The commutation classes of a permutation

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Coxeter groups

- Let W be a Coxeter group with generating set $S = \{s_i : i \in I\}$.
- For $i, j \in I$, let $m(i,j)$ be the order of $s_i s_j$ in W so that $M = [m(i,j)]_{i,j \in I}$ is the Coxeter matrix of W.
- Let I^* denote the free monoid formed by all finite words over the alphabet I
- Given $w \in W$, let $\ell(w)$ be the length of (every) reduced (*i.e.* minimal) expression $w = s_{i_1} \cdots s_{i_l}$ with $i_1, \ldots, i_l \in I$. The corresponding index sequence $a = i_1 \cdots i_l \in I^*$ is called a **reduced word** for w.
- Let $R(w)$ denote the set of all reduced word for $w \in W$.

Braid relations

- For integers $m \geq 0$ and $i, j \in I$, define

$$\langle i, j \rangle_m = \underbrace{ijij \cdots}_{m} \in I^*$$

  and let $\approx$ denote the congruence on I^* generated by the braid relation

$$\langle i, j \rangle_{m(i,j)} \approx \langle j, i \rangle_{m(i,j)}$$

  for all $i, j \in I$ such that $m(i, j) < \infty$.
- A result of Tits shows that $R(w)$ constitutes a single braid class, *i.e.* any reduced word for w is obtained from any other by means of the braid relations.

Commutation classes

- Let $\sim$ denote the congruence class on I^* generated by the relation

$$ij \sim ji$$

  for all $i, j \in I$ such that $m(i,j) = 2$. The equivalence class of $a \in I^*$ with respect to $\sim$ is called the **commutation class** of a, represented by $[a]$.

- For each $w \in W$, there is a decomposition

$$R(w) = C_1 \dot{\cup} \cdots \dot{\cup} C_l,$$

  where each C_i is a commutation class.

Symmetric group

- $\mathcal{S}_n$ denote the symmetric group on $[n] := \{1, 2, \ldots, n\}$

$$\mathrm{w} = [\mathrm{w}_1 \mathrm{w}_2 \cdots \mathrm{w}_n] \in \mathcal{S}_n$$

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- $\mathcal{S}_n$ is generated by $S = \{s_1, s_2, \ldots, s_{n-1}\}$, where $s_i = (i \;\, i{+}1)$:

$$\begin{aligned} s_i^2 &= 1, &&\text{for all } i \\ s_i s_j &= s_j s_i, &&\text{for } |i-j| > 1, \text{ and} \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1}, &&\text{for all } 1 \le i < n-1. \end{aligned}$$

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- $\ell(\mathrm{w}) = \#\{(w_j, w_i) : i < j \text{ and } \mathrm{w}_i > \mathrm{w}_j\} = \#Inv(\mathrm{w})$.

The graph of reduced words of $\mathbf{w}$

Example

$[3241] \in \mathcal{S}_4$ has length 4, $Inv(3241) = \{12, 13, 14, 23\}$, reduced compositions

$$s_2 s_1 s_2 s_3,\ s_1 s_2 s_1 s_3,\ s_1 s_2 s_3 s_1$$

and thus

$$R(3241) = \{2123,\ 1213,\ 1231\}.$$

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Definition

Given $\mathrm{w} \in \mathcal{S}_n$, denote by $G(\mathrm{w})$ the graph with vertices $R(\mathrm{w})$ and an edge between two vertices if they differ by a single braid relation.

Example

$G(3241)$: 2123 —— 1213 —— 1231

The graph $G(\mathrm{w}_0)$:

- is connected [Tits Theorem, 1969];
- $\#R(\mathrm{w}_0) = \frac{\binom{n}{2}!}{1^{n-1}3^{n-2}5^{n-3}\cdots(2n-3)^1}$ [R. Stanley, 1984];
- is bipartite [N. Bergeron, C. Ceballos and J. Labbé, 2015];
- has diameter $\frac{1}{24}(n-1)n(n+1)(3n-2)$ [V. Reiner and Y. Roichman, 2013].

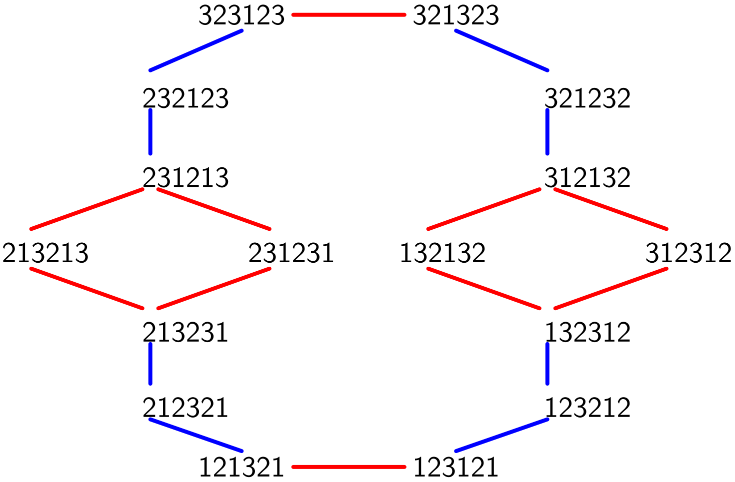

Figure: The graph $G(4321)$.

The commutation graph of w

Contracting the commutation edges of $G(\mathrm{w})$ we obtain the commutation graph $C(\mathrm{w})$, whose vertices are the commutation classes, $[a]$, $a \in R(\mathrm{w})$, and the edges are long braid relations between the classes.

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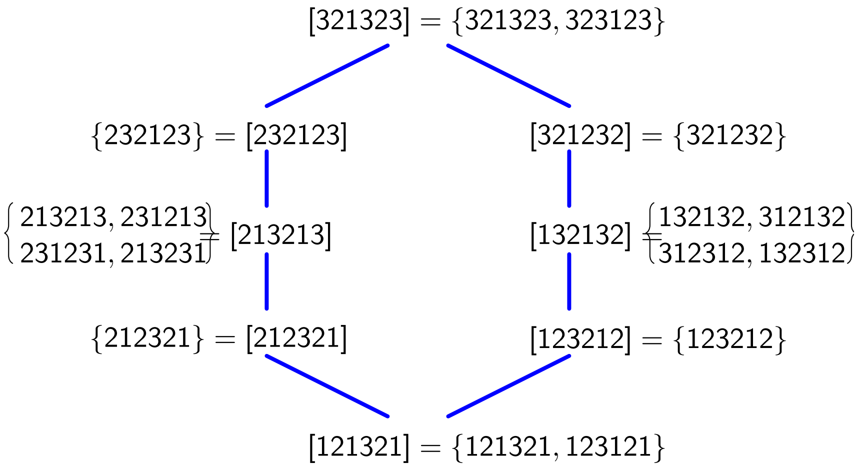

The number of commutation classes for w_0

$c_n = \#C(w_0)$, for $n \geq 1$ is sequence A006245 of The On-Line Encyclopedia of Integer Sequences

n	c_n
1	1
2	1
3	2
4	8
5	62
6	908
7	24698
8	1232944
9	112018190
10	18410581880

The current best upper-bounds: $c_n \leq 2^{0.6571n^2}$, for sufficiently large n

The diameter of graphs on reduced words

- Let $G = (V, E)$ be a simple connected graph. The **distance** $d(a, b)$ between two vertices $a, b \in V$ is the length of a shortest path joining a and b.
- The **diameter** $diam(G)$ of the graph G is the maximum value of $d(a, b)$ over all $a, b \in V$.
- A subgraph $G' = (V', E')$ of G is called an induced subgraph if it contains all edges of G that join two vertices in V'.
- If a is a reduced word for w, the commutation class $[a]$, that is the set of all reduced words obtained from a by a sequence of commutations, is an induced subgraph of $G(\mathrm{w})$

Root systems and Weyl groups

- Let $\Phi \subseteq E \cong \mathbb{R}^n$ be a finite crystallographic root system of rank n:
    - Φ is finite and spans E
    - If $\alpha \in \Phi$, then $-\alpha \in \Phi$ and $\pm\alpha$ are the only multiples of α in Φ
    - Φ is invariant under the reflection in the hyperplane orthogonal to any $\alpha \in \Phi$:
    $$s_\alpha(\beta) = \beta - 2proj_\alpha\beta = \beta - 2\alpha\frac{(\beta,\alpha)}{(\alpha,\alpha)} \in \Phi$$
    - $2\frac{(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$

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  - $2\frac{(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$
- Choose $\Phi^+ \subset \Phi$ to be a set of positive roots, whose corresponding simple roots are $\Delta = \{\alpha_1, \dots, \alpha_n\} \subset \Phi^+$.
- For each $\alpha \in \Phi^+$, s_α is the reflection across the hyperplane orthogonal to α. Then, $W = W(\Phi)$ is the Weyl group associated to Φ generated by $s_{\alpha_1}, \dots, s_{\alpha_n}$.
- The inversion set of $w \in W(\Phi)$ is $I_\Phi(w) = \{\alpha \in \Phi^+ : w\alpha \notin \phi^+\}$

Root systems of type A_{n-1}

We adopt the following conventions for root systems of type A_{n-1}, where e_i denotes the ith standard basis vector in $\mathbb{R}^n$:

- $\Phi = \{e_i - e_j : 1 \leq i \neq j \leq n\}$
- $\Phi^+ = \{e_j - e_i : 1 \leq i < j \leq n\}$
- $\Delta = \{\alpha_1 = e_2 - e_1, \alpha_2 = e_3 - e_2, \ldots, \alpha_n = e_n - e_{n-1}\}$

$W(\Phi) \cong \mathcal{S}_n$ the Symmetric Group and $I_\Phi(w) \cong Inv(w)$

$$(e_j - e_i \equiv ij)$$

Root orderings

- The **root ordering** $(I_\Phi(w), <_a)$ associated with a reduced word $a = i_1 \cdots i_l \in R(w)$ is an ordering $ro(a) = \beta_l, \ldots, \beta_1$ of $I_\Phi(w)$, where

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- Let $w^{(j)} = s_{i_1} \cdots s_{i_j}$, with $w^{(0)} = id$ and $w^{(l)} = w$
- Then, β_j is the unique positive root $\beta \in \Phi^+$ such that

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Example

Let $w = [3412] \in W(A_3)$ and $a = 2132 \in R(w)$. Then, the root ordering is $ro(a) = e_3 - e_2, e_4 - e_2, e_3 - e_1, e_4 - e_1$.

The reduced word a describes how w is reduced to id (as $ws_2 s_1 s_3 s_2 = id$):

$$w = 3412 \xrightarrow{e_4 - e_1} 3142 \xrightarrow{e_3 - e_1} 1342 \xrightarrow{e_4 - e_2} 1324 \xrightarrow{e_3 - e_2} 1234$$

A metric for $G(\mathrm{w})$

Define

$$I_2(\mathrm{w}) = \{(e_j - e_i, e_l - e_k) \in I_\Phi(w)^2 : i < k,\ \{i,j\} \cap \{k,l\} = \emptyset\},$$

$$I_3(\mathrm{w}) = \{(e_j - e_i, e_l - e_k) \in I_\Phi(w)^2 : i < j = k < l\},$$

and $L_2(\mathrm{w}) = I_2(\mathrm{w}) \cup I_3(\mathrm{w})$. For simplicity, we often write $(e_j - e_i, e_l - e_k) = (ij, kl)$ and a pair $(ij, jl) \in I_3(w)$ as a triple ijl.

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Note: If $ij\ell \in I_3(\mathrm{w})$, then $ij, i\ell, j\ell \in Inv(\mathrm{w})$

Example

For the permutation $[4321] \in S_4$ we have $I_\Phi(4321) = \{e_2 - e_1, e_3 - e_1, e_4 - e_1, e_3 - e_2, e_4 - e_2, e_4 - e_3\}$ and

$$I_2(4321) = \{(12, 34), (13, 24), (14, 23)\},$$

$$I_3(4321) = \{123,\ 124,\ 134,\ 234\}.$$

Definition

Given a permutation $\mathrm{w} \in \mathcal{S}_n$, define the family of maps $(\Gamma_a)_{a \in R(\mathrm{w})}$ on the set $L_2(\mathrm{w})$ by setting

$$\Gamma_a(ij, k\ell) = \begin{cases} 1, & \text{if } ij >_a k\ell \\ 0, & \text{if } ij <_a k\ell \end{cases},$$

where $(I_\Phi(\mathrm{w}), \leq_a)$ is the root ordering induced by a.

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Example

Let $\mathrm{w} = [4321]$ and consider the reduced words $a = 132132$ and $b = 231231$. The root orderings of $I_\Phi(4321)$ induced by a and b are:

$$12 <_a 34 <_a 14 <_a 24 <_a 13 <_a 23;$$

$$23 <_b 24 <_b 13 <_b 14 <_b 12 <_b 34.$$

We have $\Gamma_a(12, 34) = \Gamma_b(12, 34) = 0$ while $\Gamma_a(13, 34) = 1$ and $\Gamma_b(13, 34) = 0$.

Let $\mathrm{w}_{min}, \mathrm{w}_{max} \in R(\mathrm{w})$ defined as follows:

- w_{min} is the reduced word which first swaps the value 1 leftward as many times as possible to move it to the first position, and then does the same to move the value 2 to the second position, and so on.
- w_{max} is the reduced word which first swaps the value n rightward as many times as possible to move it to the last position, and then does the same to move the value $n-1$ to the penultimate position, and so on.

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Lemma

Let $\mathrm{w} \in \mathcal{S}_n$. Then, the words w_{min} and w_{max} are reduced words for w. Moreover, $\Gamma_{\mathrm{w}_{min}}(ijk) = 0$ and $\Gamma_{\mathrm{w}_{max}}(ijk) = 1$ for any triple $ijk \in I_3(\mathrm{w})$.

Proposition

Given a commutation class $[a]$, $\Gamma_a(ijk) = \Gamma_b(ijk)$ for all $ijk \in I_3(w)$ if and only if $b \in [a]$.

Definition

Given $a, b \in Red(\mathrm{w})$, let

$$T_{a,b} = \{(ij, k\ell) \in L_2(\mathrm{w}) : \Gamma_a(ij, k\ell) \neq \Gamma_b(ij, k\ell)\}$$

and $T_{[a]} = \bigcup_{b,c \in [a]} T_{b,c}$.

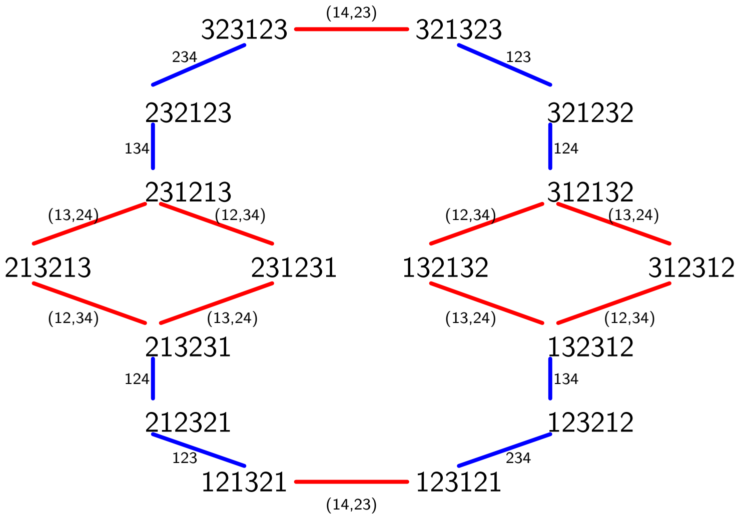
Definition

Given $a, b \in Red(\mathrm{w})$, let

$$t(a, b) = |T_{a,b}| = \sum_{(ij,k\ell) \in L_2(\mathrm{w})} \Gamma_a(ij, k\ell) \oplus_2 \Gamma_b(ij, k\ell),$$

where $\oplus_2$ represents the sum modulo 2.

It is not difficult to check that t is a metric on $R(\mathrm{w})$.



If $a = 231231$ and $b = 231213$ then $T_{a,b} = \{(12, 34)\}$ and $t_{a,b} = 1$.
If $c = 132132$ then $T_{a,c} = \{134, 234, (14, 23), 123, 124\}$ and $t_{a,c} = 5$.

The diameter of $C(\mathrm{w})$

The commutation classes of w_{min} and w_{max} are the furthest classes in the graph $C(\mathrm{w})$ and thus $diam(C(\mathrm{w})) = d([\mathrm{w}_{min}], [\mathrm{w}_{max}]) = |I_3(\mathrm{w})|$.

Theorem

Let $\mathrm{w} \in \mathcal{S}_n$. The partial order defined on the commutation classes of $C(\mathrm{w})$ given by the transitive closure of covering relations

$$[a] < [b] \ \textit{if} \ [a] \underset{L}{\sim} [b] \ \textit{and} \ t(\mathrm{w}_{min}, b) = t(\mathrm{w}_{min}, a) + 1,$$

makes $C(\mathrm{w})$ into a ranked partially ordered set with a unique minimal element $[\mathrm{w}_{\mathrm{min}}]$ and a unique maximal element $[\mathrm{w}_{\mathrm{max}}]$. Moreover, the diameter of $C(w)$ is equal to the cardinality of $I_3(\mathrm{w})$.

321-avoiding permutations

Let $\mathrm{w} = w_1 \cdots w_n \in \mathcal{S}_n$ and let $p \in \mathcal{S}_r$, for $r \leq n$. We say that w **contains the pattern** p if there exists a subsequence $w_{i_1} \cdots w_{i_r}$ whose elements are in the same relative order as the elements in p. If w does not contain p, then we say that w **avoids** p, or that w is p-avoiding.

Example

The permutation [241563] is 321-avoiding but it contains the pattern 132 (24**15**6**3**)

Theorem (Billey, Jockusch, and Stanley, 1993)

A permutation $\mathrm{w} \in \mathcal{S}_n$ is fully commutative if and only if it is 321-avoiding.

Proposition

Given $a, b \in Red(w)$, we have $t(a, b) \leq d(a, b)$.

Note: The diameter of the metric space $(R(\mathrm{w}), t)$ is a lower bound for the diameter of the graph $G(\mathrm{w})$

Proposition

Given $a, b \in Red(w)$*, we have* $t(a, b) \leq d(a, b)$*.*

Note: The diameter of the metric space $(R(\mathrm{w}), t)$ is a lower bound for the diameter of the graph $G(\mathrm{w})$

Proposition

Let $a, b \in R(\mathrm{w})$*. If* a *and* b *are in the same commutation class, then* $t(a, b) = d(a, b)$*.*

Definition

Given $\mathrm{w} \in \mathcal{S}_n$, define the following disjoint subsets of $I_2(w)$:

$$I_2^S(\mathrm{w}) := \{(ij, k\ell) \in I_2(w) : i < j < k < \ell\},$$

$$I_2^C(\mathrm{w}) := \{(ik, j\ell) \in I_2(w) : i < j < k < \ell\},$$

$$I_2^N(\mathrm{w}) := \{(i\ell, jk) \in I_2(w) : i < j < k < \ell\}.$$

Definition

Given $a \in Red(\mathrm{w})$, define the following disjoint subsets of $T_{[a]}$:

$$T_{[a]}^S := I_2^S(\mathrm{w}) \cap T_{[a]},$$

$$T_{[a]}^C := I_2^C(\mathrm{w}) \cap T_{[a]},$$

$$T_{[a]}^N := I_2^N(\mathrm{w}) \cap T_{[a]}.$$

Lemma

Let a be a reduced word of a permutation w and $(ij, k\ell) \in I_2^S(w)$*, with* $i < j < k < \ell$*. The pair* $(ij, k\ell) \in T_{[a]}^S$ *if and only if either of the following conditions hold:*

- $ij\ell \notin I_3(w)$ *and* $ik\ell \notin I_3(w)$*;*
- $ij\ell \notin I_3(w)$ *and* $\Gamma_a(ik\ell) = 1$*;*
- $\Gamma_a(ij\ell) = 0$ *and* $ik\ell \notin I_3(w)$*;*
- $\Gamma_a(ij\ell) = 0$ *and* $\Gamma_a(ik\ell) = 1$*;*
- $\Gamma_a(ij\ell) = 1$ *and* $\Gamma_a(ik\ell) = 0$*.*

	$ik\ell \notin I_3$	$\Gamma_a(ik\ell) = 0$	$\Gamma_a(ik\ell) = 1$
$ij\ell \notin I_3$	✓	✗	✓
$\Gamma_a(ij\ell) = 0$	✓	✗	✓
$\Gamma_a(ij\ell) = 1$	✗	✓	✗

Table: Separated pairs of inversions $(ij, k\ell) \in I_2^S(w)$

Theorem

Let $a \in Red(\mathrm{w})$*. The diameter of the commutation class* $[a]$ *is given by*

$$|T_{[a]}| = |T^S_{[a]}| + |T^C_{[a]}| + |T^N_{[a]}|.$$

Corollary

If w *is a fully commutative permutation, then the diameter of the graph* $G(\mathrm{w})$ *is given by*

$$|I^S_2(\mathrm{w})| + |I^C_2(\mathrm{w})|.$$

Example

The permutation $\mathrm{w} = [24517386]$ is fully commutative, since $I_3(w) = \emptyset$. The graph $G(w)$ has 344 reduced words and 1818 commutations. We have $I_\Phi(w) = \{12, 14, 15, 34, 35, 37, 67, 68\}$ and thus,

$$\begin{aligned} I_2^S(w) =& \{(12,34),(12,35),(12,37),(12,67),(12,68),(14,67),(14,68),\\ &(15,67),(15,69),(34,67),(34,68),(35,67),(35,68)\} \\ I_2^C(w) =& \{(14,35),(14,37),(15,37),(37,68)\}. \end{aligned}$$

It follows that the diameter of $G(24517386) = |I_2^S(w)| + |I_2^C(w)| = 13 + 4 = 17$.

A permutation $\mathrm{w} \in \mathcal{S}_n$ is called **Grassmannian** if it has at most one descent. Grassmannian permutations can be defined using their Lehmer code $L(\mathrm{w}) = (c_1, c_2, \ldots, c_n)$, where

$$c_i = \#\{j : j > i \text{ e } \mathrm{w}_j < \mathrm{w}_i\}.$$

It is easy to see that w is a Grassmannian permutation with descent r if and only if $c_1 \leq c_2 \leq \cdots c_r \leq n - r$ and $c_{r+1} = \cdots = c_n = 0$.

Corollary

Let $\mathrm{w} \in \mathcal{S}_n$ be a Grassmannian permutation with Lehmar code $L(w) = (c_1, \ldots, c_n)$. Then, the diameter of the graph $G(\mathrm{w})$ is given by

$$\sum_{1 \leq i < k \leq r} c_i(c_k - c_i) + \sum_{i=1}^{r-1} \binom{c_i}{2}(r - i).$$

References

- Y. Cho, J. S. Kim and E. Lee (2022). Enumeration of Gelfand-Cetlin type reduced words, The Electronic Journal of Combinatorics 29(1), P1.27.
- S. Dahlberg and Y. Kim (2023). Diameters of graphs on reduced words of 12 and 21-inflations. Journal of Combinatorics, Vol.14 (1), 69–110.
- G. Gutierres, R. Mamede and J.L. Santos (2020). Commutation classes of the reduced words for the longest element of S_n. The Electronic Journal of Combinatorics, volume 27, Issue 2, P2.21.
- G. Gutierres, R. Mamede and J.L. Santos (2022). Diameter of the commutation classes graph of a permutation. European Journal of Combinatorics, 103, paper no.103525.
- G. Gutierres and R. Mamede (2024). Diameter of a commutation class on reduced words. Discrete Mathematics 347 (2024), paper no. 114098.
- V. Reiner and Y. Roichman (2013). Diameter of graphs of reduced words and galleries. Transactions of the American Mathematical Society, 365(5), 2779–2802.

Thank you.