On chain polynomials of rank uniform posets and geometric lattices

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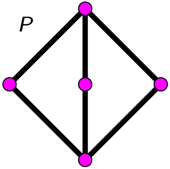
Motivation

Given a finite poset P, its chain polynomial $c_P(t)$ is defined as

$$c_P(t) := \sum_{k \geq 0} c_k(P) t^k,$$

where $c_k(P)$ is the number of k-element chains in P.



$$c_P(t) = 1 + 5t + 7t^2 + 3t^3$$

Question: For which posets is the chain polynomial real-rooted?

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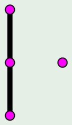
Examples

(i) Boolean lattices (the h-polynomials of the order complexes are the Eulerian polynomials);
(ii) Face lattices of simplicial polytopes (Brenti-Welker, 2008);
(iii) Posets that do not contain the following as an induced subposet (Stanley, 2009):



(iv) Subspace lattices and partition lattices of types A and B (Athanasiadis, Kalampogia-Evangelinou, 2023)

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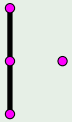

(iv) Subspace lattices and partition lattices of types A and B (Athanasiadis, Kalampogia-Evangelinou, 2023)

Counterexample

There exist finite distributive lattices with non real-rooted chain polynomials (Stembridge, 2007).

Main idea

If Δ is a simplicial complex of dimension $n-1$, then the *h-vector* of Δ is defined as the sequence $(h_0(\Delta), h_1(\Delta), \ldots, h_n(\Delta))$ for which

$$f_\Delta(t) = \sum_{k=0}^{n} h_k(\Delta) t^k (1+t)^{n-k}.$$

$h_0, \ldots, h_n \geq 0 \implies c_\Delta(t)$ is real-rooted

Idea: Given a complex Γ, find polynomials $R^\Gamma_{n,k}(t)$ such that

$$f_\Gamma(t) = \sum_{k=0}^{n} h_k(\Gamma) R^\Gamma_{n,k}(t).$$

UD-generated sequences of polynomials and total nonnegativity

$$U = (u_{ij})_{i,j=0}^{\infty} = \begin{cases} 1 & \text{if } i \leq j \\ 0 & \text{if } i > j \end{cases}$$

$$\mathrm{UD} = \{U \,\mathrm{diag}(\lambda_{n,0}, \lambda_{n,1}, \ldots, \lambda_{n,n}, 0, 0, \ldots) \colon \lambda_{n,k} \geq 0 \text{ for all } 0 \leq k \leq n\}$$

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Define $\mathrm{R}_0 = (1, 1, 1, \ldots)^T$, and $\mathrm{R}_n = (R_{n,0}(t), R_{n,1}(t), \ldots)^T$ recursively by

$$\mathrm{R}_{n+1} = t^{n+1}\mathrm{R}_0 + \mathrm{A}^{(n)}\mathrm{R}_n, \quad 0 \leq n \leq N-1. \tag{1}$$

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Definition (UD-generated sequence)

We say that $\{R_{n,k}(t)\}$ is UD-*generated* if R_n satisfies the recursion (1) where $\mathrm{A}^{(n)} \in \mathrm{UD}$ for each $0 \leq n \leq N-1$.

Definition (Totally nonnegative matrix)

A matrix is called *totally nonnegative* if all its minors are nonnegative.

Theorem (Brändén-Saud, 2024)

Let $R = (r_{n,k})_{n,k=0}^{N}$, where $N \in \mathbb{N} \cup \{\infty\}$, be a lower triangular matrix with all diagonal entries equal to one. Let further $R_n(t) = \sum_{k=0}^{n} r_{n,k} t^k$ be the row generating polynomial of the nth row. The following are equivalent:

(i) $\{R_n(t)\}_{n=0}^{N}$ *is* UD*-generated.*

(ii) *There is a matrix $(\lambda_{n,k})_{n,k=0}^{N-1}$ of nonnegative numbers, and an array of monic polynomials $(R_{n,k}(t))_{0 \le k \le n \le N}$ such that $R_{n,0}(t) = R_n(t)$, $t^k \mid R_{n,k}(t)$, and $R_{n+1,k+1}(t) = R_{n+1,k}(t) - \lambda_{n,k} R_{n,k}(t)$ for all $0 \le k \le n < N$.*

(iii) *There are linear (diagonal) operators $\alpha_i : \mathbb{R}[t] \to \mathbb{R}[t]$, $1 \le i \le N$, such that*

$$\alpha_i(t^k) = \alpha_{i,k} t^k, \text{ where } \alpha_{i,k} \ge 0 \text{ for all } i, k,$$

and

$$R_n(t) = (t + \alpha_1)(t + \alpha_2) \cdots (t + \alpha_n)1.$$

(iv) *R is* TN*.*

Moreover if (iii) is satisfied, then the polynomials $R_{n,k}(t) = (t + \alpha_1) \cdots (t + \alpha_{n-k}) t^k$ satisfy (ii).

Interlacing sequences of polynomials

Suppose f and g are real-rooted polynomials with positive leading coefficients, and that $\cdots \leq \alpha_3 \leq \alpha_2 \leq \alpha_1$ and $\cdots \leq \beta_3 \leq \beta_2 \leq \beta_1$ are the zeros of f and g, respectively. We say that f interlaces g ($f \prec g$) if

$$\cdots \leq \alpha_3 \leq \beta_3 \leq \alpha_2 \leq \beta_2 \leq \alpha_1 \leq \beta_1.$$

A sequence $\{f_i\}_{i=0}^n$ of real-rooted polynomials with nonnegative coefficients is said to be *interlacing* if $f_i \prec f_j$ for all $i < j$.

Subdivision operators on rank uniform posets

Definition (Rank uniform poset)

A poset P with a least element $\hat{0}$ is *rank uniform* if

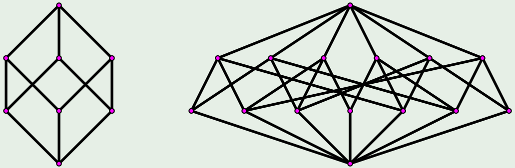
(i) P is locally finite and ranked, with a rank function $\rho : P \to \mathbb{N}$, i.e., $\rho(\hat{0}) = 0$ and $\rho(y) = \rho(x) + 1$ whenever y covers x in P, and

(ii) for any $x, y \in P$ with $\rho(x) = \rho(y)$,

$$|\{z \in [\hat{0}, x] : \rho(z) = k\}| = |\{z \in [\hat{0}, y] : \rho(z) = k\}|, \text{ for all } k \geq 0.$$

The *rank* of P is $\rho(P) = \sup\{\rho(x) : x \in P\} \in \mathbb{N} \cup \{\infty\}$.

Examples (B_3, Π_4^{op} and C^2)



Definition (Subdivision operator)

Suppose $\{R_n(t)\}_{n=0}^N$, where $N \in \mathbb{N} \cup \{\infty\}$, is a sequence of monic polynomials in $\mathbb{R}[t]$, where the degree of $R_n(t)$ is n for each n. The *subdivision operator* associated to $\{R_n(t)\}_{n=0}^N$ is the linear operator $\mathcal{E} : \mathbb{R}_N[t] \to \mathbb{R}[t]$ defined recursively by $\mathcal{E}(1) = 1$, and

$$\mathcal{E}(t^n) = t\mathcal{E}(R_n(t) - t^n), \qquad \text{if } 0 < n \leq N. \tag{2}$$

If $\{R_n(t)\}_{n=0}^N$ are the rank generating polynomials of a rank uniform poset P, we say that $\mathcal{E}$ is the subdivision operator of P.

Proposition (Brändén-Saud, 2024)

Let P be a rank uniform poset with rank generating polynomials $\{R_n(t)\}_{n=0}^N$. If $\rho(x) = n > 0$, then

$$\mathcal{E}(t^n) = \sum_{j \geq 1} |\{\hat{0} < x_1 < \cdots < x_j = x\}| \cdot t^j. \tag{3}$$

Moreover if I is a nonempty and finite order ideal of P, then

$$C_I(t) = (1+t)\mathcal{E}(f_I(t)).$$

Theorem (Brändén-Saud, 2024)

If $\{R_{n,k}\}_{n,k=0}^{\infty}$ *is* UD*-generated, then* $\{\mathcal{E}(R_{n,k})\}_{k\geq 0}$ *is an interlacing sequence of polynomials whose zeros all lie in the interval* $[-1,0]$*, for each* $n \geq 0$*.*

R-positive posets

Definition (R-positive poset)

Let P be a rank uniform poset of rank r, and let

$$\mathrm{R} = \{R_{n,k}(t)\}_{n,k=0}^{N}$$

be an array of polynomials. We say that P is R-*positive* if

(i) for each $y \in P$,
$$\sum_{x \leq y} t^{\rho(x)} = R_{\rho(y),0} \qquad \text{and,}$$

(ii) the rank generating polynomial $f_P(t)$ has a nonnegative expansion in the polynomials $\{R_{r,k}(t)\}_{k=0}^{N}$.

Theorem (Brändén-Saud, 2024)

Let R *be a* UD*-generated array. If P is an* R*-positive poset, then the chain polynomial of P is real-rooted.*

Boolean algebras

B_n = Boolean lattice on n elements
Rank polynomial: $R_n^B(t) = (1+t)^n$

$$R_{n,k}(t) := \begin{cases} t^k(1+t)^{n-k} \text{ if } 0 \leq k \leq n \\ t^n \text{ if } k > n \end{cases}$$

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$$\mathsf{A}_B^{(n)} = U\,\mathrm{diag}(\underbrace{1,\dots,1}_{n+1 \text{ times}},0,\dots) = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 0\cdots \\ 0 & 1 & 1 & \cdots & 1 & 1 & 0\cdots \\ 0 & 0 & 1 & \cdots & 1 & 1 & 0\cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0\cdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0\cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \end{pmatrix}$$

q-partition lattices

$Q_n = q$-partition Dowling lattice on $\mathbb{F}_q^n$ (Dowling, 1973)
Rank polynomial: $R_n^{Q^{\mathbf{op}}}(t) = \sum_{i=0}^n T(n,i)t^i$, where

$$T(n,i) = T(n-1,i-1) + [1+(q-1)i]T(n-1,i)$$

$$R_{n,k}^{Q^{\mathbf{op}}}(t) = R_{n,k-1}^{Q^{\mathbf{op}}}(t) - [1+(q-1)(k-1)]R_{n-1,k-1}^{Q^{\mathbf{op}}}(t)$$

$$\begin{aligned} A_{Q^{\mathbf{op}}}^{(n)} &= U \operatorname{diag}(1, q, 2q-1, \dots, nq-(n-1), 0, \dots) \\ &= \begin{pmatrix} 1 & q & 2q-1 & \cdots & nq-(n-1) & 0\cdots \\ 0 & q & 2q-1 & \cdots & nq-(n-1) & 0\cdots \\ 0 & 0 & 2q-1 & \cdots & nq-(n-1) & 0\cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\ 0 & 0 & 0 & \cdots & nq-(n-1) & 0\cdots \\ 0 & 0 & 0 & \cdots & 0 & 0\cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \end{pmatrix} \end{aligned}$$

Partition lattices of types A and B

(q = 2) Π_n^A = lattice of all partitions of the set $[n]$
(q = 3) Π_n^B = lattice of all partitions π of the set $\{n, n-1, \dots, -n\}$ ordered by reverse refinement such that

(i) $B \in \pi \implies (-B) \in \pi$,

(ii) if $\{k, -k\} \subseteq B$ for some $k \in [n]$ and some block $B \in \pi$, then $0 \in B$.

$$A^{(n)}_{(\Pi_n^A)^{\mathbf{op}}} = \begin{pmatrix} 1 & 2 & 3 & \cdots & n+1 & 0\cdots \\ 0 & 2 & 3 & \cdots & n+1 & 0\cdots \\ 0 & 0 & 3 & \cdots & n+1 & 0\cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\ 0 & 0 & 0 & \cdots & n+1 & 0\cdots \\ 0 & 0 & 0 & \cdots & 0 & 0\cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \end{pmatrix} \quad A^{(n)}_{(\Pi_n^B)^{\mathbf{op}}} = \begin{pmatrix} 1 & 3 & 5 & \cdots & 2n+1 & 0\cdots \\ 0 & 3 & 5 & \cdots & 2n+1 & 0\cdots \\ 0 & 0 & 5 & \cdots & 2n+1 & 0\cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\ 0 & 0 & 0 & \cdots & 2n+1 & 0\cdots \\ 0 & 0 & 0 & \cdots & 0 & 0\cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \end{pmatrix}$$

C^r_{n-1} = $(n-1)$-dimensional r-cubical lattice
Rank polynomial: $R^r_n(t) = 1 + t(r+t)^{n-1}$

$$R^r_{n,k}(t) := \begin{cases} R^r_n(t) \text{ if } k = 0 \\ (r-1+t)t^k(r+t)^{n-k-1} \text{ if } 0 < k < n \\ t^n \text{ if } k \geq n \end{cases} .$$

$$\mathrm{A}^{(n)}_r = U\,\mathrm{diag}(1, \underbrace{r, \ldots, r}_{n-1 \text{ times}}, r-1, 0, \ldots)$$

$$= \begin{pmatrix} 1 & r & r & \cdots & r & r-1 & 0\cdots \\ 0 & r & r & \cdots & r & r-1 & 0\cdots \\ 0 & 0 & r & \cdots & r & r-1 & 0\cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\ 0 & 0 & 0 & \cdots & 0 & r-1 & 0\cdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0\cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \end{pmatrix}$$

Subspace lattices

$\mathcal{L}_q^d$ = lattice of all subspaces of a d-dimensional vector space over $\mathbb{F}_q$
Rank polynomial: $R_n^q(t) = \sum_{k=0}^{n} \binom{n}{k}_q t^k$
$R_{n,k}^q(t) = R_{n,k-1}^q(t) - q^k R_{n-1,k-1}^q(t)$

$$\mathrm{A}_q^{(n)} = U\,\mathrm{diag}(1, q, q^2, \dots, q^n, 0, \dots)$$
$$= \begin{pmatrix} 1 & q & q^2 & \cdots & q^n & 0\cdots \\ 0 & q & q^2 & \cdots & q^n & 0\cdots \\ 0 & 0 & q^2 & \cdots & q^n & 0\cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\ 0 & 0 & 0 & \cdots & q^n & 0\cdots \\ 0 & 0 & 0 & \cdots & 0 & 0\cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \end{pmatrix}$$

What else?

Conjecture (Athanasiadis, Kalampogia-Evangelinou, 2023)

The chain polynomial $c_{\mathcal{L}}(t)$ is real-rooted for every geometric lattice (lattice of flats of a matroid) $\mathcal{L}$.

- Prove this conjecture for paving matroids (**Conjecture:** almost all matroids are paving matroids (Mayhew-Newman-Welsh-Whittle, 2011)).

- Generalize the idea of paving matroids and prove the conjecture for other matroids (still not all of them).

- Prove the conjecture for some matroids obtained by single-element extension.

Thank you very much!

Muito obrigado!

Tack så mycket!

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