Arithmetic varieties of numerical semigroups

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Joint work with;
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Introduction

- $>$ Notable elements
- $>$ The smallest arithmetic variety containing a family of numerical semigroups
- $>$ The tree associated with an arithmetic variety
- $>$ The elements of an arithmetic variety with bounded Frobenius number
- $>$ Numerical semigroups with given depth
- $>$ Main references

Notable elements

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> Numerical semigroups with given depth

> Main references

Notable elements

• Let $A \subseteq \mathbb{N}$ we will denote by $\langle A \rangle$ the submonoid of $(\mathbb{N}, +)$ generated by A, that is,

$$\langle A \rangle = \{\lambda_1 a_1 + \cdots + \lambda_n a_n \mid n \in \mathbb{N} \backslash \{0\}, a_i \in A, \lambda_i \in \mathbb{N} \text{ for all } i \in \{1, \ldots, n\}\}.$$

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• $\#\,\mathrm{msg}(S) = \mathrm{e}(S)$ is the embedding dimension of S.

Notable elements

Example

With tree coins of denominations 5,7 and 11 one can obtain
$S = \langle 5, 7, 11 \rangle = \{0, 5, 7, 10, 11, 12, 14, \rightarrow\}$ is a numerical semigroup with
$\mathrm{msg}(S) = \{5, 7, 11\}$, $\mathrm{m}(S) = 5$, $\mathrm{e}(S) = 3$, $\mathrm{F}(S) = 13$ and
$\mathrm{g}(S) = \#\{1, 2, 3, 4, 6, 8, 9, 13\} = 8$

The smallest arithmetic variety containing a family of numerical semigroups

Smallest arithmetic variety containing a family of numerical sem.

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Definition

An *arithmetic variety* is a non-empty family $\mathscr{A}$ of numerical semigroups such that

(a) if $\{S, T\} \subseteq \mathscr{A}$, then $S \cap T \in \mathscr{A}$;

(b) if $S \in \mathscr{A}$ and T is an arithmetic extension of S, then $T \in \mathscr{A}$.

In this case, we say that $\mathscr{A}$ is a *finite arithmetic variety* when $\mathscr{A}$ has finite cardinality.

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Proposition

Let $\mathscr{A}$ be a non-empty family of numerical semigroups. Then $\mathscr{A}$ is a arithmetic variety if and only if the following holds

(a) *if $\{S, T\} \subset \mathscr{A}$, then $S \cap T \in \mathscr{A}$;*

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Proposition

If $\mathcal{F}$ is a family of numerical semigroups, then $\mathscr{A}(\mathcal{F})$ is the smallest arithmetic variety containing $\mathcal{F}$.

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Proposition

If S is a numerical semigroup, then

$$\mathscr{A}(\{S\}) = \left\{ \bigcap_{i=1}^{n} \frac{S}{d_i} \mid n \in \mathbb{N} \setminus \{0\} \textit{ and } \{d_1, \ldots, d_n\} \subset \mathbb{N} \setminus S \right\} \cup \{\mathbb{N}\}.$$

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Theorem

If $\mathcal{F}$ is a non-empty family of numerical semigroups, then

$$\mathscr{A}(\mathcal{F}) = \left\{ \bigcap_{i=1}^{n} T_i \mid n \in \mathbb{N} \setminus \{0\} \text{ and } T_i \in \mathscr{A}(\{S_i\}) \text{ for some } S_i \in \mathcal{F}, i = 1, \ldots, n \right\}.$$

Algorithm

Computation of $\mathscr{A}(\mathcal{F})$. INPUT: *A finite set* $\mathcal{F} = \{S_1, \ldots, S_n\}$ *of numerical semigroups.* OUTPUT: $\mathscr{A}(\mathcal{F})$.

1. *Set* $\mathscr{A}(\mathcal{F}) = \{\mathbb{N}\}$.
2. *For each* $i \in \{1, \ldots, n\}$, *set* $\mathscr{A}_i = \mathscr{A}(\{S_i\})$.
3. *For each* $(T_1, \ldots, T_n) \in \mathscr{A}_1 \times \cdots \times \mathscr{A}_n$, *do*

$$\mathscr{A}(\mathcal{F}) = \mathscr{A}(\mathcal{F}) \cup \{T_1 \cap \ldots \cap T_n\}.$$

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Example

Let $\mathcal{F} = \{\langle 2,5\rangle, \langle 3,5,7\rangle\}$. By Algorithm AE, we have that

$$\mathscr{A}(\{\langle 2,5\rangle\} = \{\mathbb{N}, \langle 2,3\rangle, \langle 2,5\rangle\}; \;\; \mathscr{A}(\{\langle 3,5,7\rangle\} = \{\mathbb{N}, \langle 2,3\rangle, \langle 3,4,5\rangle, \langle 3,5,7\rangle\}.$$

Therefore, by previous Algorithm, we conclude that

$$\mathscr{A}(\mathcal{F}) = \{\mathbb{N}, \langle 2,3\rangle, \langle 2,5\rangle, \langle 3,4,5\rangle, \langle 3,5,7\rangle, \langle 4,5,6,7\rangle, \langle 5,6,7,8,9\rangle\}.$$

$\mathscr{A}$ −system of generators

The set $\{x \in \mathbb{N} \mid ax \mod b \leq cx\}$ is a *proportionally modular numerical semigroup*.
$\mathrm{ED}(e) = \{S \in \mathscr{L} \mid \mathrm{e}(S) = e\}$.

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Proposition

The arithmetic variety $\mathscr{A}(\mathrm{ED}(2))$ *is equal to the set of intersections of finitely many proportionally modular numerical semigroups.*

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Some open problems

By this results, one can deduce an algorithm to decide whether a numerical semigroup belongs to $\mathscr{A}(\mathrm{ED}(2))$. We propose as an open problem to formulate the corresponding algorithm for $\mathscr{A}(\mathrm{ED}(3))$ and, being optimistic, for $\mathscr{A}(\mathrm{ED}(e)),\ e \geq 4$.

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The tree associated with an arithmetic variety

Graph $\mathcal{G}_{\mathscr{A}}$, vertex set is $\mathscr{A}$ and $(S, T) \in \mathscr{A} \times \mathscr{A}$ iff $S = \frac{T}{2}$
$\Leftrightarrow$ the set of children of $S \in \mathscr{A} \setminus \{\mathbb{N}\}$ is $\mathcal{D}_2(S) \cap \mathscr{A}$, with
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If $\mathcal{A}$ is an arithmetic variety, then $\mathcal{G}_{\mathscr{A}}$ is a directed rooted tree with root $\mathbb{N}$.

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Remark

If H is upper $m-$set of S and $m \in S$ is odd
$S(m, H) = \{2s \mid s \in S\} \cup \{2s + m \mid s \in S\} \cup \{2h + m \mid h \in H\}$.
Thus, if $\text{msg}(S) = \{a_1, \ldots, a_e\}$, then
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Theorem

If $S \subsetneq \mathbb{N}$ is a numerical semigroup, then

$$\mathcal{D}_2(S) = \{S(m, H) \mid m \text{ is an odd element of } S \text{ and } H \text{ is an upper } m-\text{set of } S\}.$$

The elements of an arithmetic variety with bounded Frobenius number

The elements of a variety with bounded Frobenius number

If $\mathscr{A}$ is an arithmetic variety and F is a positive integer, we define

$$\mathscr{A}_F := \{S \in \mathscr{A} \mid \mathrm{F}(S) \leq F\}.$$

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Proposition

Let $\mathscr{A}$ be a arithmetic variety and let F be a positive integer. If $S \in \mathscr{A}_F$, then $\mathcal{D}_2(S) \cap \mathscr{A}_F = \{T \in \mathcal{D}_2(S) \mid \mathrm{F}(T) \leq F\} \cap \mathscr{A}$.

The elements of a variety with bounded Frobenius number

Proposition

Let $S \neq \mathbb{N}$ be a numerical semigroup. If m is an odd element of S and H is an upper m−set of S, then

$$\mathrm{F}(S(m,H)) = \begin{cases} \max(2\,\mathrm{F}(S), m-2) & \text{if } H = \mathbb{N} \setminus S; \\ \max(2\,\mathrm{F}(S), 2\max(\mathbb{N} \setminus S \cup H) + m)) & \text{if } H \neq \mathbb{N} \setminus S \end{cases}$$

Theorem

Let S be a numerical semigroup and let F be a positive number. If $2\,\mathrm{F}(S) \leq F$, then $\{T \in \mathcal{D}_2(S) \mid \mathrm{F}(T) \leq F\}$ is equal to the union of

$$\{S(m, \mathbb{N} \setminus S) \mid m \text{ is an odd element of } \{\mathrm{F}(S)+1, \ldots, F+2\}\} \quad \text{and}$$

$$\left\{ S(m,H) \;\middle|\; \begin{array}{l} m \text{ is an odd element of } S \text{ and} \\ H \neq \mathbb{N} \setminus S \text{ is an upper } m\text{−set} \\ \text{with } 2\max(\mathbb{N} \setminus S \cup H) + m \leq F \end{array} \right\}.$$

The elements of a variety with bounded Frobenius number

The algorithm to compute $\{T \in \mathcal{D}_2(S) \mid \mathrm{F}(T) \leq F\}$ for given $S \in \mathscr{L}$ and $F \in \mathbb{N} \setminus \{0\}$.

Algorithm

Computation of $\{T \in \mathcal{D}_2(S) \mid \mathrm{F}(T) \leq F\}$.
INPUT: *A numerical semigroup* S *and a positive integer* F.
OUTPUT: $\{T \in \mathcal{D}_2(S) \mid \mathrm{F}(T) \leq F\}$.

1. *If* $2\,\mathrm{F}(S) > F$, *then return* $\varnothing$.
2. *Set* $A = \{m \in \mathbb{N} \mid m$ *is odd and* $\mathrm{F}(S) + 1 \leq m \leq F + 2\}$.
3. *Set* $B = \{m \in S \mid\mid m$ *is odd and* $m \leq F - 2\}$.
4. *For each* $m \in B$ *define*

$$H(m) = \left\{ H \;\middle|\; \begin{array}{l} H \neq \mathbb{N} \setminus S \text{ is an upper } m\text{−set} \\ \text{such that } \max(\mathbb{N} \setminus S \cup H) \leq \frac{F-m}{2} \end{array} \right\}.$$

5. *Return* $\{S(m, \mathbb{N} \setminus S) \mid m \in A\} \cup \{S(m, H) \mid m \in B$ *and* $H \in H(m)\}$.

The elements of a variety with bounded Frobenius number

Example

Let $S = \langle 4, 5, 11 \rangle$ and $F = 15$. Using the GAP function above, we can verify, as follows, that $\{T \in \mathcal{D}_2(S) \mid \mathrm{F}(T) \leq 15\}$ is equal to

$$\{S(9, \{1,2,3,6,7\}), S(11, \{1,2,3,6,7\}), S(13, \{1,2,3,6,7\}), S(15, \{1,2,3,6,7$$
$$S(17, \{1,2,3,6,7\}), S(5, \{3,6,7\}), S(5, \{6,7\}), S(9, \{1,2,6,7\}), S(9, \{1,3,6,7$$
$$S(9, \{1,6,7\}), S(9, \{2,3,6,7\}), S(9, \{2,6,7\}), S(9, \{3,6,7\}), S(9, \{6,7\}),$$
$$S(11, \{1,3,6,7\}), S(11, \{2,3,6,7\}), S(11, \{3,6,7\}), S(13, \{2,3,6,7\})\}.$$

Now, using the GAP function *UpperMSetToNumericalSemigroup* $:=$ *function*(S, m, H) we obtain that the above set is equal to

$$\{\langle 8,9,10,11,13,15\rangle, \langle 8,10,11,13,15,17\rangle, \langle 8,10,13,15,17,19,22\rangle,$$
$$\langle 8,10,15,17,19,21,22\rangle, \langle 8,10,17,19,21,22,23\rangle, \langle 5,8,11,17\rangle, \langle 5,8,17,19\rangle,$$
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Numerical semigroups with given depth

Definition

The *depth* of S, denoted $\mathrm{depth}(S)$, is the integer number q such that $\mathrm{F}(S)+1 = q\,\mathrm{m}(S) - r$ for some integer $0 \le r < \mathrm{m}(S)$.

$$\textit{or}\ \ \mathrm{depth}(S) = \left\lfloor \frac{\mathrm{F}(S)}{\mathrm{m}(S)} \right\rfloor + 1 \ \text{ and denote } \mathscr{C}_q = \{S \in \mathscr{L} \mid \mathrm{depth}(S) \le q\}.$$

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Theorem

The set $\mathscr{C}_q$ *is an arithmetic variety for every* $q\in\mathbb{N}$*.*

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Definition

The *depth* of S, denoted $\mathtt{depth}(S)$, is the integer number q such that $\mathrm{F}(S)+1 = q\,\mathrm{m}(S) - r$ for some integer $0 \leq r < \mathrm{m}(S)$.

$$\textit{or}\ \ \mathtt{depth}(S) = \left\lfloor \frac{\mathrm{F}(S)}{\mathrm{m}(S)} \right\rfloor + 1 \ \text{ and denote } \mathscr{C}_q = \{S \in \mathscr{L} \mid \mathtt{depth}(S) \leq q\}.$$

Theorem

The set $\mathscr{C}_q$ is an arithmetic variety for every $q \in \mathbb{N}$.

We have that $\mathcal{G}_{(\mathscr{C}_q)_F}$ is a finite rooted tree with root $\mathbb{N}$ such that the set of children of $S \in (\mathscr{C}_q)_F$ is equal to

$$\{T \in \mathcal{D}_2(S) \mid \mathrm{F}(T) \leq F \text{ and } \mathtt{depth}(T) \leq q\}.$$

Therefore, we can formulate an algorithm for the computation of $(\mathscr{C}_q)_F$.

Main references

> Notable elements

> The smallest arithmetic variety containing a family of numerical semigroups

> The tree associated with an arithmetic variety

> The elements of an arithmetic variety with bounded Frobenius number

> Numerical semigroups with given depth

> Main references

Main references

- Manuel B. Branco, Ignacio Ojeda and José Carlos Rosales, Arithmetic varieties of numerical semigroups- submited

- I. Ojeda and J.C. Rosales, The arithmetic extensions of a numerical semigroup. Comm. Algebra **48** (2020), no. 9, 3707–3715.

-A.M. Robles-Pérez, J.C. Rosales, P. Vasco, The doubles of a numerical semigroup. J. Pure Appl. Algebra **213** (2009), no. 3, 387–396.