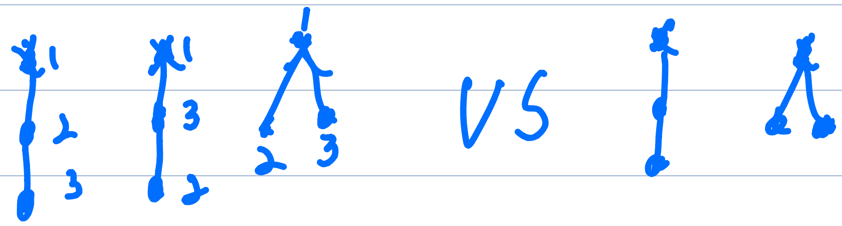
POLYA THEORY REVISITED

1. ENUMERATION UNDER SYMMETRY

- EVERYBODY KNOWS: THERE ARE n^{n-2} ROOTED, LABELED TREES ON n VERTICES



- HOW MANY UNLABELED TREES?

MORE GENERALLY

$\mathcal{X}$ - FINITE SET

G - FINITE GROUP ACTING ON $\mathcal{X}$

$\mathcal{X} = \mathcal{O}_1 \cup \mathcal{O}_2 \cdots \cup \mathcal{O}_k$, $x \in \mathcal{O}_x$

- HOW MANY ORBITS?
- WHAT ARE THEIR SIZES?
- DO THEY HAVE 'NICE NAMES'?
- HOW TO 'PICK AN ORBIT AT RANDOM?

EXAMPLE $G = \mathcal{X}$ $x^t = t^{-1} x t$

ORBITS ARE CONJUGACY CLASSES

eg $G = S_n$ CLASSES INDEXED BY PARTITIONS OF n

HOW TO PICK $\lambda \vdash n$ eg $N = 10^6$?

NOTE THESE QUESTIONS CAN BE DIFFICULT!

eg. $G = U_n(q) = \{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}, * \in \mathbb{F}_q \}$

All PARTS SEEM DIFFICULT.

BURNSIDE PROCESS (G ACTS ON $\mathcal{X}$)

- FROM $x \in \mathcal{X}$, PICK $s \in G_x = \{ s : x^s = x \}$
- FROM $s \in G$, PICK $y \in \mathcal{X}_s = \{ y : y^s = y \}$

$$K(x, y) = \frac{1}{|G_x|} \sum_{s \in G_x \cap G_y} 1/|\mathcal{X}_s|$$

MARKOV CHAIN ON $\mathcal{X}$.

CLAIM K HAS STATIONARY DISTRIBUTION

$\pi(x) = z^{-1}/|O_x|$ (z = # ORBITS)

LEMMA LET $K(x,y) = \frac{1}{|G_x|} \sum_{s \in G_x \cap G_y} 1/|\mathcal{X}_s|$

$$\pi(x) = z^{-1}/|O_x|$$

THEN: $\pi(x)K(x,y) = \pi(y)K(y,x)$

$\left(\text{SO } \sum_x \pi(x)K(x,y) = \sum_x \pi(y)K(y,x) = \pi(y)\right)$

PROOF OF COURSE $|O_x| = |G|/|G_x|$ SO

$$\pi(x)K(x,y) = \frac{z^{-1}}{|G|} \sum_{s \in G_x \cap G_y} 1/|\mathcal{X}_s| = \pi(y)K(y,x)$$

EX $\mathcal{X} = C_2^n$, $G = S_n$

$$O_i = \{x : |x| = i\} \quad 0 \le i \le n$$

- FROM x, CHOOSE s : $x^s = x$
  $G_x = S_i \times S_{n-i}$ IF $|x| = i$ } EASY

- FROM s CHOOSE y : $y^s = y$
  $s = c_1 c_2 \cdots c_j$, LABEL CYCLES 0 OR 1 PROB $1/2$ } EASY

NOT ALWAYS SO EASY

e.g. FOR TREES:

. GIVEN T PICK $x \in \mathrm{Aut}(T)$

. GIVEN x PICK T' : $(T')^2 = T'$

BUT OFTEN CAN DO IT!

EX PARTITIONS $G = \mathcal{X}$

. FROM $x \in G$ CHOOSE t : $tx = xt$

. FROM $t \in L$ CHOOSE x : $xt = tx$

ONLY NEED ONE STEP.

$G = S_n$, $\sigma \sim 1^{a_1} 2^{a_2} \cdots n^{a_n}$, $a_i(\sigma) = \#\ i\text{-CYCLES}$

$C_{S_n}(\sigma) = \prod_{i=1}^{n} C_i^{a_i} \times S_{a_i}$ EASY.

$G = U_n(q)$: GIVEN M, CHOOSE M' : $MM' = M'M$

. THESE ARE LINEAR EQUATIONS!

. CHOOSE A BASIS AND TAKE A RANDOM LINEAR COMBINATION "EASY"

OPEN QUESTION HOW MANY CONJUGACY CLASS $K(n,q)$?

Th(HIGMAN) $q^{\frac{n^2}{12}(1+o(1))} \le K(n,q) \le q^{\frac{n^2}{4}(1+o(1))}$

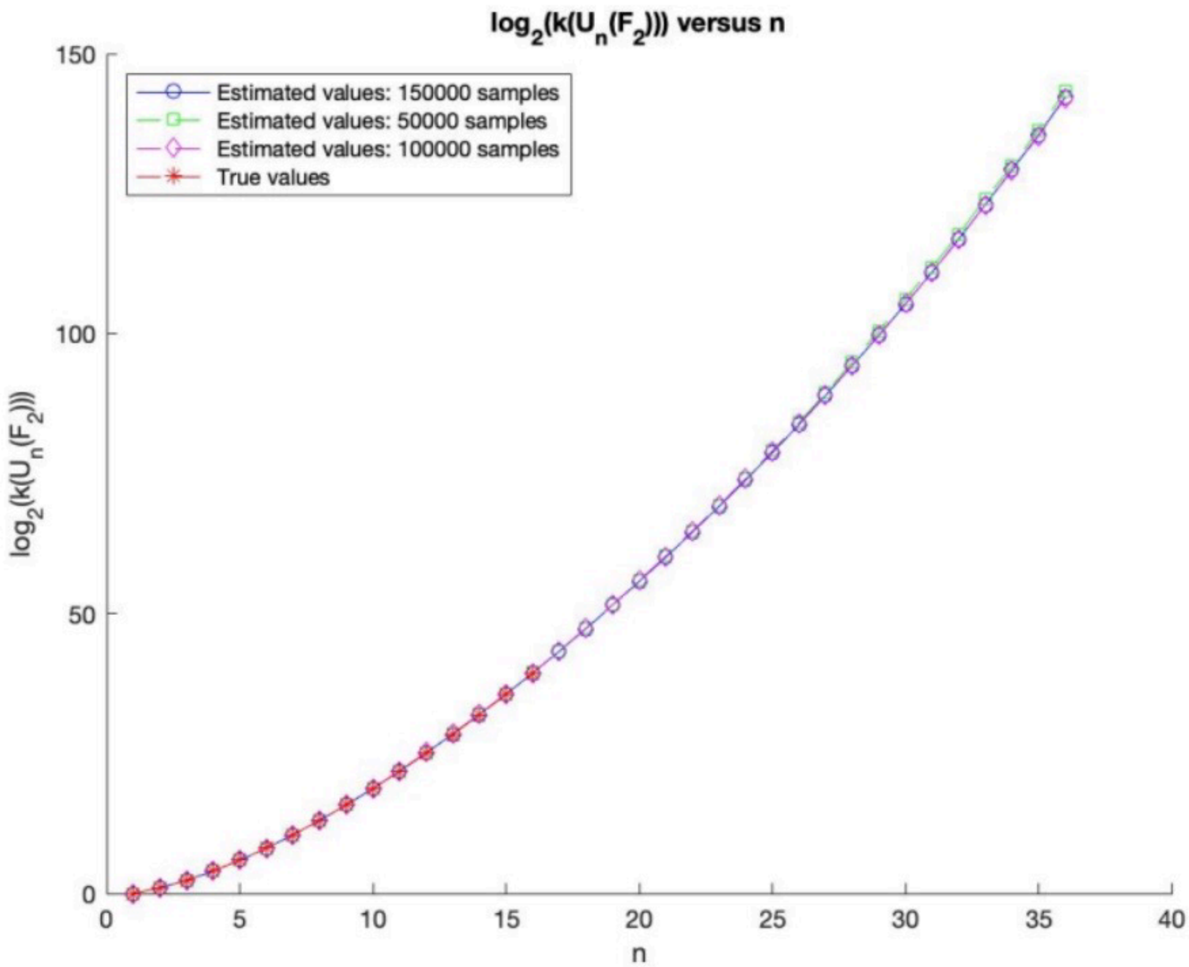


Figure 1: Plot of $\log_2(k(U_n(\mathbb{F}_2)))$ for $n = 1, \cdots, 36$

? CAN WE PROVE ANYTHING ?

EX : $\mathcal{X} = C_2^n, \quad G = S_n$

Th (WITH CHENYANG ZHONG)

$$\frac{1}{4}\left(\frac{1}{4}\right)^{\ell} \le \|K_1^{\ell} - \pi\|_{TV} \le 4\left(\frac{1}{4}\right)^{\ell}$$

SO BOUNDED NUMBER OF STEPS ARE N + S FOR ALL n !

? WHAT HAPPENED TO POLYA THEORY ?

<u>DEF</u> $G \subseteq S_n$, THE CYCLE INDEX

$$Z_G(x_1,\dots,x_n) = \frac{1}{|G|}\sum_{\sigma\in G}\prod x_i^{a_i(\sigma)}$$

<u>POLYA KNEW</u>

- $Z_{C_n}(x_1,\dots x_n) = \frac{1}{n}\sum_{j|n}\phi(j)x_j^{n/j}$

- $Z_{S_n}(x_1\dots x_n) = \sum_{\lambda\vdash n}\frac{1}{z_\lambda}\prod_{i=1}^{n}x_i^{a_i(\lambda)}, \quad z_\lambda = \prod i^{a_i}a_i!$

- $G = G_1^n \rtimes G_2, \quad G_1 \subseteq S_A, G_2 \leq S_n, G \leq S_{An}$

$$Z_G(x_1,\dots,x_{An}) = Z_{G_2}(z_1,\dots,z_n), \quad z_i = Z_{G_1}(x_i, x_{2i},\dots,x_{Ai})$$

- $\sum_{n=0}^{\infty} Z_n(x_1\dots x_n)t^n = \prod_{i=1}^{\infty} e^{\frac{t^i}{i}x_i}$

I WILL USE THE TOOLS: PERMUTATION ENUMERATION

<u>REMINDER</u>: PICK $\sigma \in S_n$ 'AT RANDOM'

WHAT DOES σ 'LOOK LIKE'?

- FIXED POINTS?
- # CYCLES?
- LENGTH OF LONGEST CYCLE?
- ORDER?

$\vdots$

$$\sigma \sim 1^{a_1} 2^{a_2} \cdots n^{a_n} \qquad \sum_{i=1}^{n} i a_i = n$$

<u>Th</u> PICK $\sigma \in S_n$ AT RANDOM, DISTRIBUTION OF $\{a_i(\sigma)\}_{i=1}^{n}$ CONVERGES TO $\{A_i\}_{i=1}^{\infty}$ AS $n \nearrow \infty$.

WHERE A_i ARE INDEPENDENT POISSON$(1/i)$

(PROB $\mathrm{Pois}_\lambda = j$ = $e^{-\lambda} \lambda^j / j!$)

SO

- $P_n(a_1(\sigma) = j) \sim e^{-1}/j!$
- $P\left\{ \dfrac{\#\text{ CYCLES} - \log n}{\sqrt{\log n}} \leq x \right\} \to \Phi(x)$

$$E\left(\frac{a_{max}}{n}\right) \sim \int_0^\infty e^{-x - \int_x^\infty (e^{-y}/y)\,dy}\,dx \doteq 0.62432997$$

$\vdots$

NEED SAME KIND OF RESULTS FOR LARGE SUBGROUPS

- $B_n = C_2^n \rtimes S_n$ HYPEROCTAHEDRAL $\leq S_{2n}$

- $C_k^n \rtimes S_n$ COMMUTATORS $\leq S_{kn}$

- $S_k \rtimes S_n$ MAXIMAL (O'NAN-SCOTT TH).

CYCLES (WORK WITH NATHAN TUNG)

<u>RECALL</u> $Z_G(x_1 \dots x_n) = \frac{1}{|G|} \sum_{\sigma \in G} \prod_i^n x_i^{a_i(\sigma)} \quad G \leq S_n$

SET ALL $x_i = x$, USE $C(\sigma) = a_1 + \dots + a_n$

$$C(x) = \sum_j P_G(C(\sigma) = j)\, x^j$$

<u>OK</u> $C_{G_1^n \rtimes G_2}(x) = Z_{G_2}(t_1, t_2, \dots, t_n), \quad t_i = C_{G_1}(x)$

SO $= \sum P_{G_2}(C(\sigma) = j)\, C_{G_1}^j(x)$

<u>COR</u> PICK $\sigma \in G_1^n \rtimes G_2$ AT RANDOM

$$C(\sigma) \stackrel{\mathcal{L}}{=} \sum_{i=1}^{N} X_i$$

X_i iid FROM $C_{G_1}(x)$, $N \sim C_{G_2}(x)$ $\perp$

A RANDOMLY STOPPED SUM

$$E_G(C(\sigma)) = E(N)E(X_1)$$

$$VAR_G(C(\sigma)) = VAR(E(X|N)) + E(VAR(X|N))$$

AND UNDER MILD CONDITIONS ON N, ANSCOMBE

Th LET $\Gamma \subseteq S_k$. $G = \Gamma^n \wr S_n$, THEN $\sigma \in G$ HAS $C(\sigma)$ CYCLES

$$\frac{C(\sigma) - \mu_n}{\sigma\sqrt{\log n}} \Rightarrow \mathcal{N}(0,1)$$

$\mu_n = \mu(\log n + \gamma + O(\frac{1}{n}))$,

μ, σ^2 MEAN + VAR OF $C(\eta)$ FROM Γ.

FURTHER POLYA THEORY OFFERS A USEFUL DESCRIPTION OF ALL CYCLES

Th LET $G_n = S_3 \wr S_n$, $\sigma \in G_n$, $\sigma \sim \prod i^{a_i(\sigma)}$

THEN $\{a_i\}_{i=1}^{3n} \Rightarrow \{A_i\}_{i=1}^{\infty}$

$i = 1(6)$ $\quad A_i \overset{L}{=} 3W_i + Z_i$ $\qquad W_i \sim Pois(1/6i)$

$i = 2(6)$ $\quad A_i \overset{L}{=} 3W_i + Z_i + Z_{i/2}$ $\qquad Z_i \sim Pois(1/2i)$

$i = 3(6)$ $\quad A_i \overset{L}{=} 3W_i + Z_i + Y_i$ $\qquad Y_i \sim Pois(1/3i$

$i = 4(6)$ $\quad A_i \overset{L}{=} 3W_i + Z_i + Z_{i/2}$ $\qquad$ ALL INDEPENDENT

$i = 5(6)$ $\quad A_i \overset{L}{=} 3W_i + Z_i$

$i = 0(6)$ $\quad A_i \overset{L}{=} 3W_i + Z_i + Z_{i/2} + Y_i$

GENERAL THEOREM FIX k AND $\Gamma \leq S_k$, SET $G_n = \Gamma^n \rtimes S_n$,

PICK $\sigma \sim \prod_{i=1}^{kn} i^{a_i} \in G_n \leq S_{kn}$, THEN $\{a_i\} \Rightarrow \{A_i\}_{i=1}^{\infty}$

$$A_i = \sum_{\substack{j \geq 1 \\ 1 \leq a < \infty \\ 1 \leq b \leq k}} \delta_i(ab)\, a_b(j)\, Z_{aj}$$

$Z_{aj} \sim \mathrm{Poiss}(P_\Gamma(j)/a)$ ALL INDEPENDENT.

CAN ALSO DO ($G_n = \Gamma^n \rtimes S_n$

- DESCENTS
- INVERSIONS
- $\ell(\sigma)$

BACK TO BIG PICTURE; FOR G ACTING ON $\mathcal{X}$

- BURNSIDE PROCESS ALLOWS
  - RANDOM GENERATION OF ORBIT
  - COUNTING # ORBITS
- CAN MAKE A HISTOGRAM OF $|O_{x_i}|$
- 'NAMING' STRUCTURE (eg $O_x = |G|/|G_x|$)

STILL OPEN.

ACTUALLY RUNNING BURNSIDE IS A RESEARCH PROBLEM

PROVING THINGS "

SUMMARY

- INTRODUCED BURNSIDE PROCESS
- COMMUTING GRAPH PROCESS
- CONJUGACY CLASSES ($U_n(q)$)
- PROOFS FOR S_n ON C_2^n
- POLYA THEORY, CYCLE INDICES
- CYCLES FOR $S^n \times H_n$
- CYCLES FOR WREATH PRODUCTS

OPEN PROBLEMS

- HOW TO ACTUALLY 'DO' BURNSIDE (TRY TREES)
- PROVE 'ANYTHING' ABOUT $K(x,y)$ (PARTITIONS)

REFERENCES

- ANDERSON-DIACONIS, 'HIT AND RUN AS A UNIFYING DEVICE'
- DIACONIS-SIMPER, 'STATISTICAL ENUMERATION OF GROUPS BY DOUBLE COSETS'
- DIACONIS-TUNG, 'POISSON LIMITS FOR LARGE PERMUTATION GAMES'
- DIACONIS-ZHONG, 'HAHN POLYNOMIALS AND THE BURNSIDE PROCESS'

ALL MY PAPERS ARE UP ON MY WEBSITE