Some partial orders on the class of $(0,1)$-matrices and related conjectures

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- $R = (r_1, \ldots, r_m)$, $S = (s_1, \ldots, s_n)$ two sequences of positive integers in weakly decreasing order having the same sum,

$$r_1 \geq \ldots \geq r_m, \qquad s_1 \geq \ldots \geq s_n,$$

$$r_1 + \ldots + r_m = s_1 + \ldots + s_n.$$

- $\mathcal{A}(R, S)$ the class of all m-by-n $(0, 1)$-matrices with row sum vector R and column sum vector S.
- $\mathcal{A}(n, k)$ the class of all n-by-n $(0, 1)$-matrices with constant row and column sums k.

EXAMPLE

$$R = (4,4,4,3),\ S = (3,3,3,3,3)$$

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix} \in \mathcal{A}(R,S)$$

$$B = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix} \in \mathcal{A}(5,3)$$

$A = [a_{ij}]$ a m-by-n real matrix, let $\Sigma(A)$ be the m-by-n matrix whose (r,s)-entry is

$$\sigma_{r,s}(A) = \sum_{i=1}^{r}\sum_{j=1}^{s} a_{ij}, \;\; 1 \leq r \leq m, \, 1 \leq s \leq n.$$

Example

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}, \quad \sigma_{2,3}(A) = 4$$

$$\Sigma(A) = \begin{bmatrix} 1 & 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 6 & 8 \\ 2 & 4 & 6 & 9 & 12 \\ 3 & 6 & 9 & 12 & 15 \end{bmatrix}.$$

BRUHAT ORDER

- $A, C \in \mathcal{A}(R,S)$ then A precedes C in the Bruhat order, written $A \preceq_B C$, provided that $\Sigma(A) \geq \Sigma(C)$ (by the entrywise order).

Example

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}$$

$$\Sigma(A) = \begin{bmatrix} 1 & 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 6 & 8 \\ 2 & 4 & 6 & 9 & 12 \\ 3 & 6 & 9 & 12 & 15 \end{bmatrix}, \quad \Sigma(C) = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 6 & 8 \\ 2 & 4 & 6 & 9 & 12 \\ 3 & 6 & 9 & 12 & 15 \end{bmatrix}.$$

$$\Sigma(A) \geq \Sigma(C) \quad \textit{and} \quad A \preceq_B C$$

An interchange consists of replacing one of the following two submatrices by the other,

$$L_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Example

$$B = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

If $A_1 \in \mathcal{A}(R,S)$ and A_2 is the matrix obtained from A_1 replacing a 2-by-2 submatrix of A_1 equal to L_2 (respectively, I_2) by I_2 (respectively, L_2) then we say that A_2 is obtained from A_1 by an $L_2 \to I_2$ (respectively, $I_2 \to L_2$)

Example

$$B = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

D is obtained from B by an $L_2 \to I_2$.

Example

$$B=\begin{bmatrix}1&1&0&1&0\\1&0&1&1&0\\1&0&1&0&1\\0&1&1&0&1\\0&1&0&1&1\end{bmatrix},\quad D=\begin{bmatrix}1&1&0&1&0\\1&1&1&0&0\\1&0&1&0&1\\0&0&1&1&1\\0&1&0&1&1\end{bmatrix}$$

D is obtained from B by an $L_2 \to I_2$.

$$B=\begin{bmatrix}1&1&0&1&0\\1&0&1&1&0\\1&0&1&0&1\\0&1&1&0&1\\0&1&0&1&1\end{bmatrix}=D+\begin{bmatrix}0&0&0&0&0\\0&-1&0&1&0\\0&0&0&0&0\\0&1&0&-1&0\\0&0&0&0&0\end{bmatrix}$$

B is obtained from D by an (1)-interchange

Example

$$B = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

D is obtained from B by an $L_2 \to I_2$.

$$D = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix} = B + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

D is obtained from B by a (-1)-interchange

SECONDARY BRUHAT ORDER

- if $A, C \in \mathcal{A}(R,S)$ then A precedes C in the secondary Bruhat order, written $A \preceq_{\hat{B}} C$, if A can be obtained from C by a finite sequence of $L_2 \to I_2$ interchanges (sequence of (-1)-interchanges).
- The only interchanges allowed are the $L_2 \to I_2$ interchanges ((-1)-interchanges)

Example

$$C = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$D_2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

$$A \preceq_{\hat{B}} C$$

Theorem

Let $A, C \in \mathcal{A}(R,S)$. If A is obtained from C by the interchange $C[\{i,j\};\{k,l\}] = L_2 \to I_2$ then $A \preceq_B C$.

Proof :

$$C = \begin{bmatrix} c_{11} & & \cdots & & c_{1n} \\ & 0 & \cdots & 1 & \\ \vdots & \vdots & & \vdots & \vdots \\ & 1 & \cdots & 0 & \\ c_{m1} & & \cdots & & c_{mn} \end{bmatrix}, \quad A = \begin{bmatrix} c_{11} & & \cdots & & c_{1n} \\ & 1 & \cdots & 0 & \\ \vdots & \vdots & & \vdots & \vdots \\ & 0 & \cdots & 1 & \\ c_{m1} & & \cdots & & c_{mn} \end{bmatrix}$$

$$\sigma_{r,s}(A) = \sum_{i=1}^{r} \sum_{j=1}^{s} a_{ij}, \quad 1 \le r \le m,\ 1 \le s \le n.$$

$$\sigma_{r,s}(A) = \begin{cases} \sigma_{r,s}(C) + 1 & \text{if } (r,s) \in \{i,\ldots,j-1\} \times \{k,\ldots,l-1\} \\ \sigma_{r,s}(C) & \text{otherwise} \end{cases}$$

□

Theorem

Let $A, C \in \mathcal{A}(R,S)$. If $A \preceq_{\hat{B}} C$ then $A \preceq_B C$.

Corollary

Let $A \in \mathcal{A}(R,S)$. If A is a minimal matrix for the Bruhat order on $\mathcal{A}(R,S)$ (matrix $\Sigma(A)$), then A is a minimal matrix for the secondary Bruhat order on $\mathcal{A}(R,S)$, (A does not have a submatrix equal to L_2).

Corollary

Let $A \in \mathcal{A}(R,S)$. If A is a minimal matrix for the Bruhat order on $\mathcal{A}(R,S)$ (matrix $\Sigma(A)$), then A is a minimal matrix for the secondary Bruhat order on $\mathcal{A}(R,S)$, (A does not have a submatrix equal to L_2).

Conjecture

The converse of this Corollary is true.
If $A \in \mathcal{A}(R,S)$ does not have a submatrix equal to L_2 then A is a minimal matrix for the Bruhat order on $\mathcal{A}(R,S)$.

This conjecture was shown to be false using a counterexample in $\mathcal{A}(R,S)$ with $R \neq S$.

Conjecture

If $A \in \mathcal{A}(n,k)$ does not have a submatrix equal to L_2 then A is a minimal matrix for the Bruhat order on $\mathcal{A}(n,k)$.

This conjecture arose other notions linked to the Bruhat orders.

INVERSION IN A (0,1)-MATRIX

- An inversion in $A = [a_{ij}] \in \mathcal{A}(R,S)$ consists of two entries $a_{ij} = a_{kl} = 1$ such that $(i-k)(j-l) < 0$.
- The total number of inversions in A is denoted by $\nu(A)$.

Example

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}$$

$$a_{13} = a_{42} = 1 \quad \textit{and} \quad (1-4)(3-2) = -3 < 0.$$

$$\nu(A) = 5+7+9+2+4+5+6+1+3+3 = 45.$$

Theorem

Let $A, C \in \mathcal{A}(R,S)$. If A is obtained from C by the interchange $C[\{i,j\};\{k,l\}] = L_2 \rightarrow I_2$ then $\nu(A) < \nu(C)$.

Proof :

$$C = \begin{bmatrix} c_{11} & & \cdots & & c_{1n} \\ & 0 & \cdots & 1 & \\ \vdots & \vdots & & \vdots & \vdots \\ & 1 & \cdots & 0 & \\ c_{m1} & & \cdots & & c_{mn} \end{bmatrix}, \quad A = \begin{bmatrix} c_{11} & & \cdots & & c_{1n} \\ & 1 & \cdots & 0 & \\ \vdots & \vdots & & \vdots & \vdots \\ & 0 & \cdots & 1 & \\ c_{m1} & & \cdots & & c_{mn} \end{bmatrix}$$

$$\nu(A) < \nu(C).$$

□

Theorem

Let $A, C \in \mathcal{A}(R,S)$. *If* $A \prec_{\hat{B}} C$ *then* $\nu(A) < \nu(C)$.

Conjecture

Let $A, C \in \mathcal{A}(R,S)$. *If* $A \prec_{B} C$ *then* $\nu(A) < \nu(C)$.
(If $\Sigma(A) > \Sigma(C)$ *then* $\nu(A) < \nu(C)$.*)*

Example

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \in \mathcal{A}(3,2)$$

$$\Sigma(A) = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 3 & 4 \\ 2 & 4 & 6 \end{bmatrix}, \quad \Sigma(C) = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 3 & 4 \\ 2 & 4 & 6 \end{bmatrix}$$

$$\Sigma(A) > \Sigma(C) \Longrightarrow A \prec_B C$$

Definition

Let $A = [a_{ij}]$ *be an* m*-by-*n *real matrix, with* $m, n \geq 2$*, and* b *be a real number. Let* $1 \leq k < l \leq m$*,* $1 \leq p < r \leq n$ *be integers and* $E^{(b)}_{\{k,l;p,r\}} = [e_{ij}]$ *be the* m*-by-*n *real matrix with all entries equal to zero, except*

$$E^{(b)}_{\{k,l;p,r\}}[\{k,l\};\{p,r\}] = \begin{bmatrix} -b & b \\ b & -b \end{bmatrix}.$$

We say that the m*-by-*n *real matrix* D *is obtained from* A *by the* b*-interchange in the submatrix* $A[\{k,l\};\{p,r\}]$ *if*

$$D = A + E^{(b)}_{\{k,l;p,r\}}.$$

Example

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \ E^{(1)}_{\{1,2;1,2\}} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} = A + E^{(1)}_{\{1,2;1,2\}} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

D is obtained from A by a 1-interchange.

Definition

Let $A = [a_{ij}]$ *be an* m*-by-*n *real matrix with* $m, n \geq 2$. *We denote by* $\xi(A)$ *the real number*

$$\xi(A) = \sum_{i=1}^{m-1} \sum_{j=2}^{n} (\sigma_{m,j-1}(A) - \sigma_{i,j-1}(A)) a_{ij},$$

where the number $\sigma_{ij}(A)$ *is defined in* $\Sigma_{ij}(A)$.

Example

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}$$

$(\sigma_{4,2}(A) - \sigma_{1,2}(A)) a_{13} = (6 - 1)1 = 5.$

Theorem

Let $A \in \mathcal{A}(R, S)$. *Then* $\xi(A) = \nu(A)$.

Theorem

Let $A = [a_{ij}]$ be an m-by-n real matrix and b be a real number. Let k, p, r be positive integers with $1 \leq k \leq m-1$, $1 \leq p < r \leq n$. Let $x = \sum_{j=p+1}^{r} a_{kj}$ and $z = \sum_{j=p}^{r-1} a_{k+1,j}$. Let D be the matrix obtained from A by the b-interchange in the submatrix $A[\{k, k+1\}; \{p, r\}]$. Then

$$\xi(D) = \xi(A) + (x + z + b)b.$$

Example

$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ -1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 1 \end{bmatrix}$ *D is obtained from A by the 2-interchange in the submatrix* $A[\{1,2\};\{2,3\}]$, *(*$b = 2,\ p = 2,\ r = 3$*)*

$$D = A + \begin{bmatrix} 0 & -2 & 2 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 1 \\ -1 & 2 & -1 & 1 \\ 2 & 1 & 1 & 1 \end{bmatrix}$$

$x = \sum_{j=p+1}^{r} a_{kj} = 0$ *and* $z = \sum_{j=p}^{r-1} a_{k+1,j} = 0$ *and* $(x + z + b)b = (0 + 0 + 2)2.$

$$\xi(D) = \xi(A) + (0 + 0 + 2)2 = \xi(A) + 4 \Rightarrow \xi(D) > \xi(A)$$

Theorem

Let $A = [a_{ij}]$ be an m-by-n real matrix and b be a real number. Let k, p, r be positive integers with $1 \leq k \leq m-1$, $1 \leq p < r \leq n$. Let $x = \sum_{j=p+1}^{r} a_{kj}$ and $z = \sum_{j=p}^{r-1} a_{k+1,j}$. Let D be the matrix obtained from A by the b-interchange in the submatrix $A[\{k, k+1\}; \{p, r\}]$. Then

$$\xi(D) = \xi(A) + (x + z + b)b.$$

Remark

In the conditions of the previous theorem we conclude that if A is a real matrix, $x \geq 0$, $z \geq 0$ and $b > 0$ then $\xi(A) < \xi(D)$.

Definition

Let $Y = [y_{ij}]$, $W = [w_{ij}] \in \mathcal{A}(R,S)$ and α be a positive integer with $1 \leq \alpha \leq m$. The (α, Y, W)-matrix is the m-by-n matrix $Z = [z_{ij}]$ such that

- *the i-row of Z is the i-row of Y, for $1 \leq i < \alpha$.*
- *the i-row of Z is the i-row of W, for $\alpha < i \leq m$.*
- *the α-row of Z is $z_{\alpha j} = s_j - \left(\sum_{l=1}^{\alpha-1} y_{lj} + \sum_{l=\alpha+1}^{m} w_{lj} \right)$, for $1 \leq j \leq n$.*

Example

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$(1, C, A) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = A, \quad (2, C, A) = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} = D$$

$$(3, C, A) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = C$$

Example

$$(1, C, A) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = A, \quad (2, C, A) = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} = D$$

D is obtained from A by the 1-interchange in the submatrix $A[\{1,2\};\{1,3\}]$, ($b = 1,\ p = 1,\ r = 3$)

$$(3, C, A) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = C$$

C is obtained from D by the 1-interchange in the submatrix $D[\{2,3\};\{1,2\}]$, ($b = 1,\ p = 1,\ r = 2$)

$$\nu(A) = \xi(A) < \xi(D) < \xi(C) = \nu(C).$$

Theorem

Let $A, C \in \mathcal{A}(R,S)$. If $A \prec_B C$ then $\nu(A) < \nu(C)$.

Corollary

Let $A, C \in \mathcal{A}(R,S)$. If $\nu(A) = \nu(C)$ then A and C are incomparable in the Bruhat order.

Theorem

Let $A, C \in \mathcal{A}(R,S)$. If $A \prec_B C$ then $\nu(A) < \nu(C)$.

The converse of this Theorem is not valid.

Example

If

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

two minimal matrices for the Bruhat order on $\mathcal{A}(7,3)$

$$\nu(A) = 20, \quad \nu(C) = 23.$$

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Thank you