Decomposition of Rectangles in Dominos

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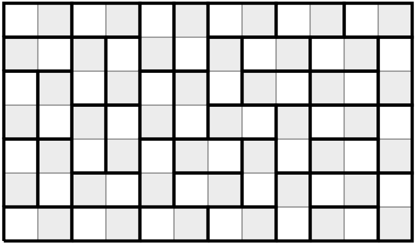
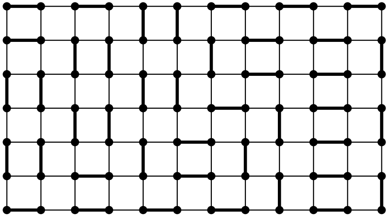
Let $\ell, k \in \mathbb{N}$. In how many ways can we cover a $\ell \times k$ "chess board" with 1×2 dominos without overlapping any two domino pieces?



Tiling randomly generated with J. Rangel-Mondragon's application for Mathematica "Random Domino Tilings", Wolfram Demonstrations Project, `http://demonstrations.wolfram.com/RandomDominoTilings/`.

Equivalently, how many different perfect matchings of the $\ell \times k$ square grid graph $(P_\ell \times P_k)$ represented below are there?



Let $T(\ell, k)$ denote that number of perfect matchings of $P_\ell \times P_k$.

It is easy to see that...

- $T(\ell, k) = T(k, \ell)$.
- $T(\ell, k) = 0$ if both ℓ and k are odd.
- $T(\ell, 1) = \begin{cases} 1 & \text{if } \ell \text{ is even;} \\ 0 & \text{if } \ell \text{ is odd.} \end{cases}$
- $T(\ell, 2) = T(\ell - 1, 2) + T(\ell - 2, 2)$, for $\ell \geqslant 3$.
  $T(1, 2) = 1 = f_2$.
  $T(2, 2) = 2 = f_3$.
  So $T(\ell, 2) = f_{\ell+1}$,
  where f_n denotes the nth Fibonacci number.

Not so is easy to see...

- $T(12,7) = 2\,188\,978\,117\,.$

Theorem (Kasteleyn's formula)

For every $\ell, k \in \mathbb{N}$,

$$T(\ell,k) = \prod_{p=1}^{\ell} \prod_{q=1}^{k} \sqrt[4]{4\cos^2\left(\frac{p\,\pi}{\ell+1}\right) + 4\cos^2\left(\frac{q\,\pi}{k+1}\right)}\,.$$

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For example, for $\ell = 4$ and $k = 3$ we have

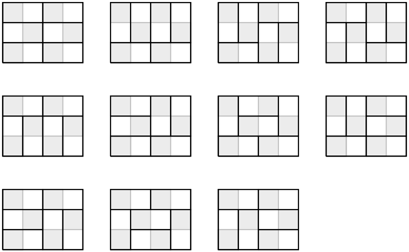
$$4\cos^2\left(\frac{\pi}{5}\right) = 4\cos^2\left(\frac{4\pi}{5}\right) = \varphi^2 = 1 + \varphi = \frac{3+\sqrt{5}}{2},$$

$$4\cos^2\left(\frac{2\pi}{5}\right) = 4\cos^2\left(\frac{3\pi}{5}\right) = 2 - \varphi = \frac{3-\sqrt{5}}{2},$$

$$4\cos^2\left(\frac{\pi}{4}\right) = 4\cos^2\left(\frac{3\pi}{4}\right) = 2 \quad , \quad 4\cos^2\left(\frac{2\pi}{4}\right) = 0,$$

Hence,

$$T(4,3) = \sqrt[4]{\left(\frac{3+\sqrt{5}}{2}\right)^2\left(\frac{3-\sqrt{5}}{2}\right)^2\left(\frac{7+\sqrt{5}}{2}\right)^4\left(\frac{7-\sqrt{5}}{2}\right)^4} = 11.$$



In what follows we assume that ℓ is even or k is even.

Label both black and white squares with integers from 1 to $d = \ell k/2$. Black squares are labeled with black labels, and white squares with red labels. The Kasteleyn Matrix is $m(\ell, k) = \left(a_{br}\right)_{1\leqslant b,r\leqslant d}$ defined by:

$$a_{br} = \begin{cases} 0, & \text{if black square } b \text{ and white square } r \\ & \text{are not adjacent;} \\ 1, & \text{if black square } b \text{ and white square } r \\ & \text{are placed side by side;} \\ i = \sqrt{-1}, & \text{if black square } b \text{ and white square } r \\ & \text{are placed one on top of the other.} \end{cases}$$

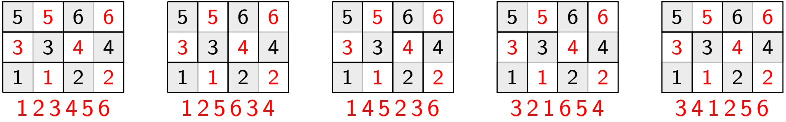
For $\ell = 4$ and $k = 3$:

5	5	6	6
3	3	4	4
1	1	2	2

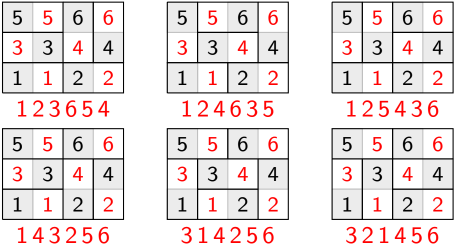
$$m(4,3) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 1 & 0 & i & 0 & 0 & 0 \\ 1 & 1 & 0 & i & 0 & 0 \\ i & 0 & 1 & 1 & i & 0 \\ 0 & i & 0 & 1 & 0 & i \\ 0 & 0 & i & 0 & 1 & 0 \\ 0 & 0 & 0 & i & 1 & 1 \end{pmatrix} \end{array}$$

To every tiling in we may assign a *permutation* $\sigma \in \mathfrak{S}_d$ defined by $\sigma(a) = b$ when the black square labeled a is paired with the white square labeled b.

When $\ell = 4$ and $k = 3$, there are five permutations with sign $+1$,



and six with sign -1,



Define, for $\tau \in \mathfrak{S}_d$,

$$s(\tau) := a_{1\tau_1} \cdot a_{2\tau_2} \cdots a_{d\tau_d} .$$

Then $s(\tau) = 0$ if and only if there is no tiling associated with τ.

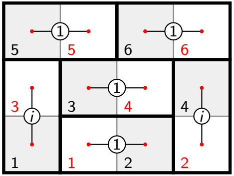
Let $\mathcal{S} = \{\tau \in \mathfrak{S}_d \mid s(\tau) \neq 0\}$. By definition,

$$\det\ m(\ell, k) = \sum_{\tau \in \mathfrak{S}_d} \mathrm{sign}(\tau)\, s(\tau) = \sum_{\tau \in \mathcal{S}} \mathrm{sign}(\tau)\, s(\tau) .$$

Then $\mathrm{sign}(\sigma)\, s(\sigma) = \mathrm{sign}(\mu)\, s(\mu)$ for any two $\sigma, \mu \in \mathcal{S}$. Since $\|\mathrm{sign}(\sigma)\, s(\sigma)\| = 1$,

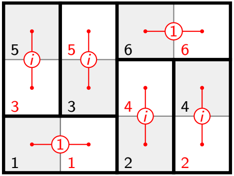
$$\begin{aligned} \|\det\ m(\ell, k)\| &= \Big\| \sum_{\tau \in \mathcal{S}} \mathrm{sign}(\tau)\, s(\tau) \Big\| \\ &= |\mathcal{S}|\, \|\mathrm{sign}(\sigma)\, s(\sigma)\| \\ &= \text{number of domino tilings of } B . \end{aligned}$$

$$\operatorname{sign}(\sigma)\, s(\sigma) = (-1)\, i^2 = \operatorname{sign}(\mu)\, s(\mu) = (+1)\, i^4 = 1\,.$$



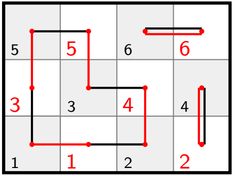
$\sigma = 3\,1\,4\,2\,5\,6 = (1\,3\,4\,2)(5)(6)$

$s(\sigma) = i^2 = -1 = \text{sign}\ \sigma$



$\mu = 1\,4\,5\,2\,3\,6 = (2\,4)(3\,5)(1)(6)$

$s(\mu) = i^4 = 1 = \text{sign}\ \mu$



$\sigma \circ \mu^{-1} = (1\,3\,5\,4)(2)(6)$

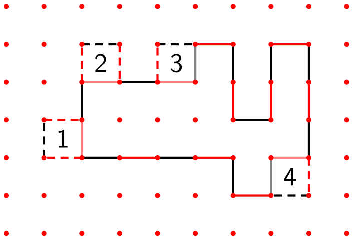


Figure: Induction argument.

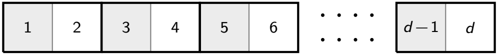
Let $\left(a_{p\,q}\right)_{1\leqslant p,q\leqslant \ell\, k}$ be the square matrix where $a_{p\,q}$ is defined as before but we do not distinguish between black and white squares. If black squares are numbered from 1 to d and white squares from $d+1$ to $2d = \ell\, k$, we obtain the following matrix, formed by four $d \times d$ blocks,

$$\begin{pmatrix} 0 & m(\ell, k) \\ m(\ell, k)^T & 0 \end{pmatrix},$$

where $m(\ell, k)$ was defined before. The determinant of this matrix is

$$(-1)^d \det\, m(\ell, k) \det\, m(\ell, k)^T = (-1)^d (\det\, m(\ell, k))^2 .$$

For $k = 1$ and $\ell = 2d$ we obtain $T(\ell, 1) = 1$.



$$M(\ell,1) = \left(\begin{array}{cccc|c} 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & 0 & 1 \\ \hline 0 & 0 & \cdots & 1 & 0 \end{array}\right), \qquad \chi_\ell(\lambda) = \det \begin{pmatrix} -\lambda & 1 & \cdots & 0 & 0 \\ 1 & -\lambda & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & -\lambda & 1 \\ 0 & 0 & \cdots & 1 & -\lambda \end{pmatrix}$$

For every $n \geqslant 3$ and $\lambda \in \mathbb{C}$,

$$\chi_\ell(\lambda) = -\lambda\, \chi_{\ell-1}(\lambda) - \chi_{\ell-2}(\lambda)\,.$$

In particular,

$$\chi_\ell(2\cos(t)) = -2\cos(t)\big(\chi_{\ell-1}(2\cos(t))\big) - \chi_{\ell-2}(2\cos(t))\,.$$

Since

$$\chi_1(2\cos(t)) = -\frac{\sin(2t)}{\sin(t)} \quad \text{and} \quad \chi_2(2\cos(t)) = \frac{\sin(3t)}{\sin(t)}\,,$$

$$\chi_\ell(2\cos(t)) = (-1)^\ell \frac{\sin\big((\ell+1)t\big)}{\sin(t)}\,.$$

The eigenvalues of $m(\ell,1)$ are, for $p = 1, 2, \ldots, \ell$, $2\cos\left(\frac{p\pi}{\ell+1}\right)$ since

$$\chi_\ell\left(2\cos\left(\frac{p\pi}{\ell+1}\right)\right) = (-1)^\ell \sin(p\pi) = 0\,.$$

In particular,

$$\left|\prod_{p=1}^{\ell} 2\cos\left(\frac{p\pi}{\ell+1}\right)\right| = \begin{cases} 1, & \text{if } \ell \text{ is even;} \\ 0, & \text{if } \ell \text{ is odd.} \end{cases}$$

Now, we consider three $(\ell\, k) \times (\ell\, k)$ matrices,
$M(\ell, k) = \big(a_{pq}\big)_{1 \leqslant p,q \leqslant \ell\, k}$ as before,
and $M_h(\ell, k) = \big(b_{pq}\big)_{1 \leqslant p,q \leqslant \ell\, k}$ and $M_v(\ell, k) = \big(c_{pq}\big)_{1 \leqslant p,q \leqslant \ell\, k}$,
defined by

$$b_{pq} = \begin{cases} 1, & \text{if squares } p \text{ and } q \text{ are placed side by side,} \\ 0, & \text{otherwise;} \end{cases}$$

$$c_{pq} = \begin{cases} i, & \text{if squares } p \text{ and } q \text{ are placed one on top of the other,} \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$M(\ell, k) = M_h(\ell, k) + M_v(\ell, k)\,.$$

If λ is an eigenvalue of $M(\ell, 1)$ and μ is an eigenvalue of $M(k, 1)$, then $\lambda + \mu\, i$ is an eigenvalue of $M(\ell, k)$.

$$\det\big(M(\ell, k)\big) = \prod_{p=1}^{\ell} \prod_{q=1}^{k} \left(2\cos\left(\frac{p\,\pi}{\ell+1}\right) + 2\cos\left(\frac{q\,\pi}{k+1}\right) i\right).$$

Finally,

$$\begin{aligned}\det\big(M(\ell,k)\big) &= \big|\det\big(M(\ell,k)\big)\big| \\ &= \prod_{p=1}^{\ell}\prod_{q=1}^{k}\left\|2\cos\left(\frac{p\,\pi}{\ell+1}\right)+2\cos\left(\frac{q\,\pi}{k+1}\right)i\right\| \\ &= \prod_{p=1}^{\ell}\prod_{q=1}^{k}\sqrt{4\cos^2\left(\frac{p\,\pi}{\ell+1}\right)+4\cos^2\left(\frac{q\,\pi}{k+1}\right)}\,.\end{aligned}$$

Thank you for your attention!