Commuting graph of a 0-Rees matrix semigroup over a group

Tânia Paulista

NOVA FCT

The 14th Combinatorics Days

This work is funded by National Funds through the FCT - Fundação para a Ciência e a Tecnologia, I.P., under the scope of the projects UIDB/00297/2020 and UIDP/00297/2020 and the studentship 2021.07002.BD.

Commuting graphs of semigroups

Let S be a finite non-commutative semigroup.

The **commuting graph** of S, denoted $\mathcal{G}(S)$, is the simple graph such that:

- $S \setminus Z(S)$ is the set of vertices, where

$$Z(S) = \{x \in S : xy = yx \text{ for all } y \in S\}.$$

- $\{x, y\}$ is an edge if $x \neq y$ and $xy = yx$.

Completely 0-simple semigroups

Theorem (Rees–Suschkewitsch Theorem)

A semigroup S is completely 0-simple if and only if there exist a group G, index sets I and Λ, and a regular $\Lambda \times I$ matrix P with entries from G^0 such that $S \simeq \mathcal{M}_0[G; I, \Lambda; P]$.

0-Rees matrix semigroup over a group

Let G be a group, I and Λ be index sets, and P be a regular $\Lambda \times I$ matrix with entries from G^0.

Let $p_{\lambda i}$ be the (λ, i)-th entry of P.

A 0-**Rees matrix semigroup over a group**, denoted $\mathcal{M}_0[G; I, \Lambda; P]$, is the set $(I \times G \times \Lambda) \cup \{0\}$ with the multiplication

$$(i, x, \lambda)(j, y, \mu) = \begin{cases} (i, xp_{\lambda j}y, \mu) & \text{if } p_{\lambda j} \neq 0, \\ 0 & \text{if } p_{\lambda j} = 0, \end{cases}$$

$$0(i, x, \lambda) = (i, x, \lambda)0 = 00 = 0.$$

Zeros are important

Lemma

Let P and Q be regular $\Lambda \times I$ matrices with entries from G^0. If for all $i \in I$ and $\lambda \in \Lambda$ $p_{\lambda i} = 0$ if and only if $q_{\lambda i} = 0$, then the graphs $\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$ and $\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; Q])$ are isomorphic.

Example

Let $e, g, h \in G$.

$$\begin{bmatrix} e & 0 & 0 \\ 0 & g & h \\ e & 0 & g \end{bmatrix} \quad \rightsquigarrow \quad \begin{bmatrix} \times & 0 & 0 \\ 0 & \times & \times \\ \times & 0 & \times \end{bmatrix}$$

Reordering columns and rows implies isomorphism

Lemma

Let Q be the matrix obtained from P by reordering the columns and rows of P. Then the graphs $\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$ and $\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; Q])$ are isomorphic.

Example

$$\begin{bmatrix} \times & 0 & \times & \times \\ 0 & \times & 0 & \times \\ \times & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} \times & \times & \times & 0 \\ 0 & \times & 0 & \times \\ \times & 0 & 0 & 0 \end{bmatrix}$$

$|I| = |\Lambda| = 1$: characterization of $\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$

Theorem (P., 2024)

- $\mathcal{M}_0[G; I, \Lambda; P] \simeq G^0$.
- *Suppose that G is non-abelian. Then the graphs $\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$ and $\mathcal{G}(G)$ are isomorphic.*

$|I| > 1$ or $|\Lambda| > 1$, P has no zeros: connectedness

- $Z(\mathcal{M}_0[G; I, \Lambda; P]) = \{0\}$.
- $\mathcal{M}_0[G; I, \Lambda; I]$ is non-commutative.

Theorem (P., 2024)

- *$\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$ is not connected.*
- *The connected components of $\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$ are the graphs induced by $\{i\} \times G \times \{\lambda\}$, $i \in I$, $\lambda \in \Lambda$.*
- *Let $\mathcal{C}$ be a connected component of $\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$. Then*

$$\mathcal{C} \simeq \begin{cases} K_{|G|} & \text{if } G \text{ is abelian,} \\ K_{|Z(G)|} \nabla \mathcal{G}(G) & \text{if } G \text{ is non-abelian.} \end{cases}$$

$|I| > 1$ or $|\Lambda| > 1$, P has zeros: connectedness

- $Z(\mathcal{M}_0[G; I, \Lambda; P]) = \{0\}$.
- $\mathcal{M}_0[G; I, \Lambda; I]$ is non-commutative.

Theorem (P., 2024)

$\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$ is connected if and only if P cannot be decomposed in one of the following ways

$$\left[\begin{array}{ccc} & A & \\ \hline \times & \cdots & \times \end{array}\right] \qquad \left[\begin{array}{c|c} & \times \\ A & \vdots \\ & \times \end{array}\right] \qquad \left[\begin{array}{ccc|ccc} & & & \times & \cdots & \times \\ & A & & \vdots & & \vdots \\ & & & \times & \cdots & \times \\ \hline \times & \cdots & \times & & & \\ \vdots & & \vdots & & B & \\ \times & \cdots & \times & & & \end{array}\right].$$

Identifying if $\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$ is connected

Example

$$\begin{bmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix}$$

Identifying if $\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$ is connected

Example

$$\begin{bmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix}$$

Identifying if $\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$ is connected

Example

$$\begin{bmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix}$$

Identifying if $\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$ is connected

Example

$$\begin{bmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix}$$

Identifying if $\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$ is connected

Example

$$\begin{bmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix}$$

Identifying if $\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$ is connected

Example

$$\begin{bmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} \boxed{0} & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} \boxed{0} & \times & \boxed{0} \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} \boxed{0} & \times & \boxed{0} \\ \times & 0 & \times \\ \times & 0 & \boxed{0} \end{bmatrix} \rightsquigarrow \begin{bmatrix} \boxed{0} & \times & \boxed{0} \\ \times & 0 & \times \\ \boxed{\times} & 0 & \boxed{0} \end{bmatrix} \rightsquigarrow \begin{bmatrix} \boxed{0} & \times & \boxed{0} \\ \times & 0 & \times \\ \boxed{\times} & \boxed{0} & \boxed{0} \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} \boxed{0} & \boxed{\times} & \boxed{0} \\ \times & 0 & \times \\ \boxed{\times} & \boxed{0} & \boxed{0} \end{bmatrix} \rightsquigarrow \begin{bmatrix} \boxed{0} & \boxed{\times} & \boxed{0} \\ \times & \boxed{0} & \times \\ \boxed{\times} & \boxed{0} & \boxed{0} \end{bmatrix} \rightsquigarrow \begin{bmatrix} \boxed{0} & \boxed{\times} & \boxed{0} \\ \boxed{\times} & \boxed{0} & \boxed{\times} \\ \boxed{\times} & \boxed{0} & \boxed{0} \end{bmatrix}$$

Identifying if $\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$ is connected

Example

$$\begin{bmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix}$$

Identifying if $\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$ is connected

Example

$$\begin{bmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix}$$

Identifying if $\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$ is connected

Example

$$\begin{bmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \leadsto \begin{bmatrix} \boxed{0} & \times & 0 \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \leadsto \begin{bmatrix} \boxed{0} & \times & \boxed{0} \\ \times & 0 & \times \\ \times & 0 & 0 \end{bmatrix}$$

$$\leadsto \begin{bmatrix} \boxed{0} & \times & \boxed{0} \\ \times & 0 & \times \\ \times & 0 & \boxed{0} \end{bmatrix} \leadsto \begin{bmatrix} \boxed{0} & \times & \boxed{0} \\ \times & 0 & \times \\ \boxed{\times} & 0 & \boxed{0} \end{bmatrix} \leadsto \begin{bmatrix} \boxed{0} & \times & \boxed{0} \\ \times & 0 & \times \\ \boxed{\times} & \boxed{0} & \boxed{0} \end{bmatrix}$$

$$\leadsto \begin{bmatrix} \boxed{0} & \boxed{\times} & \boxed{0} \\ \times & 0 & \times \\ \boxed{\times} & \boxed{0} & \boxed{0} \end{bmatrix} \leadsto \begin{bmatrix} \boxed{0} & \boxed{\times} & \boxed{0} \\ \times & \boxed{0} & \times \\ \boxed{\times} & \boxed{0} & \boxed{0} \end{bmatrix} \leadsto \begin{bmatrix} \boxed{0} & \boxed{\times} & \boxed{0} \\ \boxed{\times} & \boxed{0} & \boxed{\times} \\ \boxed{\times} & \boxed{0} & \boxed{0} \end{bmatrix}$$

Identifying connected components of $\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$

Example

	1	2	3	4	5
1	$\times$	$\times$	$\times$	0	$\times$
2	0	$\times$	$\times$	$\times$	$\times$
3	$\times$	0	$\times$	$\times$	$\times$
4	$\times$	0	$\times$	0	$\times$
5	$\times$	$\times$	0	$\times$	0
6	$\times$	$\times$	$\times$	$\times$	0
7	$\times$	$\times$	$\times$	$\times$	$\times$

$\rightsquigarrow$

	1	2	3	4	5
1	$\times$	$\times$	$\times$	0	$\times$
2	0	$\times$	$\times$	$\times$	$\times$
3	$\times$	0	$\times$	$\times$	$\times$
4	$\times$	0	$\times$	0	$\times$
5	$\times$	$\times$	0	$\times$	0
6	$\times$	$\times$	$\times$	$\times$	0
7	$\times$	$\times$	$\times$	$\times$	$\times$

Identifying connected components of $\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$

Example

$$
\begin{array}{c}
 \\ \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{4} \\ \mathbf{5} \\ \mathbf{6} \\ \mathbf{7}
\end{array}
\begin{array}{c}
\begin{array}{ccccc} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} \end{array} \\
\begin{bmatrix}
\times & \times & \times & 0 & \times \\
0 & \times & \times & \times & \times \\
\times & 0 & \times & \times & \times \\
\times & 0 & \times & 0 & \times \\
\times & \times & 0 & \times & 0 \\
\times & \times & \times & \times & 0 \\
\times & \times & \times & \times & \times
\end{bmatrix}
\end{array}
\quad \rightsquigarrow \quad
\begin{array}{c}
 \\ \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{4} \\ \mathbf{5} \\ \mathbf{6} \\ \mathbf{7}
\end{array}
\begin{array}{c}
\begin{array}{ccccc} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} \end{array} \\
\begin{bmatrix}
\times & \boxed{\times} & \times & \boxed{0} & \times \\
0 & \times & \times & \times & \times \\
\times & \boxed{0} & \times & \boxed{\times} & \times \\
\times & \boxed{0} & \times & \boxed{0} & \times \\
\times & \times & 0 & \times & 0 \\
\times & \times & \times & \times & 0 \\
\times & \times & \times & \times & \times
\end{bmatrix}
\end{array}
$$

Identifying connected components of $\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$

Example (cont.)

$\rightsquigarrow$

	2	4	1	3	5
1	×	0	×	×	×
3	0	×	×	×	×
4	0	0	×	×	×
2	×	×	0	×	×
5	×	×	×	0	0
6	×	×	×	×	0
7	×	×	×	×	×

$\rightsquigarrow$

	2	4	1	3	5
1	×	0	×	×	×
3	0	×	×	×	×
4	0	0	×	×	×
2	×	×	0	×	×
5	×	×	×	0	0
6	×	×	×	×	0
7	×	×	×	×	×

Identifying connected components of $\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$

Example (cont.)

$\rightsquigarrow$

	2	4	1	3	5
1	×	0	×	×	×
3	0	×	×	×	×
4	0	0	×	×	×
2	×	×	0	×	×
5	×	×	×	0	0
6	×	×	×	×	0
7	×	×	×	×	×

$\rightsquigarrow$

	2	4	1	3	5
1	×	0	×	×	×
3	0	×	×	×	×
4	0	0	×	×	×
2	×	×	0	×	×
5	×	×	×	0	0
6	×	×	×	×	0
7	×	×	×	×	×

Example (cont.)

$\rightsquigarrow$

	2	4	1	3	5
1	$\times$	0	$\times$	$\times$	$\times$
3	0	$\times$	$\times$	$\times$	$\times$
4	0	0	$\times$	$\times$	$\times$
2	$\times$	$\times$	0	$\times$	$\times$
5	$\times$	$\times$	$\times$	0	0
6	$\times$	$\times$	$\times$	$\times$	0
7	$\times$	$\times$	$\times$	$\times$	$\times$

The subgraph induced by $\{2,4\} \times G \times \{1,3,4\}$ is a connected component of $\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$.

Identifying connected components of $\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$

Example (cont.)

$\rightsquigarrow$

	2	**4**	**1**	**3**	**5**
1	$\times$	0	$\times$	$\times$	$\times$
3	0	$\times$	$\times$	$\times$	$\times$
4	0	0	$\times$	$\times$	$\times$
2	$\times$	$\times$	0	$\times$	$\times$
5	$\times$	$\times$	$\times$	0	0
6	$\times$	$\times$	$\times$	$\times$	0
7	$\times$	$\times$	$\times$	$\times$	$\times$

The subgraph induced by $\{2, 4\} \times G \times \{1, 3, 4\}$ is a connected component of $\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$.

Identifying connected components of $\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$

Example (cont.)

	2	4	1	3	5
1	×	0	×	×	×
3	0	×	×	×	×
4	0	0	×	×	×
2	×	×	0	×	×
5	×	×	×	0	0
6	×	×	×	×	0
7	×	×	×	×	×

The subgraph induced by $(\{1\} \times G \times \{1,3,4\})$ $\cup(\{2,4\} \times G \times \{2\})$ is a connected component of $\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$.

Identifying connected components of $\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$

Example (cont.)

	2	4	1	3	5
1	$\times$	0	$\times$	$\times$	$\times$
3	0	$\times$	$\times$	$\times$	$\times$
4	0	0	$\times$	$\times$	$\times$
2	$\times$	$\times$	0	$\times$	$\times$
5	$\times$	$\times$	$\times$	0	0
6	$\times$	$\times$	$\times$	$\times$	0
7	$\times$	$\times$	$\times$	$\times$	$\times$

The subgraph induced by $(\{1\} \times G \times \{1, 3, 4\}) \cup (\{2, 4\} \times G \times \{2\})$ is a connected component of $\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$.

Identifying connected components of $\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$

Example (cont.)

	2	**4**	**1**	**3**	**5**
1	$\times$	0	$\times$	$\times$	$\times$
3	0	$\times$	$\times$	$\times$	$\times$
4	0	0	$\times$	$\times$	$\times$
2	$\times$	$\times$	0	$\times$	$\times$
5	$\times$	$\times$	$\times$	0	0
6	$\times$	$\times$	$\times$	$\times$	0
7	$\times$	$\times$	$\times$	$\times$	$\times$

The subgraph induced by $\{2\} \times G \times \{7\}$ is a connected component of $\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$.

Identifying connected components of $\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$

Example (cont.)

	2	4	1	3	5
1	×	0	×	×	×
3	0	×	×	×	×
4	0	0	×	×	×
2	×	×	0	×	×
5	×	×	×	0	0
6	×	×	×	×	0
7	×	×	×	×	×

The subgraph induced by $\{2\} \times G \times \{7\}$ is a connected component of $\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$.

$|I| > 1$ or $|\Lambda| > 1$, P has no zeros: clique number

Let $H = (V, E)$ be a simple graph.

- A **clique** is a subset $K \subseteq V$ such that $\{u, v\} \in E$, for all distinct $u, v \in K$. The **clique number** of H, denoted $\omega(H)$, is the largest integer r such that H has a clique K such that $|K| = r$.

Theorem (P., 2024)

$$\omega(\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])) = \begin{cases} |G| & \textit{if } G \textit{ is abelian,} \\ \omega(\mathcal{G}(G)) + |Z(G)| & \textit{if } G \textit{ is non-abelian.} \end{cases}$$

$|I| > 1$ or $|\Lambda| > 1$, P has zeros: clique number

Theorem (P., 2024)

- *Suppose that G is abelian. If*

$$\begin{bmatrix} \times & 0 & 0 \\ 0 & \times & 0 \\ 0 & 0 & \times \end{bmatrix}$$

is a submatrix of P and

$$\begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

are not submatrices of P, then

$$\omega(\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])) = 3|G|.$$

$|I| > 1$ or $|\Lambda| > 1$, P has zeros: clique number

Theorem (cont.) (P., 2024)

- *Suppose that G is abelian. If*

$$\begin{bmatrix} \times & 0 \\ 0 & \times \end{bmatrix}$$

is a submatrix of P and

$$\begin{bmatrix} 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

are not submatrices of P, then

$$\omega(\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])) = 2|G|.$$

$|I| > 1$ or $|\Lambda| > 1$, P has zeros: clique number

Theorem (cont.) (P., 2024)

- *For the remaining cases, we have*

$$\omega(\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])) = |G| \cdot \max\{nm : 0_{n\times m} \text{ is a submatrix of } P\}.$$

$|I| > 1$ or $|\Lambda| > 1$, P has no zeros: girth

Let $H = (V, E)$ be a simple graph.

- The **girth** of H, denoted $\mathrm{girth}(H)$, is the length of a shortest cycle contained in H.

Theorem (P., 2024)

- *If $|G| \leqslant 2$, then $\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$ has no cycles.*
- *If $|G| \geqslant 3$, then $\mathrm{girth}(\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])) = 3$.*

$|I| > 1$ or $|\Lambda| > 1$, P has zeros: girth

Theorem (P., 2024)

- *Suppose that* $|G| \geqslant 3$. *Then*

$$girth(\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])) = 3.$$

- *Suppose that* $|G| = 2$. *If* P *contains only one zero entry, then* $\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$ *has no cycles.*

- *Suppose that* $|G| = 2$. *If* P *contains more than one zero entry, then*

$$girth(\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])) = 3.$$

$|I| > 1$ or $|\Lambda| > 1$, P has zeros: girth

Theorem (cont.) (P., 2024)

- *Suppose that* $|G| = 1$. *If*

$$\begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} \times & 0 & 0 \\ 0 & \times & 0 \\ 0 & 0 & \times \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & \times & \times \\ \times & \times & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & \times \\ 0 & \times \\ \times & 0 \\ \times & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & \times \\ 0 & 0 \\ \times & 0 \end{bmatrix}$$

are not submatrices of P, *then* $\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$ *has no cycles.*

$|I| > 1$ or $|\Lambda| > 1$, P has zeros: girth

Theorem (cont.) (P., 2024)

- *Suppose that* $|G| = 1$. *If at least one of the matrices*

$$\begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} \times & 0 & 0 \\ 0 & \times & 0 \\ 0 & 0 & \times \end{bmatrix}$$

is a submatrix of P, *then*

$$\mathit{girth}(\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])) = 3.$$

$|I| > 1$ or $|\Lambda| > 1$, P has zeros: girth

Theorem (cont.) (P., 2024)

- *Suppose that* $|G| = 1$. *If at least one of the matrices*

$$\begin{bmatrix} 0 & 0 & \times & \times \\ \times & \times & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & \times \\ 0 & \times \\ \times & 0 \\ \times & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & \times \\ \times & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & \times \\ 0 & 0 \\ \times & 0 \end{bmatrix}$$

is a submatrix of P *and*

$$\begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} \times & 0 & 0 \\ 0 & \times & 0 \\ 0 & 0 & \times \end{bmatrix}$$

are not submatrices of P, *then*

$$girth(\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])) = 4.$$

$|I| > 1$ or $|\Lambda| > 1$, P has no zeros: chromatic number

Let $H = (V, E)$ be a simple graph.

- The **chromatic number** of H, denoted $\chi(H)$, is the minimum number of colours necessary to colour the vertices of H in such a way that no adjacent vertices have the same colour.

Theorem (P., 2024)

$$\chi(\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])) = \begin{cases} |G| & \text{if } G \text{ is abelian,} \\ \chi(\mathcal{G}(G)) + |Z(G)| & \text{if } G \text{ is non-abelian.} \end{cases}$$

$|I| > 1$ or $|\Lambda| > 1$, P has no zeros: knit degree

Let S be a finite non-commutative semigroup.

- A path $a_1 - a_2 - \cdots - a_k$ in $\mathcal{G}(S)$ is called a **left path** if $a_1 \neq a_k$ and $a_1 a_i = a_k a_i$, for all $i \in \{1, \ldots, k\}$.
- Suppose $\mathcal{G}(S)$ has a left path. The **knit degree** of S, denoted $\mathrm{kd}(S)$, is the length of a shortest left path in $\mathcal{G}(S)$.

Theorem (P., 2024)

$\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$ *contains no left paths.*

$|I| > 1$ or $|\Lambda| > 1$, P has zeros: knit degree

Theorem (P., 2024)

- *Suppose that* $|G| = 1$. *If*

$$\begin{bmatrix} 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

are not submatrices of P, *then* $\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])$ *has no left paths.*

- *For the remaining cases, we have*

$$kd(\mathcal{G}(\mathcal{M}_0[G; I, \Lambda; P])) = 1.$$

Completely 0-simple semigroups

Theorem (Rees–Suschkewitsch Theorem)

A semigroup S is completely 0-simple if and only if there exist a group G, index sets I and Λ, and a regular $\Lambda \times I$ matrix P with entries from G^0 such that $S \simeq \mathcal{M}_0[G; I, \Lambda; P]$.

Commuting graph of completely 0-simple semigroups

Theorem (P., 2024)

- *For each* $n \in \mathbb{N}$*, there is a completely* 0*-simple semigroup whose commuting graph has clique number equal to* n*.*
- *For each* $n \in \mathbb{N}$*, there is a completely* 0*-simple semigroup whose commuting graph has chromatic number equal to* n*.*
- *For each* $n \in \mathbb{N}$*, there is a completely* 0*-simple semigroup whose commuting graph has diameter greater than* n*.*
- *Let* S *be a finite non-commutative completely* 0*-simple semigroup. Then* $\mathcal{G}(S)$ *has no cycles or the girth of* $\mathcal{G}(S)$ *is either* 3 *or* 4*.*
- *Let* S *be a finite non-commutative completely* 0*-simple semigroup. Then* $\mathcal{G}(S)$ *has no left paths or the knit degree of* $\mathcal{G}(S)$ *is* 1*.*

Thank you!