r"] \arrow[d, "f"'] \arrow[dr, phantom, "\Sigma"] & \cdot \arrow[d, "b"] \\ \cdot \arrow[r, "s"'] & \cdot \end{tikzcd}, \quad \exists \quad \begin{tikzcd} \cdot \arrow[r, "s"] \arrow[d, equal] \arrow[dr, phantom, "\Sigma"] & \cdot \arrow[d, "d"] \\ \cdot \arrow[r, "ds"'] & \cdot \end{tikzcd}$$

with $da = db$

(5) EQUINSERTION.
$$\forall \quad \begin{tikzcd} \cdot \arrow[r, "r"] \arrow[d, "f"'] \arrow[dr, phantom, "\Sigma"] \arrow[rr, bend left, "r"] & \cdot \arrow[d, "f'"] \arrow[r, phantom, "\Downarrow\alpha"] & \cdot \arrow[dl, bend left, "g"] \\ \cdot \arrow[r, "s"'] & \cdot & \end{tikzcd}$$

$\exists d, \alpha'$
$$\begin{tikzcd} \cdot \arrow[r, "s"] \arrow[d, equal] \arrow[dr, phantom, "\Sigma"] & \cdot \arrow[d, "d"] \\ \cdot \arrow[r, "ds"'] & \cdot \end{tikzcd} \quad = \quad \begin{tikzcd} \cdot \arrow[r, "r"] \arrow[d, "f"'] \arrow[dr, phantom, "\Sigma"] & \cdot \arrow[d, "f'"] \arrow[r, bend left, "g"] \arrow[r, phantom, "\Downarrow\alpha'"] & \cdot \arrow[dl, "d"] \\ \cdot \arrow[r, "s"] \arrow[d, equal] \arrow[dr, phantom, "\Sigma"] & \cdot \arrow[d, "d"] & \\ \cdot \arrow[r, "ds"'] & \cdot & \end{tikzcd}$$

Σ subcat. of $\mathcal{X}^{\rightarrow}$ admits a left calculus of lax fractions:

(1) IDENTITY. $\forall x,\ 1_x \in \Sigma$

$\forall s \in \Sigma,$
$$\begin{tikzcd} \cdot \arrow[r, "1_x"] \arrow[d, "1_x"'] \arrow[dr, phantom, "\Sigma"] & \cdot \arrow[d, "s"] \\ \cdot \arrow[r, "s"'] & \cdot \end{tikzcd}$$

(2) COMPOSITION. In
$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "1"] & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "2"] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \arrow[r] & \cdot \end{tikzcd}$$

①,② Σ-sqs. $\Rightarrow$ ①+② Σ-sq.

(3) SQUARE. $\forall$
$$\begin{tikzcd} \cdot \arrow[r, "s\in\Sigma"] \arrow[d, "f"'] \arrow[dr, phantom, "\Sigma"] & \cdot \arrow[d, "f'"] \\ \cdot \arrow[r, "s'\in\Sigma"'] & \cdot \end{tikzcd}$$
$\exists s', f'$

(4) EQUALISATION.
$$\forall \begin{tikzcd} \cdot \arrow[r, "r"] \arrow[d, "f"'] \arrow[dr, phantom, "\Sigma"] & \cdot \arrow[d, "a"] \\ \cdot \arrow[r, "s"'] & \cdot \end{tikzcd}, \begin{tikzcd} \cdot \arrow[r, "r"] \arrow[d, "f"'] \arrow[dr, phantom, "\Sigma"] & \cdot \arrow[d, "b"] \\ \cdot \arrow[r, "s"'] & \cdot \end{tikzcd}, \quad \exists \begin{tikzcd} \cdot \arrow[r, "s"] \arrow[d, equal] \arrow[dr, phantom, "\Sigma"] & \cdot \arrow[d, "d"] \\ \cdot \arrow[r, "ds"'] & \cdot \end{tikzcd}$$
with $da = db$

(5) EQUINSERTION.
$$\forall \begin{tikzcd} \cdot \arrow[r, "r"] \arrow[d, "f"'] \arrow[dr, phantom, "\Sigma"] \arrow[rr, bend left, "r"] & \cdot \arrow[d, "f'"] \arrow[r, phantom, "\Rightarrow\alpha"] & \cdot \arrow[dl, bend left, "g"] \\ \cdot \arrow[r, "s"'] & \cdot & \end{tikzcd}$$
$\exists d, \alpha'$
$$\begin{tikzcd} \cdot \arrow[r, "s"] \arrow[d, equal] \arrow[dr, phantom, "\Sigma"] & \cdot \arrow[d, "d"] \\ \cdot \arrow[r, "ds"'] & \cdot \end{tikzcd} = \begin{tikzcd} \cdot \arrow[r, "r"] \arrow[d, "f"'] \arrow[dr, phantom, "\Sigma"] & \cdot \arrow[d, "f'"] \arrow[dr, bend left, "g"] \arrow[d, phantom, shift left=4, "\Rightarrow\alpha'"] & \\ \cdot \arrow[r, "s"] \arrow[d, equal] \arrow[dr, phantom, "\Sigma"] & \cdot \arrow[d, "d"] & \cdot \arrow[dl, "d"] \\ \cdot \arrow[r, "ds"'] & \cdot & \end{tikzcd}$$

(6) EQUIFICATION.
$$\forall \begin{tikzcd} \cdot \arrow[r, "r"] \arrow[d, "f"'] \arrow[dr, phantom, "\Sigma"] & \cdot \arrow[d, bend right, "f'"'] \arrow[d, bend left=80, "g"] \arrow[d, phantom, shift left=3, "\overset{\alpha}{\Rightarrow}\underset{\beta}{\Rightarrow}"] \\ \cdot \arrow[r, "s"'] & \cdot \end{tikzcd}$$
with $\alpha r = \beta r$

$$\exists \begin{tikzcd} \cdot \arrow[r, "s"] \arrow[d, equal] \arrow[dr, phantom, "\Sigma"] & \cdot \arrow[d, "d"] \\ \cdot \arrow[r, "ds"'] & \cdot \end{tikzcd}$$
with $d\alpha = d\beta$

Examples.

- Laris and squares with the Beck-Chevalley condition

- Let $\mathcal{X}$ be an ordinary cat. and $\Sigma \subseteq \mathrm{Mor}(\mathcal{X})$ admitting a left calculus of fractions.
  Σ, seen as a full subcategory of $\mathcal{X}^{\to}$, for $\mathcal{X}$ seen as a 2-Cat., admits a left calculus of lax fractions.

Let $\mathcal{A}$ be a locally full 2-subcategory of $\mathcal{X}$.
Then, in each one of the following cases,

$$\mathcal{A}^{\mathrm{LInj}}$$

admits a left calculus of lax fractions:

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- $\mathcal{X}$ has 2-colimits. (3), (5), (6) in [DiLiberti, Lobbia, S., 2023]

Let $\mathcal{A}$ be a locally full 2-subcategory of $\mathcal{K}$.
Then, in each one of the following cases,

$$\mathcal{A}^{\mathrm{LInj}}$$

admits a left calculus of lax fractions:

- $\mathcal{K}$ has 2-colimits. (3), (5), (6) in [Di Liberti, Lobbia, S., 2023]
- $\mathcal{A} \hookrightarrow \mathcal{K}$ is the inclusion of the 2-category of algebras of a lax-idempotent monad.

Examples.

In **Pos**

- embeddings
- comm. squares
$$\begin{tikzcd} X \arrow[r, "m"] \arrow[d, "u"'] & Y \arrow[d, "v"] \\ Z \arrow[r, "n"'] & W \end{tikzcd}$$

s.t. $\forall y \in Y\ z \in Z,\ n(z) \leq v(y) \Rightarrow \exists x \in X,\ z \leq u(x)\ \&\ m(x) \leq y$

admits a left calculus of lax fractions

[S., 2017]

Examples.

In **Pos** for $\mathcal{A}_e = \left\{ \boxed{\begin{smallmatrix} \bullet\, 1 \\ | \\ \bullet\, 0 \end{smallmatrix}} \right\}$, $\mathcal{A}_e^{\mathrm{LInj}}$ consists of

- embeddings
- comm. squares $\begin{tikzcd} X \arrow[r, "m"] \arrow[d, "u"'] & Y \arrow[d, "v"] \\ Z \arrow[r, "n"'] & W \end{tikzcd}$

s.t. $\forall y \in Y\ z \in Z,\ n(z) \leq v(y) \Rightarrow \exists x \in X,\ z \leq u(x) \;\&\; m(x) \leq y$

admits a left calculus of lax fractions

[S., 2017]

In **Loc**, the following admit a left calculus of lax fractions:

- embeddings

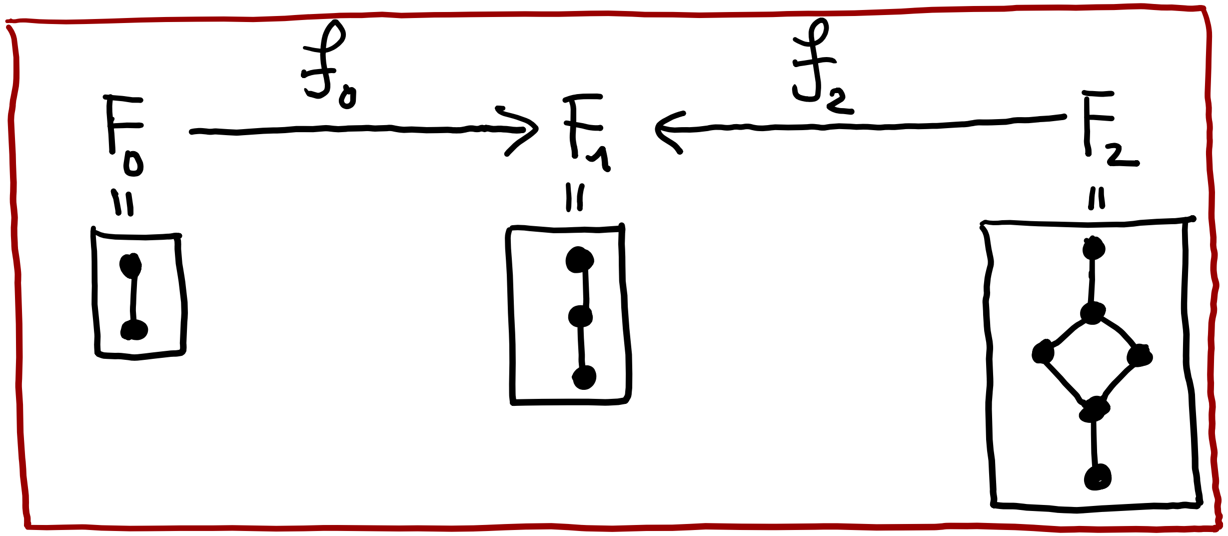
  squares
$$\begin{tikzcd} \cdot \arrow[r, "m"] \arrow[d, "u"'] \arrow[dr, phantom, "="] & \cdot \arrow[d, "v"] \\ \cdot \arrow[r, "n"'] & \cdot \end{tikzcd}$$
  with
$$\begin{tikzcd} \cdot \arrow[r, "m"] \arrow[dr, phantom, "="] & \cdot \\ \cdot \arrow[r, "n"'] \arrow[u, "u^*"] & \cdot \arrow[u, "v^*"'] \end{tikzcd}$$

- the full subcategory of $\mathcal{F}_0$ on all dense embeddings

- " " " " " " " " flat embeddings

[S., 2017], [Carvalho, S., 2015]

In **Loc**, for $\mathcal{A} =$



we have:

- $\mathcal{F}_0 = \mathcal{F}_0^{\mathrm{LInj}}$ consists of embeddings and squares $\begin{tikzcd} \cdot \arrow[r, "m"] \arrow[d, "u"'] \arrow[dr, phantom, "="] & \cdot \arrow[d, "v"] \\ \cdot \arrow[r, "n"'] & \cdot \end{tikzcd}$ with $\begin{tikzcd} \cdot \arrow[r, "m"] \arrow[dr, phantom, "="] & \cdot \\ \cdot \arrow[u, "u^*"] \arrow[r, "n"'] & \cdot \arrow[u, "v^*"'] \end{tikzcd}$
- $\boxed{\mathcal{F}_0 \xrightarrow{f_0} \mathcal{F}_1}^{\mathrm{LInj}}$ is the full subcategory of $\mathcal{F}_0$ of all dense embeddings
- $\mathcal{A}^{\mathrm{LInj}}$ " " " " " " " " flat embeddings

[S., 2017], [Carvalho, S., 2015]

In **Cat**,

Embeddings with squares
$$\begin{tikzcd} A \arrow[r, "E"] \arrow[d, "G"'] & B \arrow[d, "H"] \\ C \arrow[r, "F"'] & D \end{tikzcd}$$

such that
$$\begin{tikzcd} & & B \arrow[dd, bend left=60, "\mathrm{Lan}_E(YG)"] \\ C \arrow[d, "Y"'] \arrow[r, "F"] & D \arrow[dl, "\mathrm{Lan}_F Y"] \arrow[r, phantom, "="] & {} \\ {[C^{op}, \mathrm{Set}]} & & {} \end{tikzcd}$$

admit a left calculus of lax fractions.

In $\mathbf{Cat}(\mathbf{Mon})$, i.e., (strict) monoidal categories and (strict) monoidal functors

Σ { Monoidal functors whose underlying functor has a fully faithful right adjoint with comm. squares whose underlying functors have the Beck-Chevalley condition

admit a right calculus of lax fractions.

[Vitale, CTGDC, 2010]:

Internal groupoids in Grp and monoidal functors form a 2-category of fractions of internal groupoids and internal functors in Grp with respect to weak equivalences.

Monoidal categories, lax monoidal functors and monoidal transformations

is a 2-category of right lax fractions

of $\mathrm{Cat}(\mathrm{Mon})$ with respect to

Σ monoidal functors whose underlying functor has a fully faithful right adjoint and the commutative squares whose functors have the Beck-Chevalley condition.

$\mathcal{X}$ 2-category, Σ subcategory of $\mathcal{X}^{\rightarrow}$

Σ-cospan from A to B: $A \xrightarrow{f} I \xleftarrow{r \,\in \Sigma} B$

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2-morphism from (f, r) to (g, s):

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dd, equal] \arrow[ddr, phantom, "\alpha\Swarrow"] & I \arrow[d, "x_1"'] & B \arrow[l, "r"'] \arrow[d, equal] \\
 & X & B \arrow[l, "t"'] \arrow[d, equal] \\
A \arrow[r, "g"'] & J \arrow[u, "x_2"] & B \arrow[l, "s"]
\end{tikzcd}$$

$(\alpha, x_1, x_2) : (f, r) \Rightarrow (g, s)$

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$(\alpha, x_1, x_2) : (f,r) \Longrightarrow (g,s)$

A Σ-extension of (α, x_1, x_2):

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dddd, equal] & I \arrow[d, "x_1"'] \arrow[dr, phantom, "\cong"] & B \arrow[l, "r"'] \arrow[d, equal] \\
 & X \arrow[d, "d"'] \arrow[dr, phantom, "\cong"] & B \arrow[l, "t"'] \arrow[d, equal] \\
\arrow[r, phantom, "\Swarrow d\alpha"] & D & B \arrow[l] \arrow[d, equal] \\
 & X \arrow[u, "d"] \arrow[ur, phantom, "\cong"] & B \arrow[l, "t"'] \arrow[d, equal] \\
A \arrow[r, "g"'] & J \arrow[u] \arrow[ur, phantom, "\cong"] & B \arrow[l, "s"]
\end{tikzcd}$$

$(d\alpha, dx_1, dx_2) : (f,r) \Rightarrow (g,s)$

For $\mathcal{X}$ a 2-cat. and Σ a subcat. of $\mathcal{X}^{\rightarrow}$ admitting a left calculus of lax fractions, we have a bicategory of left lax fractions $\mathcal{X}[\Sigma_*]$ such that:

objs: those of $\mathcal{X}$

1-cells: Σ-cospans

2-cells: $\sim$-equivalence classes, where 2-morphisms are $\sim$-related if they have a common Σ-extension.