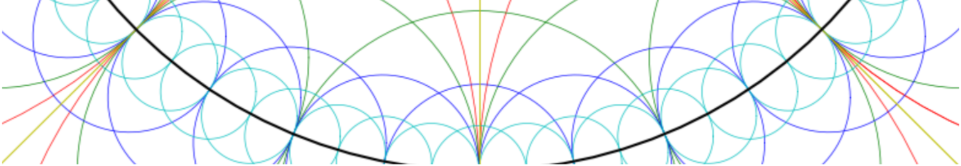

A Schur ring approach to supercharacters of adjoint groups and related subgroups

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Let G be a finite group.

- The complex group algebra of G,

$$\mathbb{C}[G] := \{\alpha : \alpha \text{ is a map from } G \text{ to } \mathbb{C}\}$$

- The convolution product,

$$(\alpha\beta)(g) := \sum_{h \in G} \alpha(h)\beta(h^{-1}g)$$

- The Hadamard product,

$$(\alpha \cdot \beta)(g) := \alpha(g)\beta(g)$$

- The anti-involution $\star$,

$$\alpha^{\star}(g) = \overline{\alpha(g^{-1})}$$

for all $\alpha, \beta \in \mathbb{C}[G]$ and all $g \in G$.

- The characteristic map:

$$\delta_K(g) := \begin{cases} 1, & \text{if } g \in K \\ 0, & \text{otherwise} \end{cases}$$

  for every $K \subseteq G$ and all $g \in G$.

- δ_1 is the identity of $\mathbb{C}[G]$ with respect to the convolution product.

- δ_G is the identity of $\mathbb{C}[G]$ with respect to the Hadamard product.

$\mathbb{C}[G]$ and the center of $\mathbb{C}[G]$ are unital $\star$-algebras over $\mathbb{C}$ with respect to the convolution product and the Hadamard product.

Schur rings

Definition [Wielandt, 1964]

A partition $\mathcal{K}$ of G is called a Schur partition of G if the following conditions hold:

- $\{1\} \in \mathcal{K}$
- $K^{-1} := \{g^{-1} : g \in K\} \in \mathcal{K}$, for all $K \in \mathcal{K}$
- The convolution product of characteristic maps of $\mathcal{K}$ is a linear combination of characteristic maps of $\mathcal{K}$.

If $\mathcal{K}$ is a Schur partition of G, then

$$\mathcal{S}_{\mathcal{K}} := \operatorname{span}_{\mathbb{C}}\{\delta_K : K \in \mathcal{K}\}$$

is the Schur ring over G afforded by $\mathcal{K}$.

What about characters of G?

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What about characters of G?

- Semisimplicity of unital $\star$-subalgebras of $\mathbb{C}[G]$.
- Uniqueness of a central partition of unity in this cases.

$$\downarrow$$

Theorem [Muzychuk, 1994]

A vector subspace of $\mathbb{C}[G]$ is a Schur ring over G if and only if it is a unital $\star$-subalgebra of $\mathbb{C}[G]$ with respect to the convolution product and the Hadamard product.

Unique central partition of unity with respect to the convolution product $\longrightarrow$ supercharacters

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Supercharacter theory

- The set of irreducible characters of G, $\mathrm{Irr}(G) \subseteq \mathbb{C}[G]$.
- If $X \subseteq \mathrm{Irr}(G)$, then we consider the character of G

$$\xi_X := \sum_{\chi \in X} \chi(1)\chi \in \mathbb{C}[G]$$

$$\downarrow$$

Definition [Diaconis & Isaacs, 2008]

A pair $(\mathcal{K}, \mathcal{X})$, where $\mathcal{K}$ is a partition of G and $\mathcal{X}$ is a partition of $\mathrm{Irr}(G)$, is called a (canonical) supercharacter theory of G if the following two conditions hold:

1. $|\mathcal{K}| = |\mathcal{X}|$
2. $X \in \mathcal{X} \Rightarrow \xi_X$ is constant on each $K \in \mathcal{K}$

If $(\mathcal{K}, \mathcal{X})$ is a supercharacter theory of G, then each $K \in \mathcal{K}$ is called a superclass of G and each ξ_X, with $X \in \mathcal{X}$, is called a supercharacter of G.

Theorem [Hendrickson, 2012]

There is a one-to-one correspondence between

$$\left\{\begin{matrix}\text{supercharacter theories} \\ \text{of } G\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{central Schur ring} \\ \text{over } G\end{matrix}\right\}$$

$$(\mathcal{K}, \mathcal{X}) \longmapsto \mathcal{S}_{\mathcal{K}}$$

Let $\mathcal{K}$ be a partition of G, $\mathcal{X}$ be a partition of $\mathrm{Irr}(G)$ and denote by $\mathrm{cf}(G)$ the center of $\mathbb{C}[G]$ (class functions of G).

Theorem]

The following are equivalent:

1. $(\mathcal{K}, \mathcal{X})$ is a supercharacter theory for G;
2. $\mathcal{S}_{\mathcal{K}}$ is a central Schur ring;
3. $\mathrm{span}_{\mathbb{C}}\{\xi_X : X \in \mathcal{X}\}$ is a subalgebra of $\mathrm{cf}(G)$ (with identity) with respect to the Hadamard product;
4. $\mathrm{span}_{\mathbb{C}}\{\delta_K : K \in \mathcal{K}\}$ is a subalgebra of $\mathrm{cf}(G)$ (with identity) with respect to the convolution product;
5. $|\mathcal{K}| = |\mathcal{X}|$ and $\mathrm{span}_{\mathbb{C}}\{\xi_X : X \in \mathcal{X}\}$ is the set of all $\mathcal{K}$-superclass functions of G.

Adjoint Group

Let R be a finite radical ring and let $R^\circ = 1 + R$ be the adjoint group of R.

- The map
$$\nu : \begin{array}{ccc} R^\circ & \longrightarrow & R^+ \\ g & \mapsto & g - 1 \end{array}$$
for all $g \in R^\circ$, is a bijection.
- $\mathcal{S}_{\mathcal{K}^+} := \mathrm{span}_{\mathbb{C}}\{\delta_{R^\circ x R^\circ} : x \in R\}$ is an orbit Schur ring over R^+.
- $\nu : \mathrm{span}_{\mathbb{C}}\{\delta_{1+R^\circ x R^\circ} : x \in R\} \rightarrowtail\!\!\!\!\!\rightarrow \mathcal{S}_{\mathcal{K}^+}$, linear extension of ν.

Theorem

The set
$$\mathcal{K}^\circ := \{1 + R^\circ x R^\circ : x \in R\}$$
is a central Schur partition of R° (affording the standard Schur ring over R°, denoted by $\mathcal{S}_{\mathcal{K}^\circ}$).

Let $\mathcal{S}_{\mathcal{K}}$ be the standard Schur ring over R°.

- $\nu : \mathcal{S}_{\mathcal{K}^{\circ}} \longrightarrow \mathcal{S}_{\mathcal{K}^{+}}$ is an isomorphism of central Schur rings.

$$\downarrow$$

Theorem [André & Nicolás, 2008] (with a new proof)

If ξ is a standard supercharacter of R°, then there exists $\vartheta \in \operatorname{Irr}(R^{+})$ such that

$$\vartheta(g) = \sum_{\vartheta' \in R^{\circ}\vartheta R^{\circ}} \vartheta'(\nu(g))$$

for all $g \in R^{\circ}$.

Let R be a finite radical ring of odd characteristic endowed with an anti-involution σ.

- The map

$$\begin{array}{cccc} \sigma^\circ : & R^\circ & \longrightarrow & R^\circ \\ & g & \mapsto & 1+\sigma(\nu(g^{-1})) \end{array}$$

defines an involution on the adjoint group R°.

Definition [S., 2023]

The subgroup of the adjoint group R°, defined by

$$R^\circ_\sigma := \{g \in R^\circ : \sigma^\circ(g) = g\},$$

is called the involutive adjoint group of R with respect to σ.

Let K be an element of the standard Schur partition of R°.

- $\sigma^\circ(K) = K$ if and only if K has a fixed element by σ°.
- We consider the subalgebra of $\mathbb{C}[R^\circ]$

$$\begin{aligned}\mathbb{C}[R^\circ]_\sigma &= \{\alpha \in \mathbb{C}[R^\circ] : \sigma^\circ(\alpha) = \alpha\} \\ &= \mathbb{C}[R^\circ_\sigma] \oplus \mathrm{span}_{\mathbb{C}}\{\delta_{g,\sigma^\circ(g)} : g \neq R^\circ_\sigma\}.\end{aligned}$$

- $\pi : \mathbb{C}[R^\circ]_\sigma \to \mathbb{C}[R^\circ_\sigma]$ is an algebra homomorphism and $\pi(\delta_K) = \delta_{K \cap R^\circ_\sigma}$.

$$\downarrow$$

Theorem [S., 2023]

The set

$$\mathcal{K}^\circ_\sigma := \{K \cap R^\circ_\sigma : K \in \mathcal{K}^\circ \text{ and } \sigma^\circ(K) = K\}$$

is a central Schur partition of R°_σ (affording the standard Schur ring over R°_σ, denoted by $\mathcal{S}_{\mathcal{K}^\circ_\sigma}$).

Let $\mathcal{S}_{\mathcal{K}_\sigma^\circ}$ be the standard Schur ring over R_σ° and consider the subgroup of R^+, $R_\sigma^+ := \{x \in R^+ : \sigma(x) = -x\}$.

- The action of R° on R_σ^+ defined by

$$g \odot x := gx\sigma^\circ(g^{-1})$$

  for all $g \in R^\circ$ and all $x \in R_\sigma^+$, affords an orbit Schur ring over R_σ^+, denoted by $\mathcal{S}_{\mathcal{K}_\sigma^+}$.

- The map (Cayley transform)

$$\begin{array}{rccc} \Phi : & R^+ & \longrightarrow & R^\circ \\ & x & \mapsto & (1+x)(1-x)^{-1} \end{array}$$

  for all $x \in R_\sigma^+$, is a bijection.

Let $\mathcal{S}_{\mathcal{K}_\sigma^\circ}$ be the standard Schur ring over R_σ°.

- $\Psi : \mathcal{S}_{\mathcal{K}_\sigma^\circ} \to \mathcal{S}_{\mathcal{K}_\sigma^+}$ is an isomorphism of Schur rings, where Ψ is the linear extension of the inverse of the Cayley transform.

$$\downarrow$$

Theorem [S. et al., 2024]

If ξ is a standard supercharacter of R_σ°, then there exists $\vartheta \in \mathrm{Irr}(R_\sigma^+)$ such that

$$\vartheta(g) = \sum_{\vartheta' \in R^\circ \odot \vartheta} \vartheta'(\Psi(g))$$

for all $g \in R^\circ$.

Main references

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Thank you!