Extreme supercharacters of the infinite unitriangular group

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A BRANCHING SCHEME is a pair

$$(\Gamma, \kappa)$$

consisting of

- a *graded graph* with
  - vertex set: $\Gamma = \bigcup_{n\geq 0} \Gamma_n$
  - edge set: $\mathcal{A}(\Gamma) \subseteq \bigcup_{n\geq 0} (\Gamma_n \times \Gamma_{n+1})$

  where the edges only connect vertices from neighbouring levels,
- a *multiplicity function*

$$\kappa\colon \mathcal{A}(\Gamma) \to \mathbb{N}, \qquad (\lambda, \Lambda) \mapsto \kappa(\lambda, \Lambda).$$

We assume that there is a single vertex of level 0 and we denote it by $\emptyset$; thus, $\Gamma_0 = \{\emptyset\}$.

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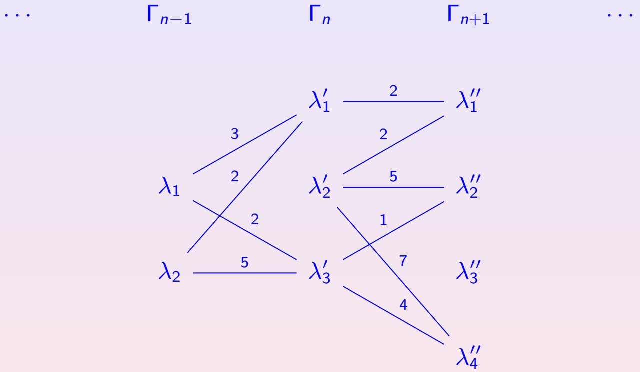
$\cdots$ Γ_{n-1} Γ_n Γ_{n+1} $\cdots$
λ_1
λ_2
λ'_1
λ'_2
λ'_3
λ''_1
λ''_2
λ''_3
λ''_4
3
2
2
5
2
2
5
1
7
4

Young scheme

- $\Gamma_n = \{\text{Young diagrams with } n \text{ boxes}\}, \qquad n \in \mathbb{N}$.
- There is an edge between the vertices $\lambda \in \Gamma_n$ and $\Lambda \in \Gamma_{n+1}$ if Λ is obtained from λ by adjunction of a box.
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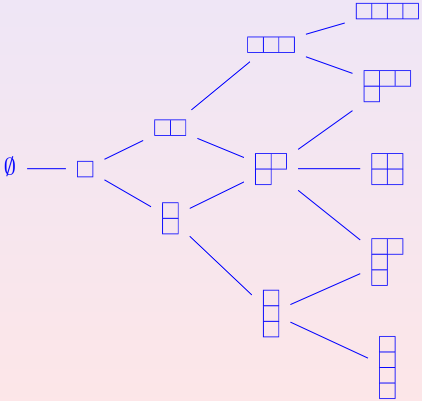

The Young's scheme (Γ, κ) is associated with the irreducible characters of the symmetric groups S_n (for $n \in \mathbb{N}$).

- Each vertex $\lambda \in \Gamma_n$ corresponds to an irreducible character χ_λ of S_n (and vice versa).
- For $\lambda \in \Gamma_{n-1}$ and $\Lambda \in \Gamma_n$, the multiplicity $\kappa(\lambda), \Lambda)$ is determined by restriction

$$\mathrm{Res}^{S_n}_{S_{n-1}}(\chi_\Lambda) = \sum_{\lambda \in \Gamma_{n-1}} \kappa(\lambda, \Lambda)\chi_\lambda.$$

Or, equivalently, by induction

$$\mathrm{Ind}^{S_n}_{S_{n-1}}(\chi_\lambda) = \sum_{\Lambda \in \Gamma_n} \kappa(\lambda, \Lambda)\chi_\Lambda.$$

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For each $n \in \mathbb{N}$, we consider

$$\Gamma_n = \{\text{partitions of the set } [n]\}$$

where $[n] = \{1, 2, \ldots, n\}$..

- If λ is a partition of $[n]$ and $1 \leq i < j \leq n$, the pair (i, j) is a *arc of* λ if i and j occur in the same block B of λ and there is no $k \in B$ with $i < k < j$; we denote by $D(\lambda)$ the set of arcs of λ.
- The *standard representation* of λ is the graph with vertices $1, 2, \ldots, n$ and edges $D(\lambda)$. For example, $\lambda = 157/3/4/689$ has a standard representation

A scheme with set partitions)

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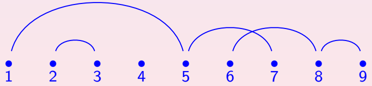

Let:

- $\mathbb{F}_q$ the finite field with q elements.
- $\mathfrak{u}_n(q)$ the nilpotent $\mathbb{F}_q$-algebra nilpotente consisting of all (strictly) upper-triangular $n \times n$ matrices with coefficients in $\mathbb{F}_q$.
- $U_n(q) = 1 + \mathfrak{u}_n(q)$ the UNITRIANGULAR GROUP over $\mathbb{F}_q$.

To each partition λ of $[n]$ we associate the matrix

$$e_\lambda = \sum_{(i,j)\in D(\lambda)} e_{i,j}$$

where

$$\{e_{i,j} : 1 \leq i < j \leq n\}$$

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For example, for $\lambda = 157/23/4/689$,

$$e_\lambda = \begin{pmatrix} 0 & & & & 1 & & & & \\ & 0 & 1 & & & & & & \\ & & 0 & & & & & & \\ & & & 0 & & & & & \\ & & & & 0 & & 1 & & \\ & & & & & 0 & & 1 & \\ & & & & & & 0 & & \\ & & & & & & & 0 & 1 \\ & & & & & & & & 0 \end{pmatrix}$$

For each partition λ of $[n]$, we define the SUPERCLASS

$$\mathcal{K}_\lambda = 1 + B_n(q) e_\lambda B_n(q) \subseteq U_n(q)$$

where

$$B_n(q) = \{\text{invertible upper-triangular matrices with coefficients in } \mathbb{F}_q\}.$$

It is clear that $U_n(q)$ is the disjoint union

$$U_n(q) = \bigcup_{\lambda \vdash [n]} \mathcal{K}_\lambda,$$

and that each superclass is a union of conjugacy classes of $U_n(q)$.

A function $\xi \colon U_n(q) \to \mathbb{C}$ is said to be a SUPERCLASS FUNCTION if ξ is constant in each superclass of $U_n(q)$. We define

$$SC_n = \{\text{superclass functions of } U_n(q)\}.$$

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SC_n is a (complex) vector space with a base $\{\kappa_\lambda \colon \lambda \vdash [n]\}$ formed by the characteristic functions of the superclasses.

Another base is made up of the supercharacters of $U_n(q)$.

A function $\xi \in SC_n$ is a SUPERCHARACTER OF $U_n(q)$ if:

- ξ is *supercentral*: $\xi(g) = \xi(e_\lambda)$ whenever $g \in \mathcal{K}_\lambda$ for $\lambda \vdash [n]$.
- ξ is *normalised*: $\xi(1) = 1$.
- ξ is *positive definite*:
  - $\xi(g^{-1}) = \overline{\xi(g)}$ for all $g \in G$.
  - For any $g_1, \ldots, g_k \in G$, the hermitic matrix

$$\left[\phi(g_i g_j^{-1})\right]_{1 \leq i,j \leq k}$$

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We consider $U_{n-1}(q)$ as a subgroup of $U_n(q)$ in a natural way:

$$U_{n-1}(q) \cong \begin{pmatrix} U_{n-1}(q) & 0 \\ 0 & 1 \end{pmatrix} \leq U_n(q).$$

For each $\Lambda \vdash [n]$, the *restriction*

$$\mathrm{Res}^{U_n(q)}_{U_{n-1}(q)}(\xi_\Lambda)$$

is a superclass function of $U_{n-1}(q)$.

In fact,

$$\mathrm{Res}^{U_n(q)}_{U_{n-1}(q)}(\xi_\Lambda) = \sum_{\lambda \vdash [n-1]} \kappa(\lambda, \Lambda)\xi_\lambda, \qquad \kappa(\lambda, \Lambda) \in \mathbb{N}_0.$$

In this way, we obtain the multiplicity function

$$\kappa \colon \Gamma \times \Gamma \to \mathbb{N}_0$$

which defines a scheme (Γ, κ).

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Example

For $1 \leq i < j \leq n$, we define $\xi_{i,j} \colon U_n(q) \to \mathbb{C}$ as follows.

If $g \in K_\lambda$ for $\lambda \vdash [n]$, we put

$$d_{i,j}(\lambda) = \#\{i < k < j \colon (k,l) \in D(\lambda) \text{ for some } i < k < l < j\}.$$

Then,

$$\xi_{i,j}(g) = \begin{cases} 0, & \text{if } \{(i,k),(k,j)\} \cap D(\lambda) \neq \emptyset \text{ for some } i < k < j, \\ -q^{d_{i,j}(\lambda)}, & \text{if } (i,j) \in D(\lambda), \\ q^{d_{i,j}(\lambda)}, & \text{otherwise} \end{cases}$$

$\xi_{i,j}$ is the supercharacter associated with the partition $\lambda \vdash [n]$ such that $D(\lambda) = \{(i,j)\}$.

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Then,

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We can prove that

$$\mathrm{Res}^{U_n(q)}_{U_{n-1}(q)}(\xi_{i,j}) = \begin{cases} \xi_{i,j}, & \text{if } j < n, \\ 1_{U_{n-1}(q)} + \xi_{i,i+1} + \cdots + \xi_{i,n-1}, & \text{otherwise.} \end{cases}$$

Note that the trivial character

$$1_{U_n(q)} = \xi_{1/2/\dots/n}$$

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We denote by

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With respect to the topology of pointwise convergence, the space $\mathcal{H}(\Gamma,\kappa)$ is convex, compact and metrizable. We denote by

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Harmonic functions occur naturally associated with extreme characters, or supercharacters, of certain group chains.

For example,

- The chain

$$S_1 \subseteq S_2 \subseteq \cdots \subseteq S_n \subseteq S_{n+1} \subseteq \cdots$$

  is associated with the *extreme characters* of the *infinite symmetric group*

$$S_\infty = \bigcup_{n \in \mathbb{N}} S_n.$$

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- ξ is *normalised*.
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The set SCh of supercharacters of ∞ is convex. An extreme point of SCh is called an EXTREME SUPERCHARACTER of $U_\infty(q)$.

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Similar constructions can be obtained for other discrete algebra groups, for example for $U_n(\Bbbk)$ where $\Bbbk$ is the algebraic closure of a finite field.

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For $1 \leq i \leq n$, we define $\xi_i \colon U_\infty(q) \to \mathbb{C}$ as follows.

If $g \in U_\infty(q)$, we choose $n \in \mathbb{N}$ such that $g \in U_n(q)$. Then, $g \in \mathcal{K}_\lambda$ for some $\lambda \vdash [n]$.

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we define

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Indeed:

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For any extreme supercharacter ξ of $U_\infty(q)$, there exists a sequence

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Therefore, the coefficients $\varphi(\lambda)$ satisfy

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$$D^+(\Gamma, \kappa) \subseteq D(\Gamma, \kappa)$$

to be the set of non-negative functions in $D(\Gamma, \kappa)$.

The following properties hold:

- If $\psi \in D^+(\Gamma, \kappa)$, then $n\psi \in D^+(\Gamma, \kappa)$ for all $n \in \mathbb{N}$.
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to be the set of non-negative functions in $D(\Gamma, \kappa)$.

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To each finite path $p = (\emptyset, \lambda_1, \ldots, \lambda_n)$ in Γ, we associate the product

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and define the DIMENSION OF A VERTEX $\lambda \in \Lambda$ by

$$\dim(\lambda) = \sum_{\text{paths } p \text{ from } \emptyset \text{ to } \lambda} \kappa(p).$$

The correspondence $\lambda \mapsto \dim(\lambda)$ defines a special element $e \in D^+(\Gamma, \kappa)$ that satisfies the *archimedean property*:

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In fact, the study of the space $\mathcal{H}(\Gamma, \kappa)$ reduces to the study of the space $\mathcal{F}_e(\Gamma, \kappa)$ consisting of all homomorphisms of ordered groups $\vartheta\colon D(\Gamma, \kappa) \to \mathbb{R}$. In particular, the space SCh also reduces to the study of $\mathcal{F}_e(\Gamma, \kappa)$.

For some branching schemes, the group $D(\Gamma, \kappa)$ has a structure of *Riesz ring*, and this guarantees (for example) that the boundary $\partial(\Gamma, \kappa)$ is a closed subset of $\mathcal{H}(\Gamma, \kappa)$.

$D(\Gamma, \kappa)$ is a RIESZ RING if there exists a multiplication in $D(\Gamma, \kappa)$ that is compatible with the order $\leq$ and such that the order identity $e \in D^+(\Gamma, \kappa)$ is the identity of $D(\Gamma, \kappa)$ with respect to this multiplication.

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We say that the branching scheme (Γ, κ) is MULTIPLICATIVE if there exists a commutative graded ring with identity

$$R = \bigoplus_{n \in \mathbb{N}_0} R_n$$

satisfying:

- There exists a basis $\{a_\lambda : \lambda \in \Gamma\}$ consisting of homogeneous elements such that for all $n \in \mathbb{N}_0$, $\{a_\lambda : \lambda \in \Gamma_n\}$ is a basis of R_n.
- There is an element $a_1 \in R_1$ such that:
  - $a_1 = \sum_{\lambda \in \Gamma_1} m_\lambda a_\lambda, \qquad m_\lambda \geq 0$ to $\lambda \in \Gamma_1$,
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Example

The Young's scheme is *multiplicative*:

- For $R = \bigoplus_{n \in \mathbb{N}_0} R_n$, we consider the ring of symmetric functions in commutative indeterminates $X_1, X_2, \ldots, X_n, \ldots$,
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In general, we have

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If (Γ, κ) is a multiplicative branching scheme and $\varphi \in \mathcal{H}(\Gamma, \kappa)$, we define $\psi\colon R \to \mathbb{R}$ by

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Then φ is an extreme harmonic function if and only if ψ is a ring homomorphism.

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If (Γ, κ) is a multiplicative branching scheme, then the boundary $\partial(\Gamma, \kappa)$ is closed under $\mathcal{H}(\Gamma, \kappa)$.

In the case of the group $U_\infty(q)$, the $\mathbb{Z}$-module

$$\mathrm{SC} = \bigoplus_{n\in\mathbb{N}} \mathrm{SC}_n$$

has a graded ring structure with respect to multiplication defined by

$$\xi \cdot \xi' = \mathrm{SInd}_{U_m(q)\times U_n(q)}^{U_{m+n}(q)}(\xi \times \xi'), \qquad \xi \in \mathrm{SC}_m,\ \zeta \in \mathrm{SC}_n\,.$$

Here, the direct product $U_m(q) \times U_n(q)$ is identified with the subgroup

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SUPERINDUCTION $\mathrm{SInd}_{U_m(q)\times U_n(q)}^{U_{m+n}(q)}(\xi \times \xi')$ is uniquely determined by the formula

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where $\langle \cdot, \cdot \rangle$ is the usual Frobenius product.

In particular, if ξ_1 is the unique supercharacter of $U_1(q)$, then

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Thus, the base of the supercharacters $\{\xi_\lambda : \lambda \in \Gamma\}$ satisfies the conditions required for (Γ, κ) to be a multiplicative scheme.

However, SC is a **non-commutative** ring.

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For each $J \subseteq [n]$, let $U_J(q)$ be the subgroup of $U_n(q)$ consisting of all matrices whose entries (i,j), for $1 \leq i < j \leq n$ with $i,j \notin J$, are zero. Of course, if $|J| = m$, then there is an isomorphism

$$\mathrm{st}_J \colon U_J(q) \to U_m(q).$$

For example, for $J = \{2,4,6\} \subseteq [6]$, we have

$$U_J(q) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & * & 0 & * \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cong \begin{bmatrix} 1 & * & * & 0 & 0 & 0 \\ 0 & 1 & * & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cong U_3(q).$$

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$$U_J(q) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & * & 0 & * \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cong \begin{bmatrix} 1 & * & * & 0 & 0 & 0 \\ 0 & 1 & * & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cong U_3(q).$$

For $J \subseteq [n]$ with $|J| = m$ and $\xi \in \mathsf{SC}_n$, we define the *J-restriction*

$$^{J}\mathrm{Res}^{U_n(q)}_{U_m(q)\times U_{n-m}(q)}(\xi) = \mathrm{Res}^{U_n(q)}_{U_{J/J^c}(q)}(\xi) \circ \mathrm{st}^{-1}_{J/J^c}$$

where $J^c = [n] \setminus J$, $U_{J/J^c}(q) = U_J(q)U_{J^c}(q)$ and

$$\mathrm{st}_{J/J^c}(gg') = \begin{pmatrix} g & 0 \\ 0 & g' \end{pmatrix}, \qquad g \in U_J(q),\ g' \in U_{J^c}(q).$$

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In particular, we obtain a map $\mathrm{Res}'^{U_n(q)}_{U_{n-1}(q)} \colon \mathrm{SC}_n \to \mathrm{SC}_{n-1}$ defined by

$$\mathrm{Res}'^{U_n(q)}_{U_{n-1}(q)}(\xi) = \sum_{J \subseteq [n],\ |J|=n-1} {}^J\,\mathrm{Res}^{U_n(q)}_{U_{n-1}(q)\times U_1(q)}(\xi).$$

We define a new *branching scheme* (Γ, κ') in which the multiplicity function is defined by the decomposition

$$\mathrm{Res}'^{U_n(q)}_{U_{n-1}(q)}(\xi_\Lambda) = \sum_{\lambda \in \Gamma_{n-1}} \kappa'(\lambda, \Lambda)\xi_\lambda, \qquad \Lambda \in \Gamma_n.$$

On the other hand, if $J \subseteq [n]$ and $|J| = m$, we define the *J-superinduction* by

$${}^J\,\mathrm{SInd}^{U_{m+n}(q)}_{U_m(q)\times U_n(q)}(\xi \times \xi') = \mathrm{SInd}^{U_{n+m}(q)}_{U_{J/J^c}(q)}((\xi \times \xi') \circ \mathrm{st}_{J/J^c}).$$

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We thus obtain a commutative multiplication in $\mathsf{SC} = \bigoplus_{n\in\mathbb{N}} \mathsf{SC}_n$ by setting

$$\xi \cdot \xi' = \sum_{J\subseteq m+n,\ |J|=m} {}^{J}\,\mathsf{SInd}_{U_m(q)\times U_n(q)}^{U_{m+n}(q)}(\xi \times \xi').$$

For this multiplication, we have

$$\xi_1 \cdot \xi_\lambda = \sum_{\Lambda\in\Gamma_{n+1}} \kappa'(\lambda,\Lambda)\xi_\Lambda, \qquad \lambda \in \Gamma_n,\ n \in \mathbb{N},$$

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$$\xi(g) = \lim_{n\to\infty} \xi_n(g), \qquad g \in U_\infty(q),$$

is an extreme supercharacter of $U_\infty(q)$.

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Thank you!!!