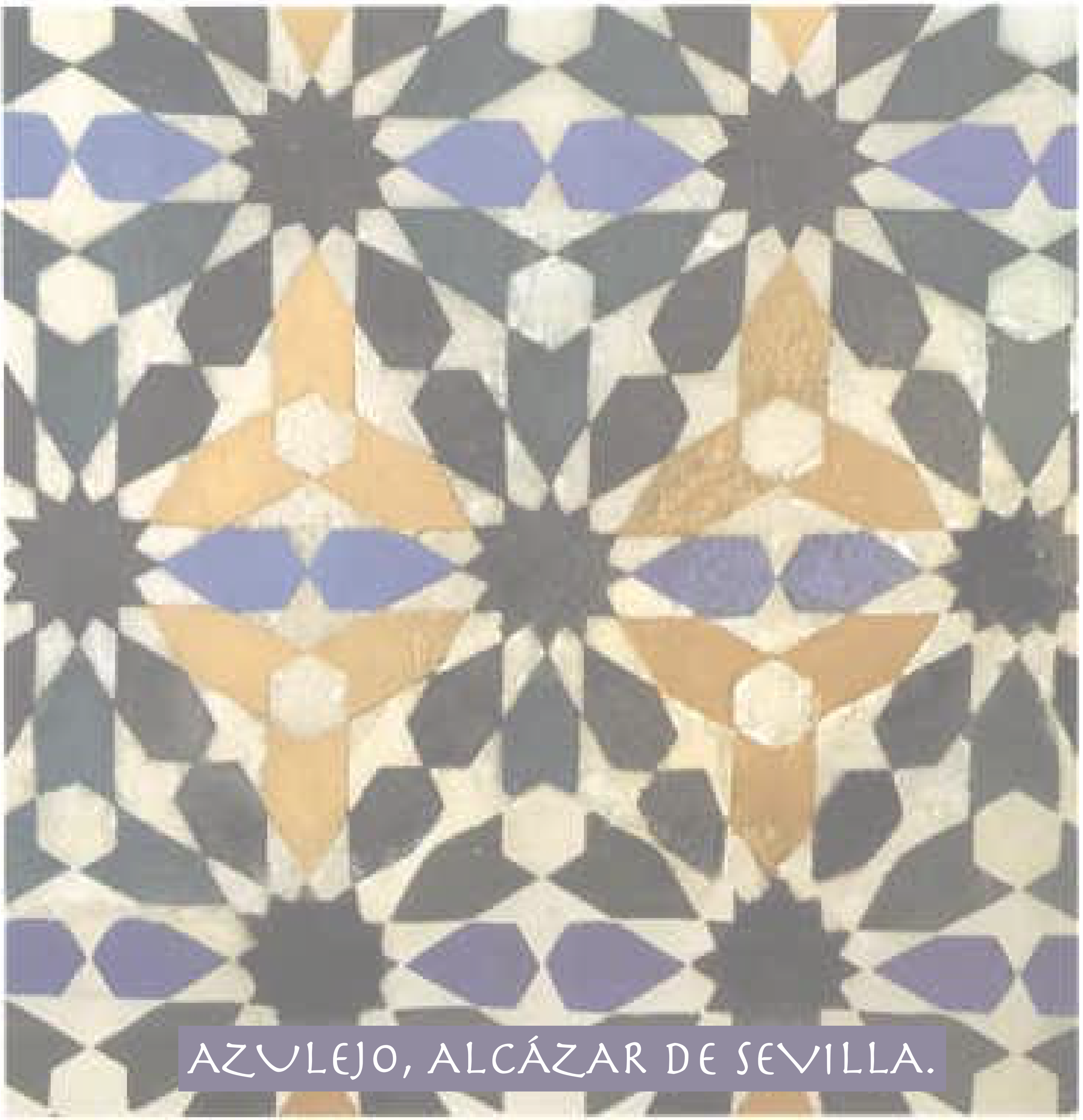
Vector partitions and representations



Mercedes Rosas
Universidad de Sevilla

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U(1) THE UNITARY GROUP $AA^* = A^*A = 1$

COMPLEX NUMBERS OF NORM ONE

LIE GROUP:

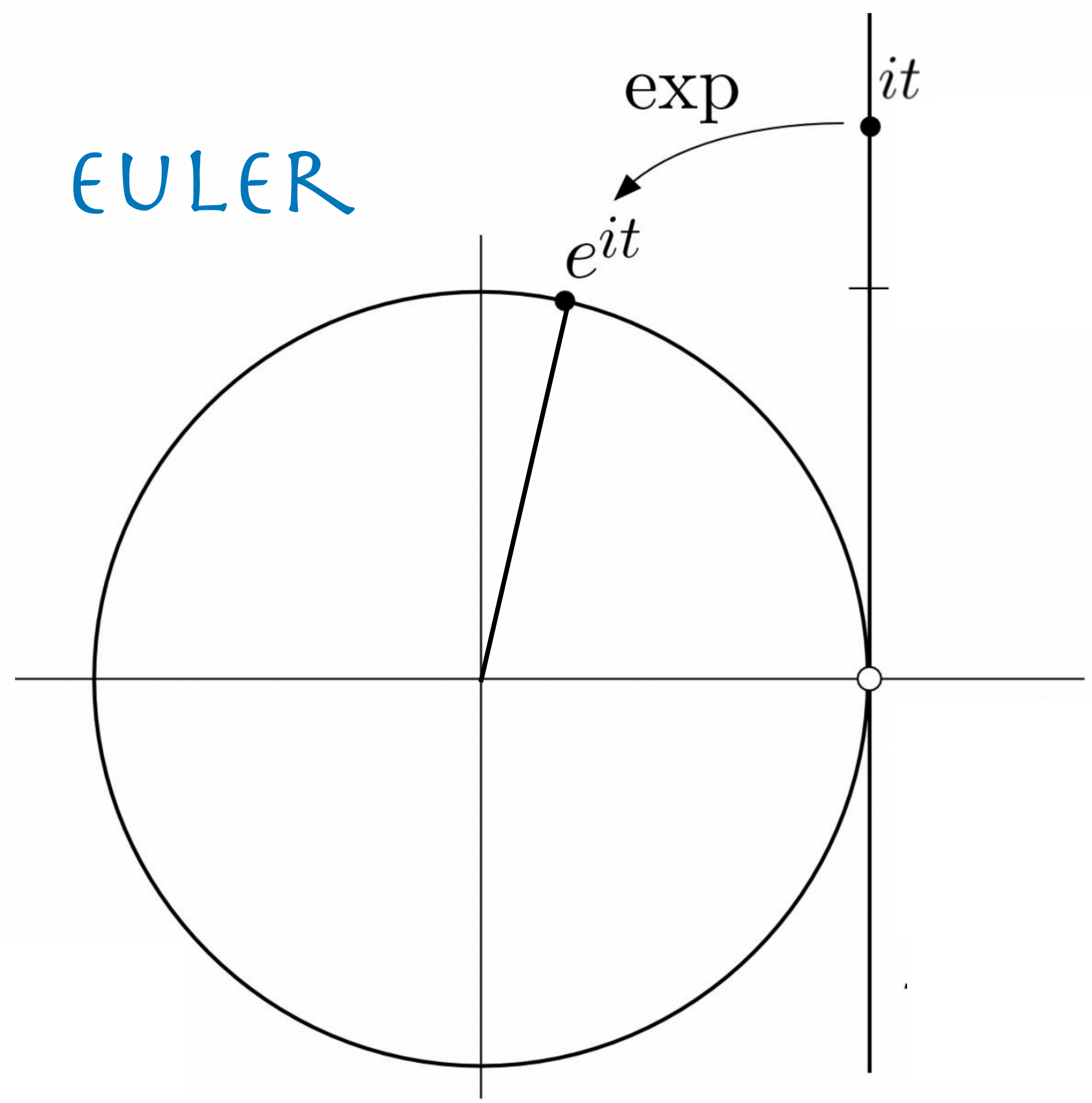
— A 1-SPHERE (CIRCUMFERENCE)

$$U(1)=\{ (e^{i\theta}) \mid \theta \in R\}$$

— AN (ABELIAN) GROUP.

$$e^{i\theta} \, e^{i\theta'} = e^{i(\theta+\theta')}$$

EULER



REPRESENTATIONS / ACTIONS

A LIE GROUP REPRESENTATION IS A DIFFERENTIABLE GROUP MORPHISM

$$\Pi : G \to GL(V)$$

WE ASK THAT V IS A FINITE DIMENSIONAL VECTOR SPACE (REAL/COMPLEX).

G ACTS —LINEARLY— ON V

AN ACTION OF U(1) ON THE REAL PLANE

$$e^{i\theta} \mapsto \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

THE MATRIX OF A ROTATION ON THE REAL PLANE GL(2,R)

NO REAL EIGENVALUE

EX : A COMPLEX 1-DIM REPRESENTATIONS:

★ $$\Pi(\exp i\theta) = (\exp in\theta)$$

THE WEIGHT OF THE REPRESENTATION, n, IS ALWAYS AN INTEGER.

THM : GIVEN $\Pi : U(1) \to GL(V)$, THERE EXISTS A BASIS OF SIMULTANEOUS EIGENVECTORS IN WHICH:

★ $$\Pi(\exp i\theta) = \begin{pmatrix} \exp in_1\theta & 0 & & 0 \\ 0 & \exp in_2\theta & & 0 \\ & & \cdots & \\ 0 & 0 & & \exp in_\ell\theta \end{pmatrix}$$

SPECTRAL THEOREM: UNITARY MATRICES ARE DIAGONALIZABLE, AND ITS EIGENVALUES ARE COMPLEX NUMBERS OF NORM 1.

(* WITH ORTHOGONAL EIGENVECTORS).

Some basic notions

Let V a space on which a group G is acting

A subspace W is invariant if, for all g in the G:

$$gW = \{\Pi(g)w \mid w \in W\} \qquad gW \subseteq W$$

In this situation we say that G acts on W.

The restriction of the representation $\Pi : U(1) \to GL(V)$ to W is a representation of G on GL(W).

A representation is irreducible if it does not have any nontrivial invariant subspace.

Theorem

Given any representation of U(1), there always exists a basis of simultaneous eigenvectors such that, for all θ real

$$\Pi(\exp i\theta) = \begin{pmatrix} \exp in_1\theta & 0 & & 0 \\ 0 & \exp in_2\theta & & 0 \\ & & \cdots & \\ 0 & 0 & & \exp in_\ell\theta \end{pmatrix}$$

Weights of the representation $n_1, n_2, \cdots, n_\ell$ integers

Commuting diagonalizable matrices are simultaneously diagonalizable.

PROOF

1) ANY REPRESENTATION OF U(1) CAN BREAKS AS A SUM OF IRREDUCIBLE REPRESENTATIONS.

FIND AN INVARIANT SUBSPACE W, THEN ITS ORTHOGONAL COMPLEMENT IS ALSO INVARIANT

WE NEED AN INVARIANT HERMITIAN PRODUCT

$$\langle v|u\rangle_{\text{inv}} = \frac{1}{2\pi}\int_0^{2\pi} \langle \Pi(e^{i\theta})v|\Pi(e^{i\theta})u\rangle d\theta$$

(DONE IN THE BLACKBOARD)

Proof

2) Any irreducible representation of U(1) has dimension one.

Schur's lemma

— simultaneous diagonalization

complex numbers

(Done in the blackboard)

3) Finally, weights are always integers.

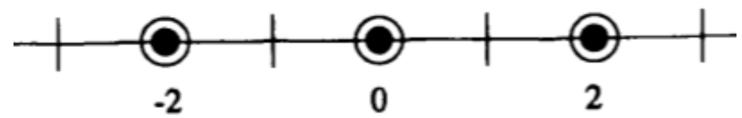
For one dimensional representations, it suffices to observe that 1 should be sent to 1 by any representation.

Then, argue by restriction to the invariant subspaces.

Ex: Two reducible representations:

A direct sum of three irreducible representations of U(1)

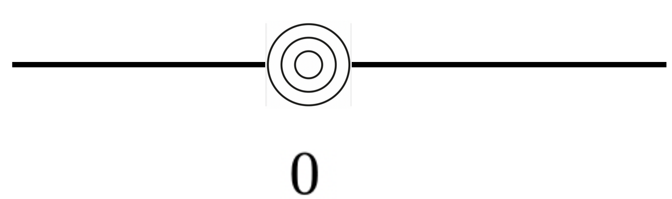
$$\exp(i\theta) \mapsto \begin{pmatrix} \exp(2i\theta) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \exp(-2i\theta) \end{pmatrix}$$



Weights diagram

Another direct sum of three irreducible representations of U(1)

$$\exp(i\theta) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



Weights diagram

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3. REPRESENTATIONS OF $\mathfrak{sl}(2, \mathbb{C})$
4. REPRESENTATIONS OF $\mathfrak{sl}(3, \mathbb{C})$
5. KOSTANT PARTITION FUNCTION AND VECTOR PARTITION FUNCTIONS.
6. LINEAR SYMMETRIES FOR THE SL(3,C) TRIPLE MULTIPLICITIES.

SU(2) The Special Unitary Group

Unitary: $$A = \begin{pmatrix} \alpha & -\beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} \qquad \alpha = \alpha_1 + i\alpha_2 \ \ \& \ \ \beta = \beta_1 + i\beta_2.$$

Special: $$\det A = \alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2 = 1$$

Lie Group:

— A 3-sphere.

Compact differentiable manifold.
Simply connected

— A non-abelian group under matrix multiplication.

SU(2) A 3-SPHERE INSIDE THE 4-DIMENSIONAL SPACE OF LINEAR COMBINATIONS

$$a\mathbf{1} + b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$$

UNIT QUATERNIONS

$$\mathbf{i} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{j} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad \mathbf{k} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

$$uv = -u \cdot v + u \times v.$$

LET su(2) BY THE REAL VECTOR SPACE GENERATED BY THE QUATERNIONS :

THUS su(2) IS THE SPACE OF ANTI-HERMITIAN MATRICES $X + X^* = 0$

THEOREM

su(2) IS THE TANGENT SPACE TO SU(2) AT THE IDENTITY.

su(2) IS A REAL 3-DIMENSIONAL VECTOR SPACE

THEOREM. THE SPACE su(2) IS THE TANGENT SPACE TO SU(2) AT THE IDENTITY.

ONE PARAMETER GROUP IN SU(2)

$$U(t) \text{ differentiable in } [-\epsilon, \epsilon] \text{ and } U(0) = 1$$

WITH $U(t)U^*(t) = 1$

THEN
$$\left.\frac{d}{dt}\right|_{t=0} (U(t)U^*(t)) =$$
$$(U'(t)U^*(t) + U(t)(U^*(t))') \,|_{t=0} =$$
$$X + X^* = 0$$
$$\text{where } X = U'(0)$$

Wanted: an operation on the Lie algebra that reflects the non-commutative group operation

su(2) is not closed under matrix multiplication:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{i}\,\mathbf{j}\,\mathbf{k} = -\mathbf{1}$$

Hamilton's group

Neither the sum nor the product of matrices in su(2) can reflect the group structure of SU(2)

The Lie Bracket

The action by conjugation of SU(2) on itself.

Two one-parameter groups in SU(2) $u(s) \;\; \& \;\; v(t)$

Set $U = u'(0) \;\; \& \;\; V = v'(0)$

Differentiating and evaluating at 0, twice we obtain the Lie bracket

$$\left.\frac{d}{ds}\right|_{s=0} \left.\frac{d}{dt}\right|_{t=0} \left(u(s)\; v(t) u(s)^{-1}\right) =$$

$$\left.\frac{d}{ds}\right|_{s=0} \left(u(s)\; v'(0) u(s)^{-1}\right) =$$

$$\left.\frac{d}{ds}\right|_{s=0} u'(s)\; v'(0) u(s)^{-1} - \left.\frac{d}{ds}\right|_{s=0} u(s)\; v'(0) u(s)^{-2} u'(s)$$

$$= UV - VU = [U, V]$$

From a Lie algebra to its Lie group

The exponential map

$$\mathfrak{g} \to G$$

$$X \mapsto e^X = \sum_{k \geq 0} \frac{X^k}{k!}$$

A Lie group homomorphism induces a unique real Lie algebra homomorphism satisfying

$$\Phi(e^X) = e^{\phi(X)}$$

Indeed, $\phi(X) = \frac{d}{dt}\Phi(e^{tX})\Big|_{t=0}$, *for all* $X \in \mathfrak{g}$.

INDEX

Examples of representation of the Lie group SU(2).

The trivial representation :
sends all elements of SU(2) to 1.

The standard representation V :
sends any element of SU(2) to itself.

A direct sum of copies these irreducible representations.

Ex:

$$\begin{pmatrix} \alpha & -\beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} \mapsto \begin{pmatrix} \alpha & -\beta & 0 & 0 \\ \overline{\beta} & \overline{\alpha} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

THE ADJOINT REPRESENTATION OF SU(2)

THE LIE GROUP SU(2) ACTS ON ITS LIE ALGEBRA su(2) BY CONJUGATION:

SU(2) —> GL(su(2))

$$A \mapsto Ad_A : X \mapsto AXA^{-1}$$

su(2) is a real vector 3- space.

$$\boxed{\det A = \alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2 = 1}$$

THEOREM

IF $A = \cos\theta + U \sin\theta$ IS A UNIT VECTOR IN SU(2), THEN CONJUGATION BY A DEFINES A ROTATION IN su(2)

AROUND AXIS U AND OF ANGLE OF 2θ

IT IS A 2-1 MAP BECAUSE A AND -A INDUCE THE SAME ROTATION.

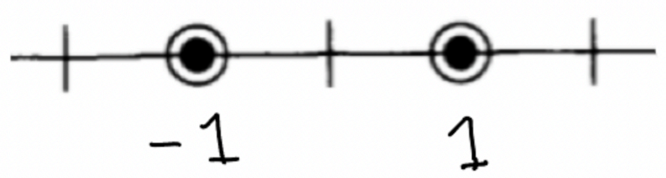
TO UNDERSTAND A LIE GROUP REPRESENTATION OF SU(2)
WE FIRST ANALYZE ITS RESTRICTION TO THE TORUS (DIAGONAL MATRICES)

THE RESTRICTION TO THE TORUS OF THE STANDARD LIE GROUP REPRESENTATION

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

WEIGHTS: (1) AND (-1)

WEIGHT DIAGRAM:



IN THE LANGUAGE OF SYMMETRIC FUNCTIONS

$$tr \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} = e^{i\theta} + e^{-i\theta} = h_1(e^{i\theta}, e^{-i\theta})$$

RESTRICTION TO THE TORUS OF THE SYMMETRIC SQUARE OF THE STANDARD REPRESENTATION

DIAGONAL ACTION

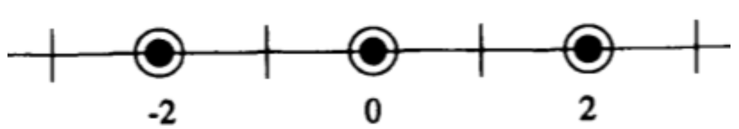
$$v_1 \odot v_1 \mapsto e^{i\theta} v_1 \odot e^{i\theta} v_1 = e^{2i\theta} v_1 \odot v_1$$

$$v_1 \odot v_2 \mapsto e^{i\theta} v_1 \odot e^{-i\theta} v_2 = e^{0i\theta} v_1 \odot v_2$$

$$v_2 \odot v_2 \mapsto e^{-i\theta} v_1 \odot e^{-i\theta} v_1 = e^{-2i\theta} v_1 \odot v_1$$

WEIGHTS: (2), (0) AND (-2)

WEIGHT DIAGRAM:



CHARACTER

$$tr \begin{pmatrix} e^{2i\theta} & 0 & 0 \\ 0 & e^{0i\theta} & 0 \\ 0 & 0 & e^{-2i\theta} \end{pmatrix} t = e^{2i\theta} + e^{0i\theta} + e^{-2i\theta} = h_2(e^{i\theta}, e^{-i\theta})$$

FROM THE REPRESENTATIONS OF A LIE GROUP TO THE REPRESENTATIONS OF ITS LIE ALGEBRA

LIE GROUP REPRESENTATION $\Pi : G \to GL(V)$

LIE ALGEBRA REPRESENTATION. $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$
(LINEAR MAP THAT RESPECT THE BRACKET)

THEOREM
A LIE GROUP REPRESENTATION DEFINES A UNIQUE LIE ALGEBRA HOMOMORPHISM

$$\Pi(e^X) = e^{\pi(X)}$$

THAT CAN BE COMPUTED AS

$$\pi(X) = \frac{d}{dt}\Pi(e^{tX})\bigg|_{t=0}$$

IF G IS CONNECTED IT IS EQUIVALENT TO ASK ABOUT IRREDUCIBILITY AND EQUIVALENCE IN EITHER SETTING, IF G IS SIMPLY CONNECTED A LIE ALGEBRA REPRESENTATION CAN BE LIFTED TO A LIE GROUP REPRESENTATION.

REPRESENTATIONS OF THE LIE ALGEBRA su(2) (real Lie algebra).

MOVE TO sl(2, C) (FUNDAMENTAL THEOREM OF ALGEBRA).

sl(2, C) SPACE OF MATRICES OF TRACE ZERO.
COMPLEX VECTOR SPACE OF DIMENSION 3

BASIS FOR sl(2, C)

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

THE IMAGE OF Hθ, THE CARTAN SUB ALGEBRA OF su(2), UNDER THE EXPONENTIAL MAP IS THE TORUS OF su(2)

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

THE ADJOINT REPRESENTATION OF sl(2, C).

THE ACTION OF CONJUGATION OF A LIE GROUP TRANSLATES TO THE ADJOINT REPRESENTATION OF ITS LIE ALGEBRA:

$$Ad : sl(2, C) \longrightarrow GL(sl(2, C))$$

$$Ad(X) = [X, \ \ ]$$

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

THEN

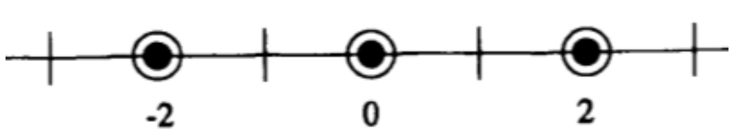
$$Ad(H)(H) = [H, H] = 0$$

$$Ad(H)(X) = [H, X] = 2X$$

$$Ad(H)(Y) = [H, Y] = -2Y$$

THE ROOTS X AND Y

THE NONZERO EIGENVECTORS OF THE ADJOINT REPRESENTATION



WEIGHTS -2, 0, 2.

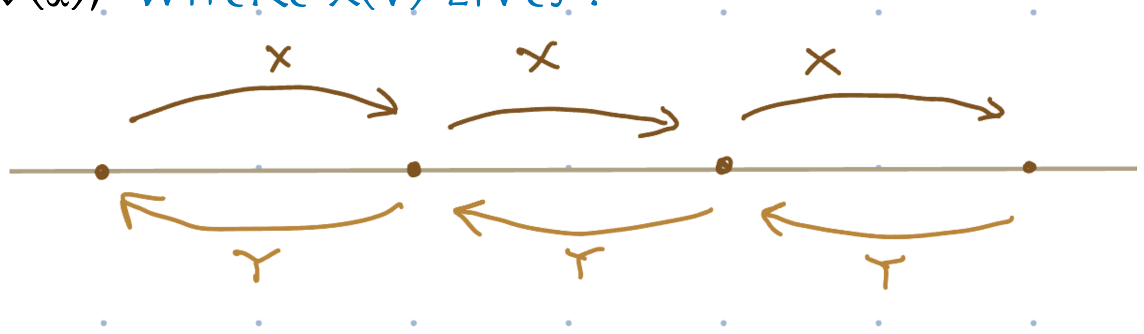
CHARACTER

$$tr \begin{pmatrix} e^{2i\theta} & 0 & 0 \\ 0 & e^{0i\theta} & 0 \\ 0 & 0 & e^{-2i\theta} \end{pmatrix} t = e^{2i\theta} + e^{0i\theta} + e^{-2i\theta} = h_2(e^{i\theta}, e^{-i\theta})$$

THE FUNDAMENTAL CALCULATION FOR su(2).

THE ACTION OF X AND Y ON THE WEIGHT SPACES.

LET V BE ANY REPRESENTATION OF su(2). GIVEN $\mathbf{v}$ IN $V(\alpha)$, WHERE $X(V)$ LIVES ?



SINCE $[H,X] = HX - XH$.

$$H(X(\mathbf{v})) = X(H(\mathbf{v})) + [H, X](\mathbf{v})$$
$$= X(\alpha\, \mathbf{v}) + 2\, X(\mathbf{v})$$
$$= (\alpha + 2)\, X(\mathbf{v})$$

SIMILARLY,

$$H(Y(\mathbf{v})) = (\alpha - 2)\, Y(\mathbf{v}).$$

THE IRREDUCIBLE REPRESENTATIONS OF su(2)

- ANY IRREDUCIBLE REPRESENTATION OF su(2) IS ISOMORPHIC TO A SYMMETRIC POWER OF THE STANDARD REPRESENTATION

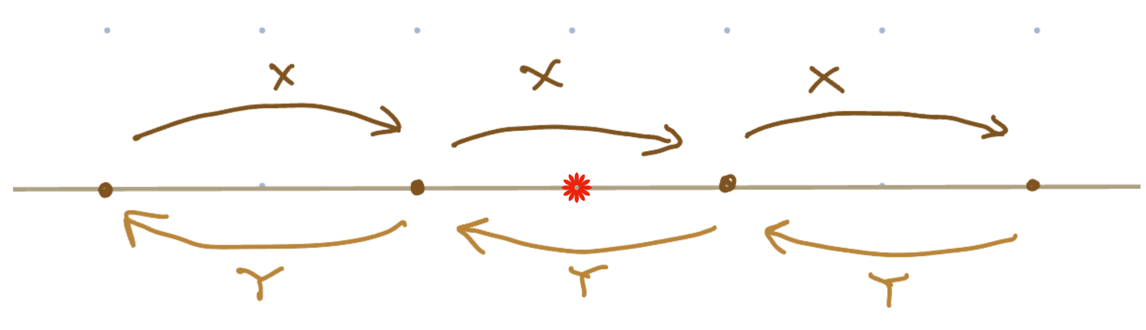
$$Sym^k V$$

WEIGHT SPACES ARE ONE DIMENSIONAL

FOR SOME K NON-NEGATIVE.

- SYMMETRIES OF THE WEIGHT SPACES

CENTRAL SYMMETRY



- Decomposing a representation into irreducibles

$$Sym^k V \otimes Sym^l V$$

(done in the blackboard)

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Representations of sl(3, C)

The Cartan sub-algebra of sl(3, C)

Diagonal 3x3 matrices of trace zero.

$$\mathfrak{h} = \left\{ \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} : a_1 + a_2 + a_3 = 0 \right\}$$

Dual

$$\mathfrak{h}^* = \mathbb{C}\{L_1, L_2, L_3\}/(L_1 + L_2 + L_3 = 0)\},$$

$$L_i \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} = a_i.$$

Thm: Any complex finite dimensional representation of sl(3, C) can be decomposed as a finite sum of weight spaces

$$V = \bigoplus V_\alpha$$

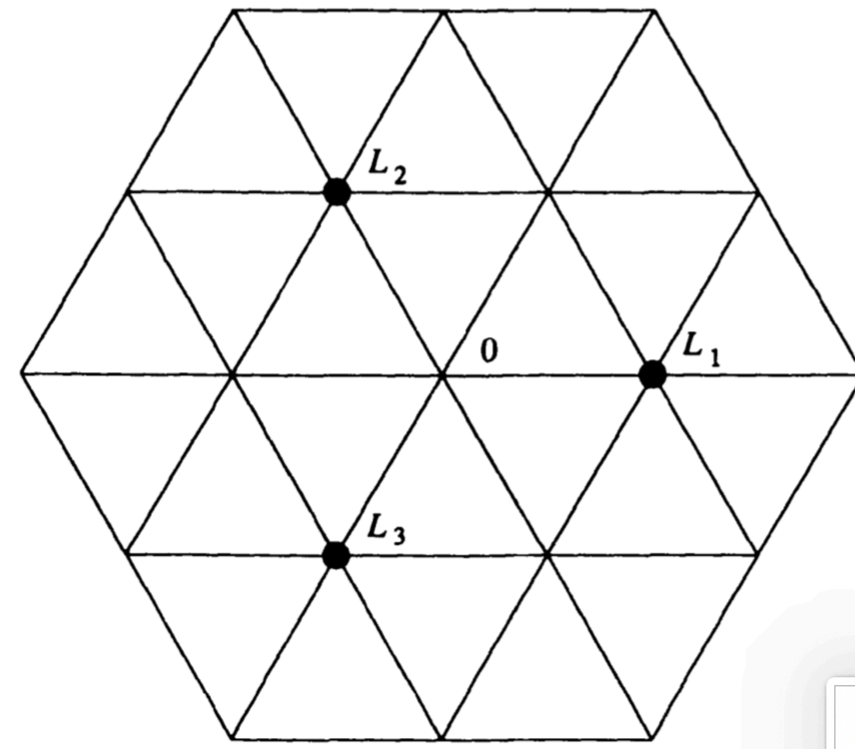
The sum is taken over a finite subset of $\mathfrak{h}^*$

In particular, the adjoint representation can be decomposed as

$$\mathfrak{sl}_3\mathbb{C} = \mathfrak{h} \oplus \left(\bigoplus \mathfrak{g}_\alpha\right),$$

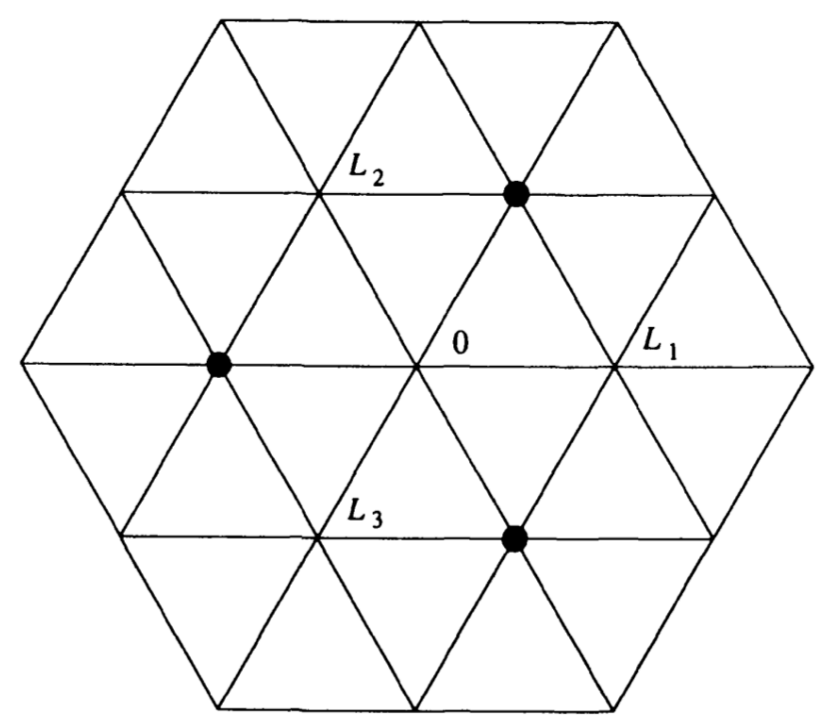
The Standard Representation

$$\begin{pmatrix} \exp i\theta_1 & 0 & 0 \\ 0 & \exp i\theta_2 & 0 \\ 0 & 0 & \exp -i(\theta_1 + \theta_2) \end{pmatrix}$$



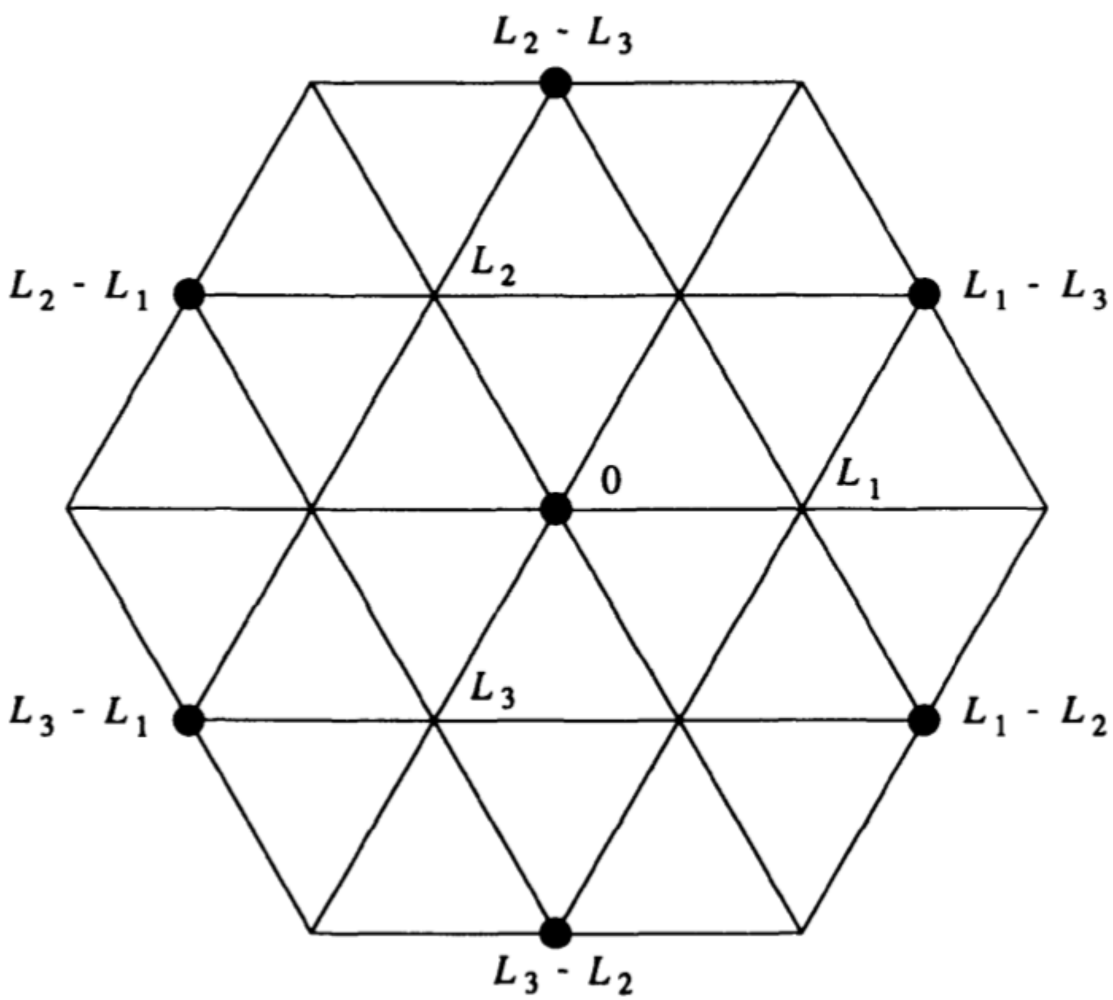
The Dual of The Standard Representation

$$X \mapsto \pi(X)$$
$$X^* \mapsto -\pi(X)^t$$



the fundamental calculation for $\mathfrak{sl}(3, \mathbb{C})$.

$$
\begin{aligned}
[H, [X, Y]] &= [X, [H, Y]] + [[H, X], Y] \\
&= [X, \beta(H) \cdot Y] + [\alpha(H) \cdot X, Y] \\
&= (\alpha(H) + \beta(H)) \cdot [X, Y].
\end{aligned}
$$



$$\mathrm{ad}(\mathfrak{g}_\alpha): \mathfrak{g}_\beta \to \mathfrak{g}_{\alpha+\beta}$$

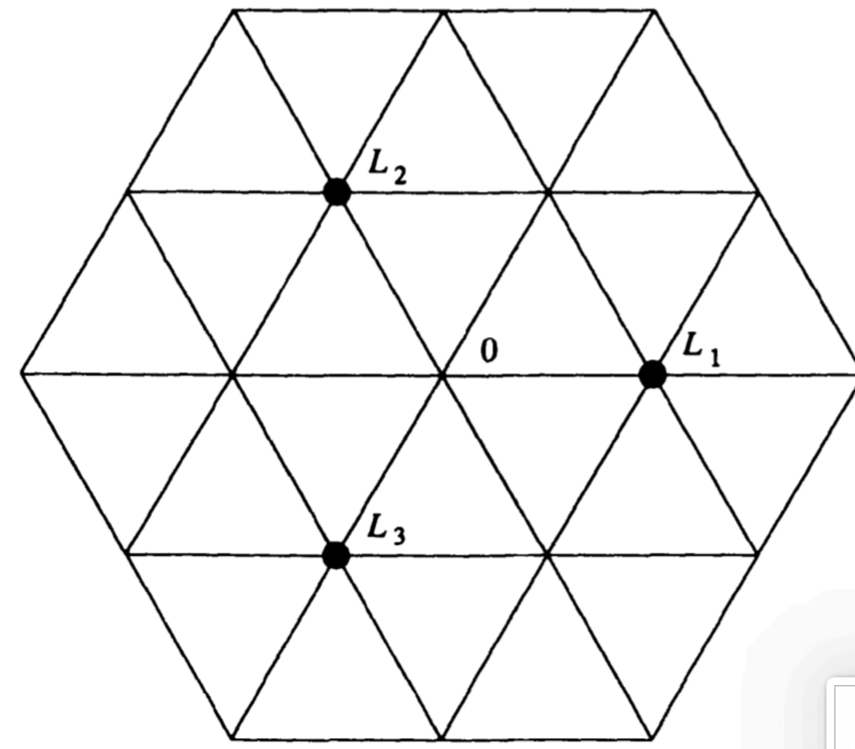
the directions of the three long diagonals of the rhombi

SECOND DAY

ALCÁZAR DE SEVILLA

The Standard Representation

$$\begin{pmatrix} \exp i\theta_1 & 0 & 0 \\ 0 & \exp i\theta_2 & 0 \\ 0 & 0 & \exp -i(\theta_1+\theta_2) \end{pmatrix}$$



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$$X \mapsto \pi(X)$$
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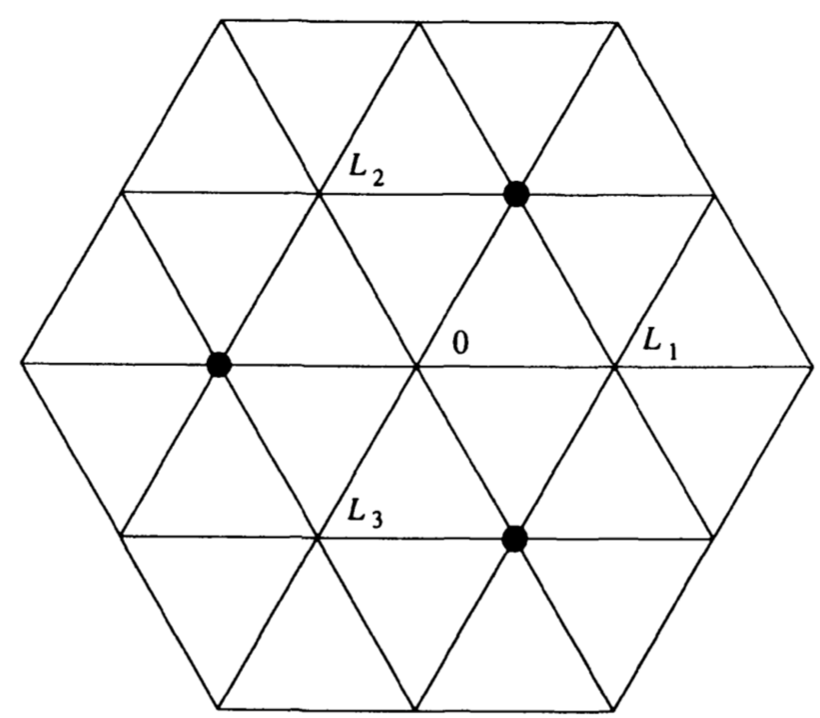
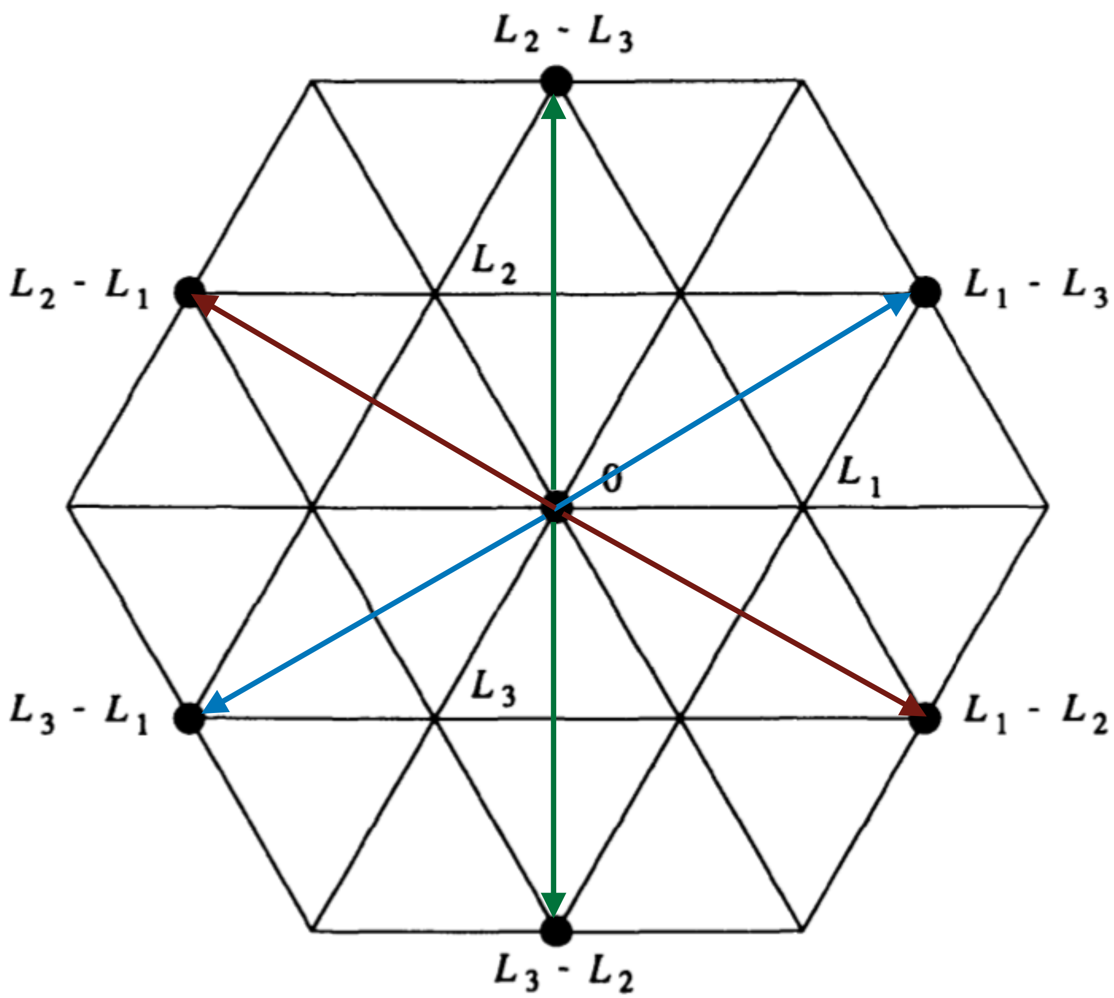
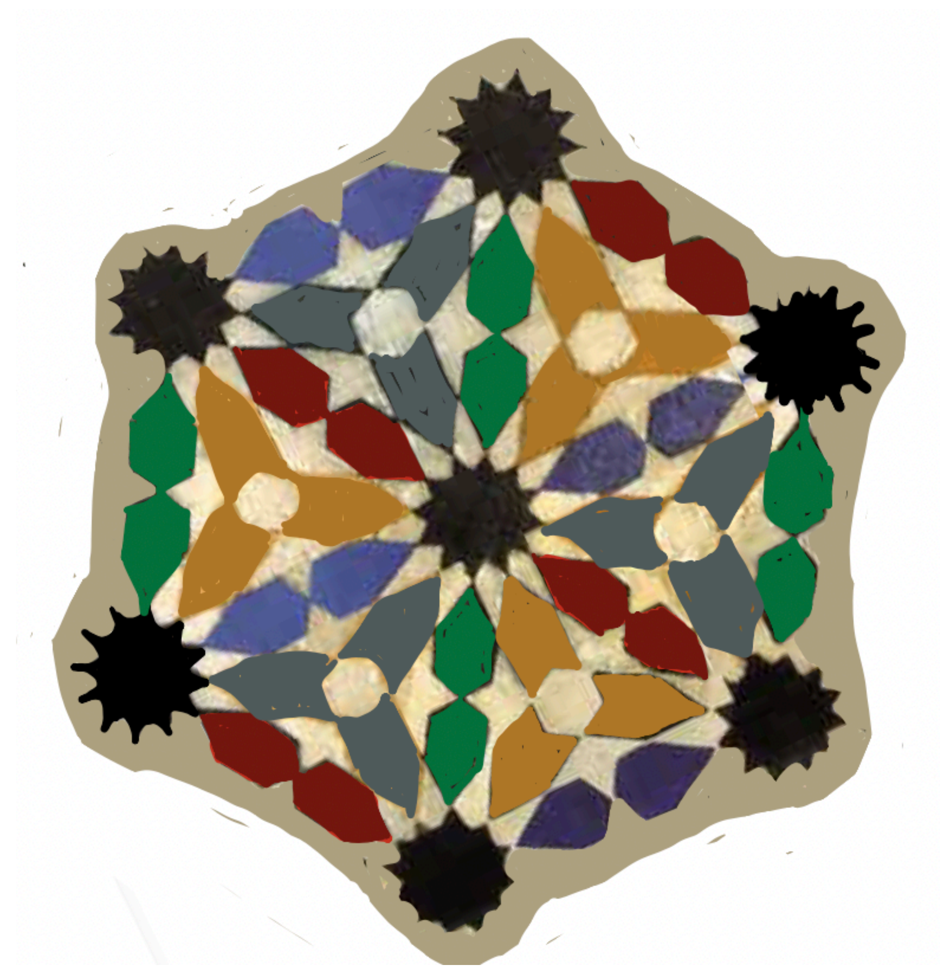


The adjoint representation of $sl(3, \mathbb{C})$.

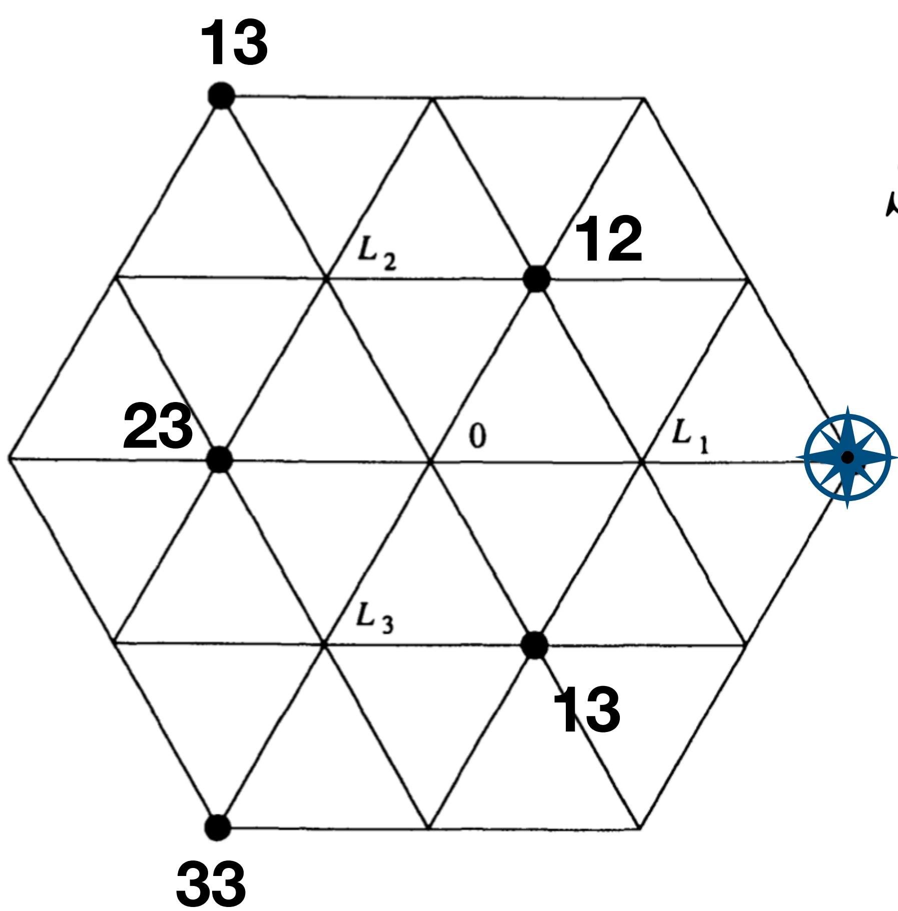


The roots allows us to
move in the directions of the
three long diagonals
of the rhombi

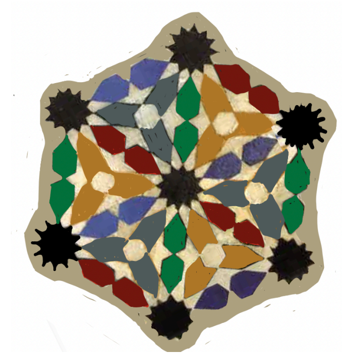


Alcázar de Sevilla

Symmetric powers of the standard representation

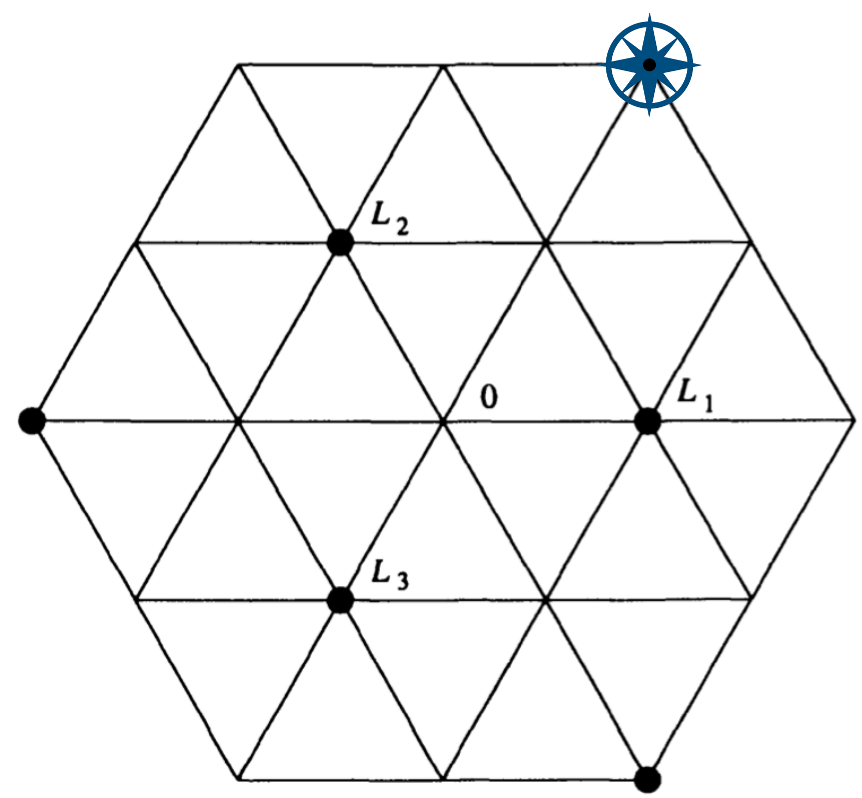


Sym^2V



Maximal weights

Sym^2V^*

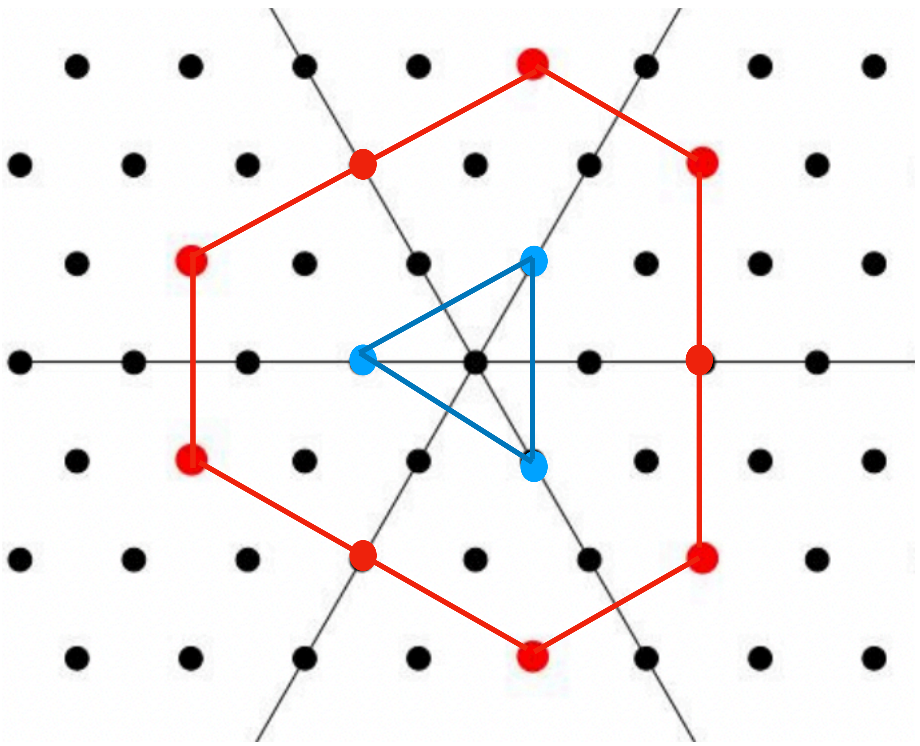


Long diagonals
of the rhombi

THE IRREDUCIBLE REPRESENTATION OF $\mathfrak{sl}(3, \mathbb{C})$.

$$\mathrm{Sym}^n V = \Gamma_{n,0} \quad \text{and} \quad \mathrm{Sym}^n V^* = \Gamma_{0,n}.$$

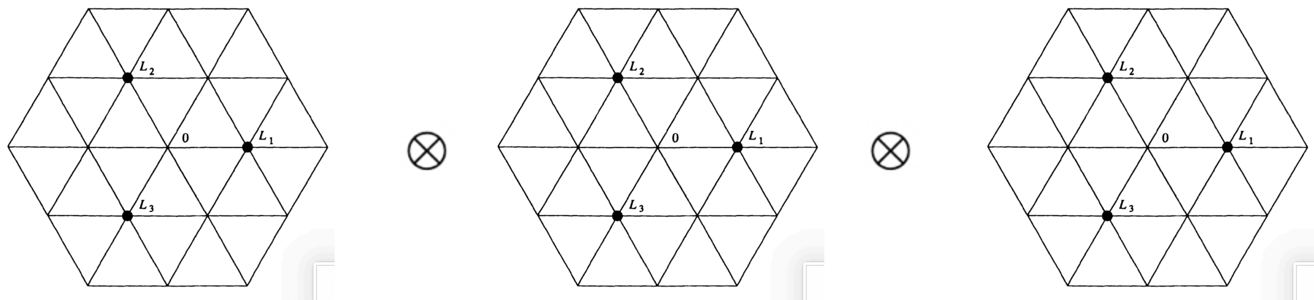
TRIANGLES



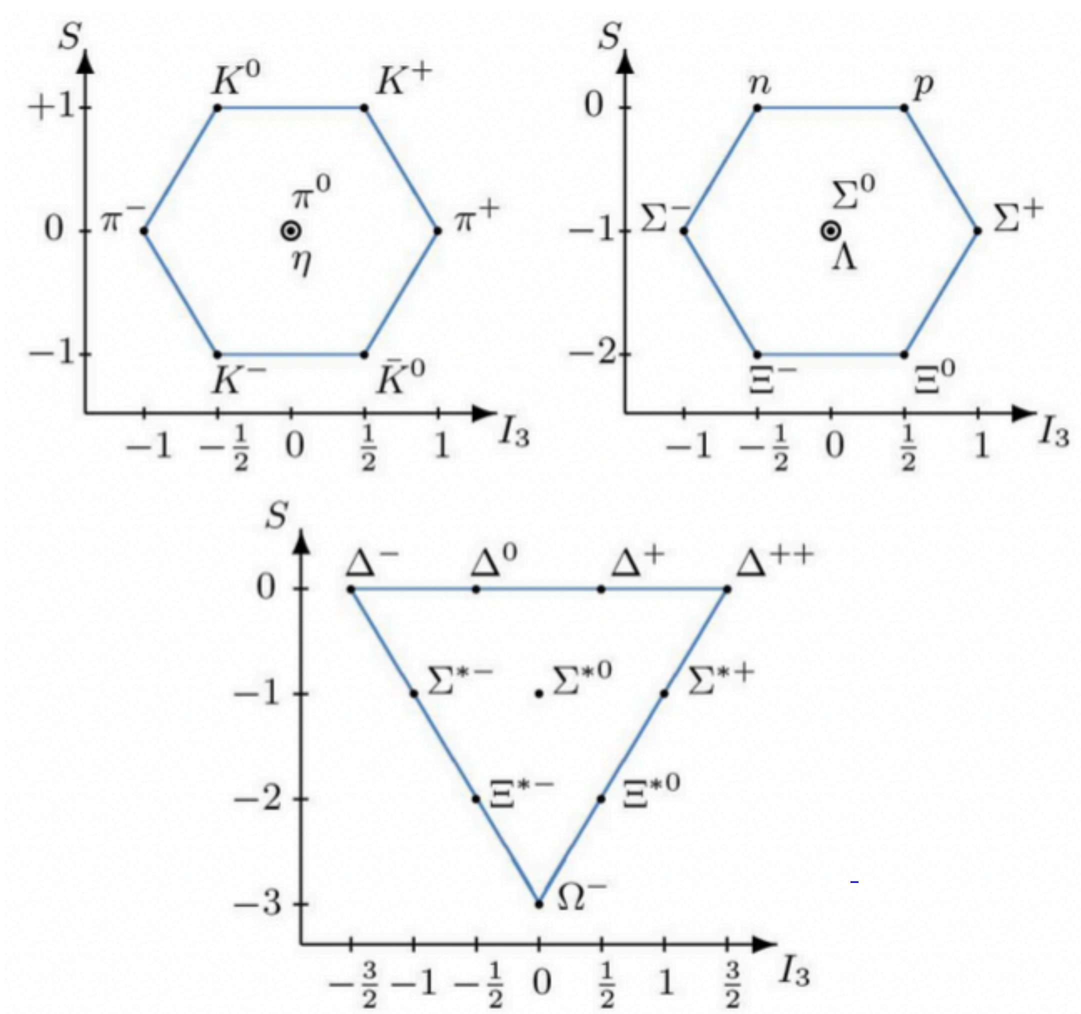
$\Gamma_{(1,2)}$

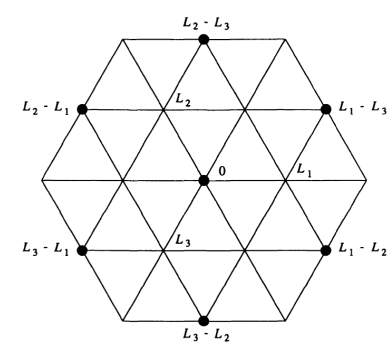
OUTER SHAPE
HEXAGON

Gell-Mann and Ne'eman Eight-fold Way

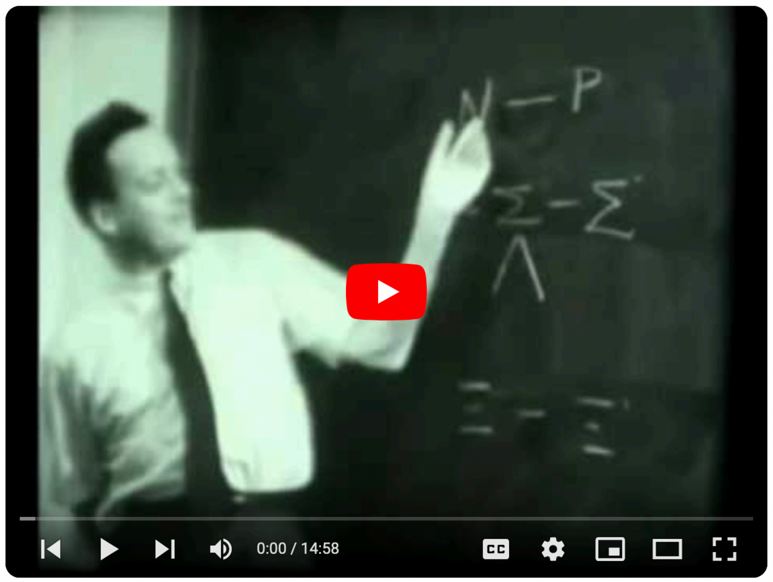


$$\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$$





$$Sym^3 V$$



Richard Feynman, Murray Gell-Mann, Juval Ne'eman: Strangeness Minus Three (BBC Horizon 1964) I

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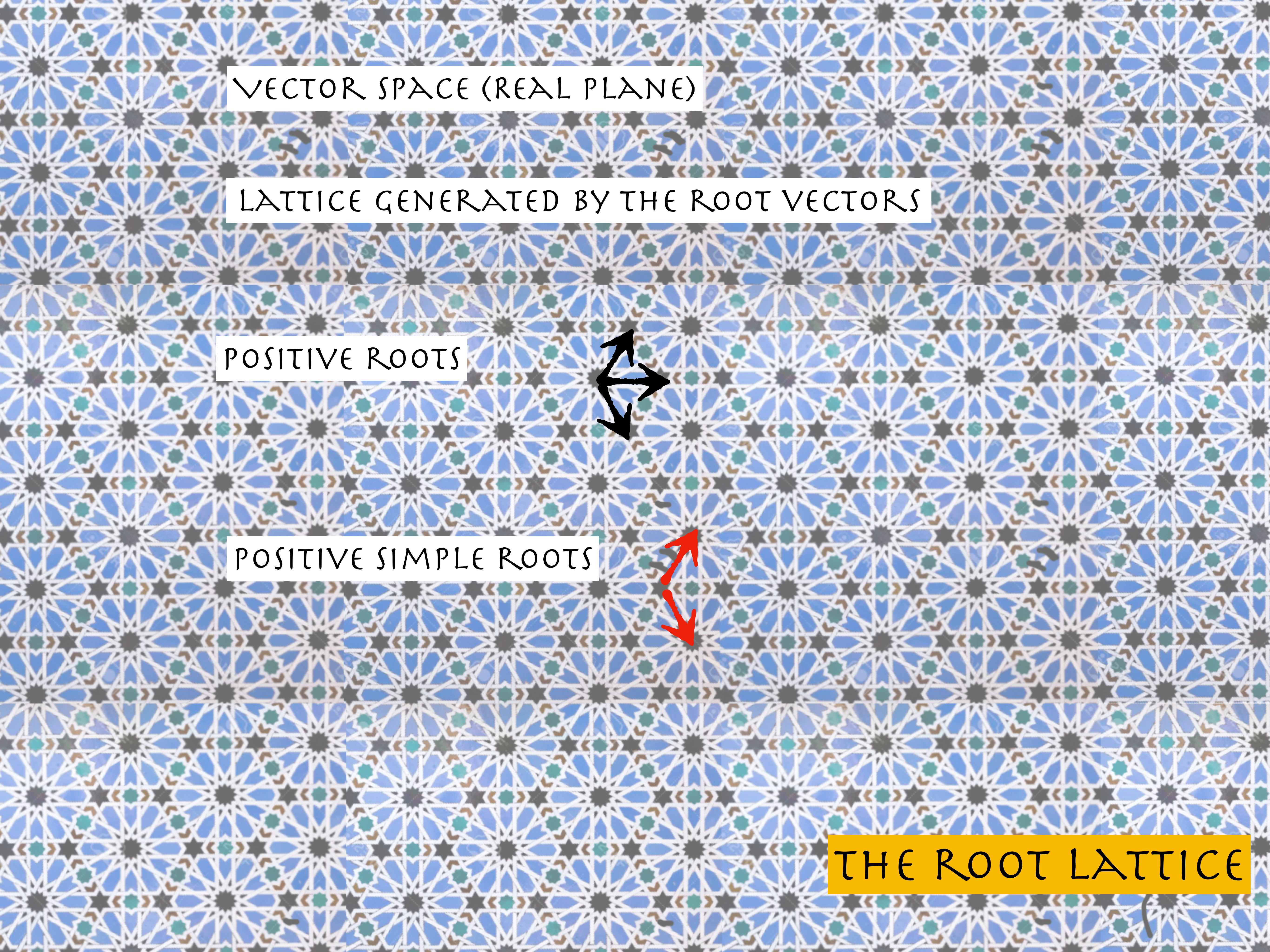
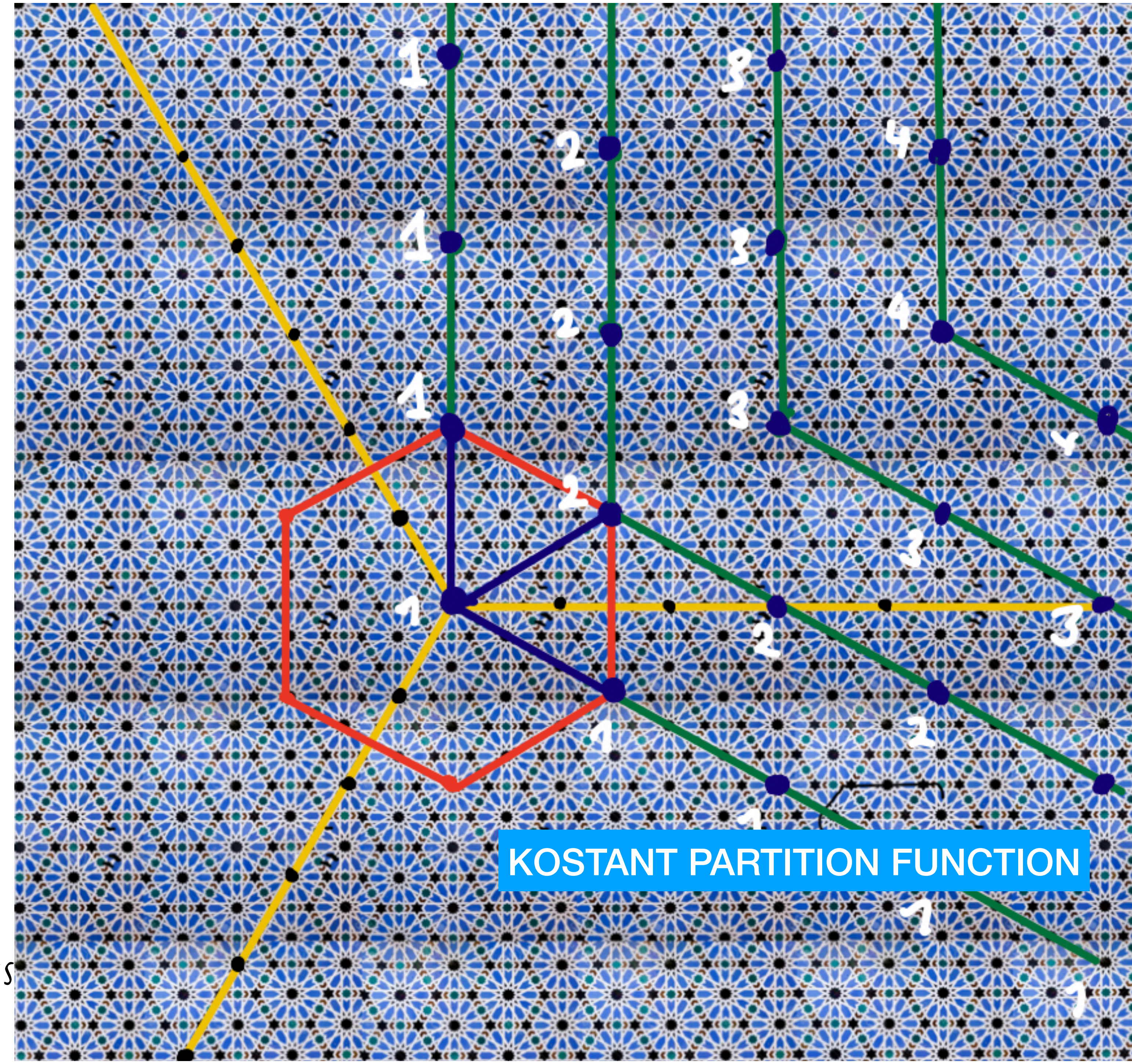

Vector space (real plane)
Lattice generated by the root vectors
Positive roots
Positive simple roots
The Root Lattice

KONSTANT PARTITION FUNCTION

POSITIVE ROOTS

$\boxed{\alpha_1, \alpha_2}$ $\boxed{\alpha_3 = \alpha_1 + \alpha_2}$



$P(\mu)$ = THE NUMBER OF WAYS OF WRITING μ AS A SUM OF POSITIVE ROOT

$$\mu = n_1\alpha_1 + n_2\alpha_2$$

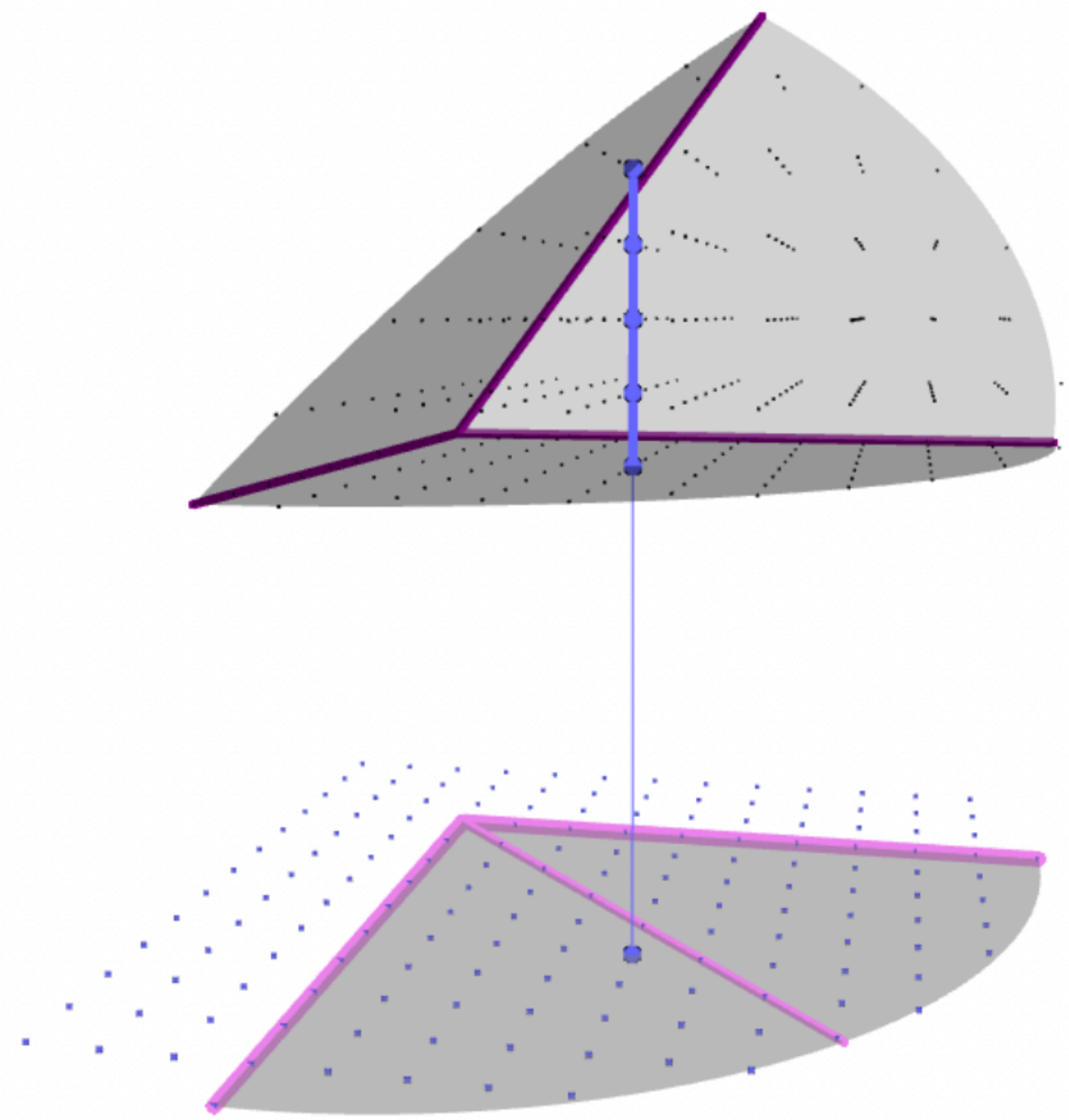
POINTED CONE :

NON-NEGATIVE REAL COEFFICIENTS.

LATTICE CONDITION

NON-NEGATIVE INTEGER COEFFICIENTS

$$p(n_1\alpha_1 + n_2\alpha_2) = 1 + \min(n_1, n_2).$$



KONSTANT PARTITION FUNCTION

FIX A SET OF POSITIVE ROOTS

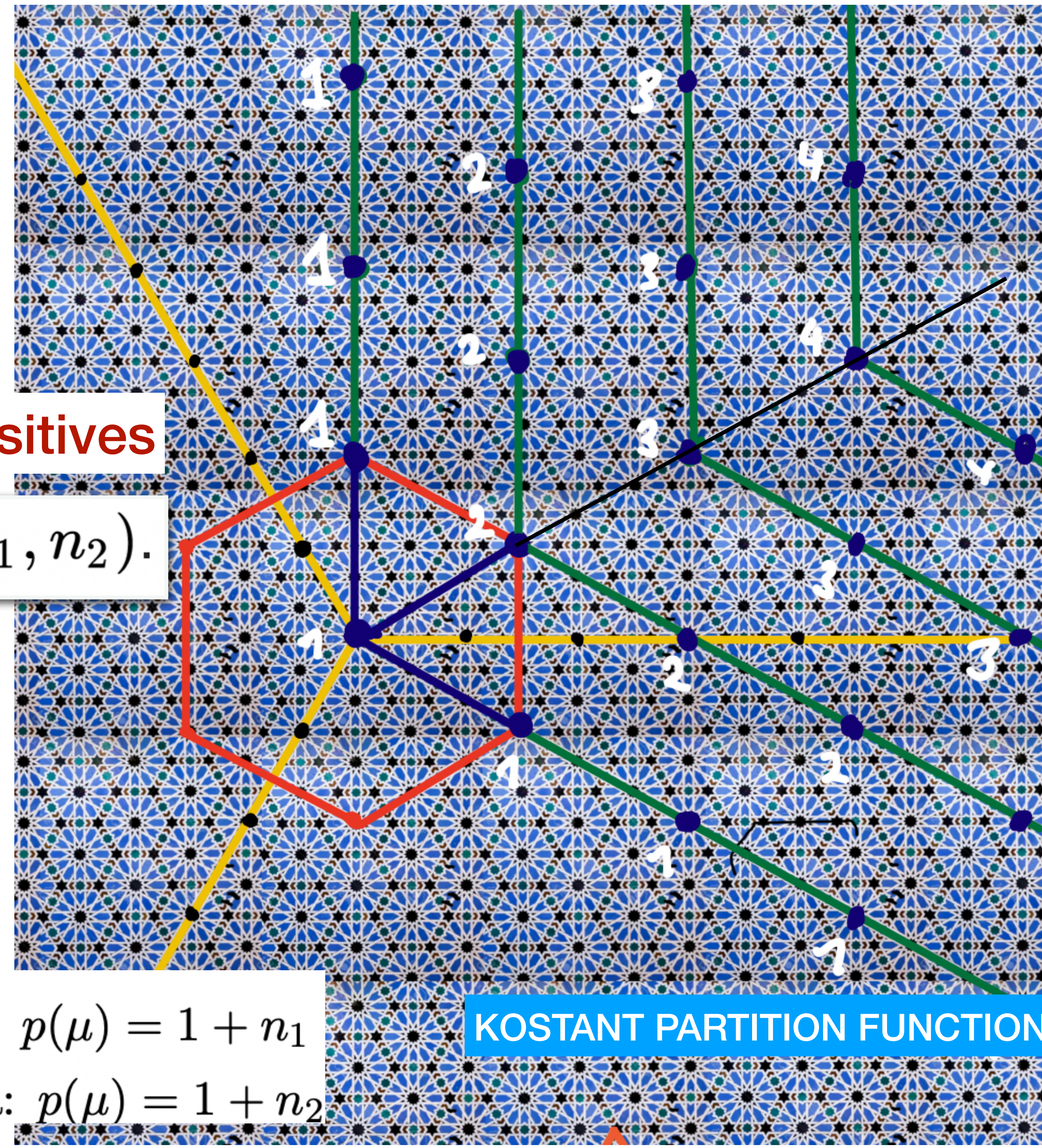
α_1, α_2 $\alpha_3 = \alpha_1 + \alpha_2$

WRITE

$$\mu = n_1\alpha_1 + n_2\alpha_2$$

THEN

$$p(n_1\alpha_1 + n_2\alpha_2) = 1 + \min(n_1, n_2).$$



POINTED CONE :

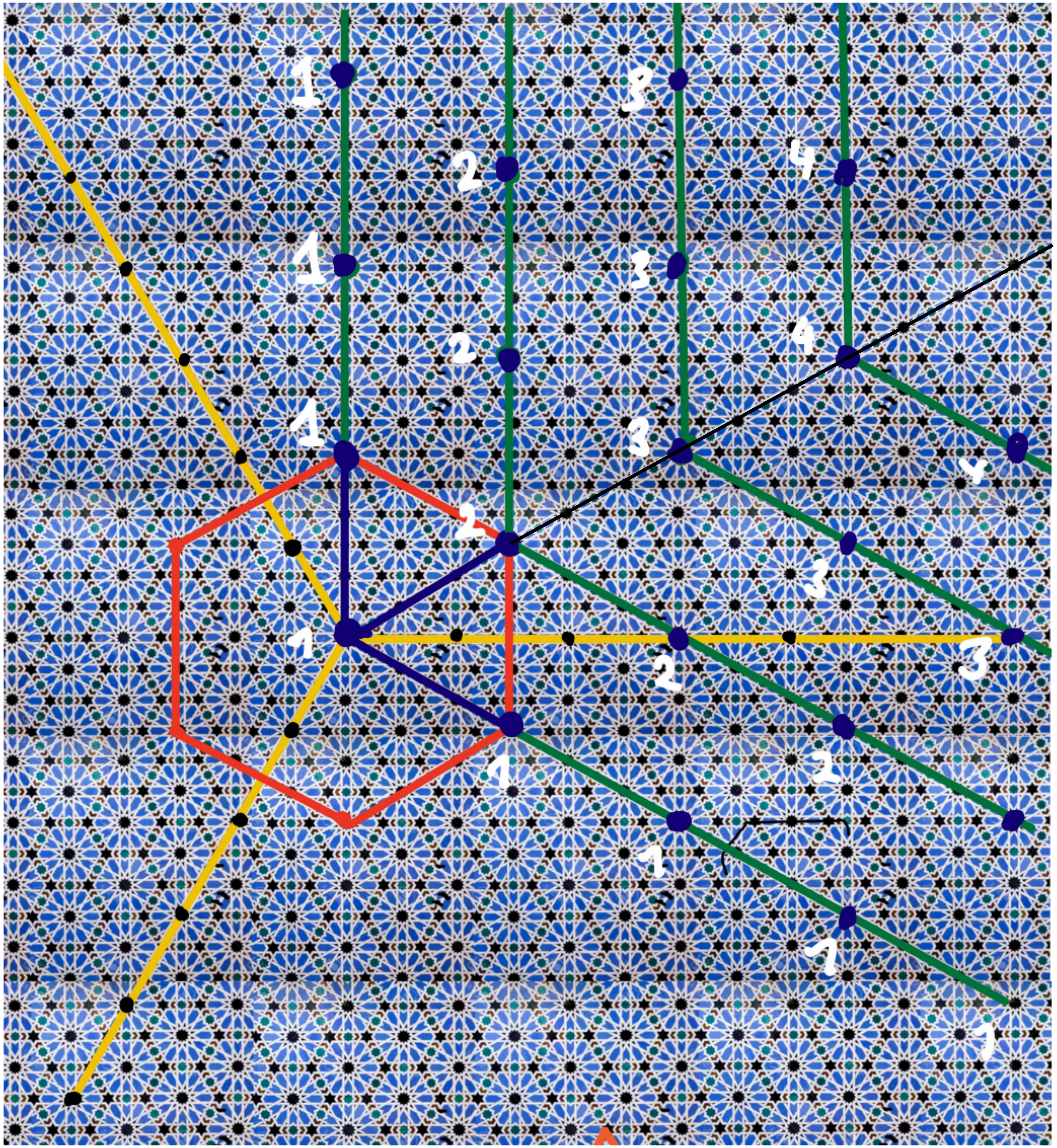
NON-NEGATIVE REAL COEFFICIENTS.

VECTOR PARTITION FUNCTION:

chamber 1: $0 \leq n_1 \leq n_2$, formula: $p(\mu) = 1 + n_1$

chamber 2: $0 \leq n_2 \leq n_1$, formula: $p(\mu) = 1 + n_2$

Symmetry and Stability



Symmetry

Reflection around the line generated by

$$\alpha_3 = \alpha_1 + \alpha_2$$

Cyclic group of order two

Stability

Kostant multiplicity formula

$$\mathrm{mult}(\mu) = \sum_{w \in W} (-1)^{\ell(w)} p(w \cdot (\lambda + \rho) - (\mu + \rho)).$$

It is a daunting task to deal with this signed sum.

INDEX

1. THE LIE GROUP $U(1)$ AND THE NOTION OF WEIGHT. ✔
2. THE LIE GROUP $SU(2)$ AND ITS LIE ALGEBRA ✔
3. REPRESENTATIONS OF $\mathfrak{sl}(2, \mathbb{C})$ ✔
4. REPRESENTATIONS OF $\mathfrak{sl}(3, \mathbb{C})$ ✔
5. KOSTANT PARTITION FUNCTION AND VECTOR PARTITION FUNCTIONS. ✔
6. LINEAR SYMMETRIES FOR THE $SL(3,\mathbb{C})$ TRIPLE MULTIPLICITIES.

SET $\ell = (\ell_1, \ell_2)$, $m = (m_1, m_2)$ and $n = (n_1, n_2)$.

THE LITTLEWOOD-RICHARDSON COEFFICIENTS

THE MULTIPLICITY OF V_n IN THE TENSOR PRODUCT

$$V_\ell \otimes V_m$$

IRREP OF $\mathfrak{sl}(3, \mathbb{C})$

SCHUR FUNCTIONS

$$[s_\lambda]\ s_\mu s_\nu$$

THE TRIPLE MULTIPLICITIES

DIMENSION OF $(V_\ell \otimes V_m \otimes V_n^*)^{SU(3)}$

THE TRIPLE MULTIPLICITIES

$$c(\ell; m; n) = \dim \left(V_\ell \otimes V_m \otimes V_n\right)^{SU(3)}$$

THUS $c^{\lambda}_{\mu.\nu}$ EQUALS $c(\ell; m; n^*)$

THE SUPPORT OF THE TRIPLE MULTIPLICITIES

SET OF DYNKIN LABELS WITH $c(\ell; m; n) \neq 0$

GENERATES A SUB LATTICE Λ_{TM} OF $\mathbb{Z}^6$

$$\ell_1 + m_1 + n_1 \equiv \ell_2 + m_2 + n_2 \mod 3.$$

Λ_{TM}

A linear symmetry for the triple multiplicities

Linear automorphism of Λ_{TM}

$$c(\theta(\ell, m, n)) = c(\ell; m; n)$$

Permutations of the Dynkin labels

$$c(\ell; m; n) = \dim\left(V_\ell \otimes V_m \otimes V_n\right)^{SU(3)}$$

Duality symmetry

$$(\ell, m, n) \leftrightarrow (\ell^*, m^*, n^*)$$

Group of symmetries of the triple multiplicities $SU(k)$

$\mathfrak{S}_2 \times \mathfrak{S}_3$ order 12 $k \geq 3$

Littlewood-Richardson coefficients
o rather triple multiplicities

Berenstein & Zelevinsky, 1991

Triple Multiplicities for $s\ell(r+1)$ and the Spectrum of the Exterior Algebra of the Adjoint Representation

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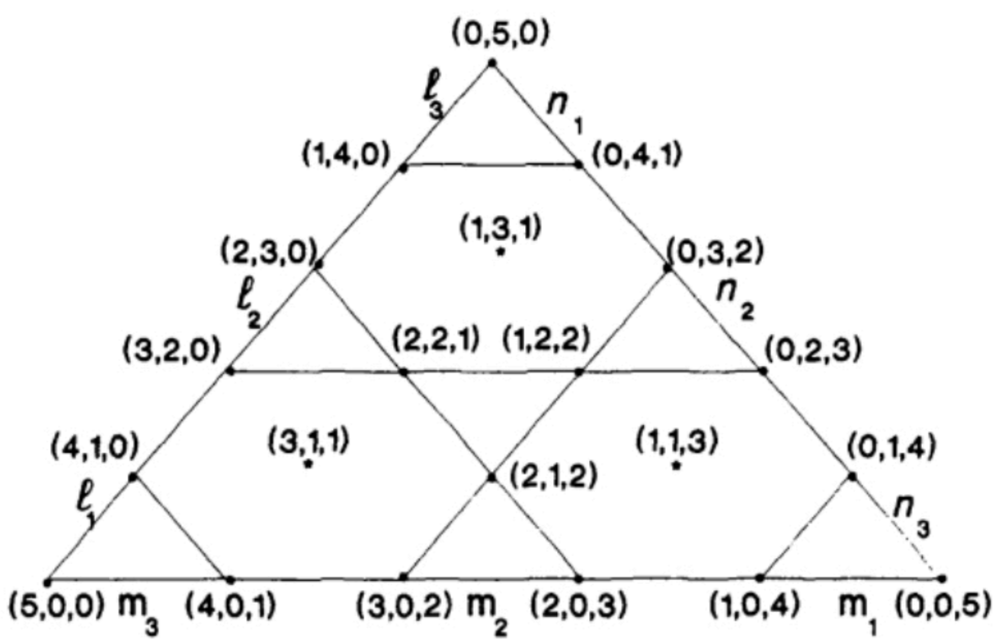
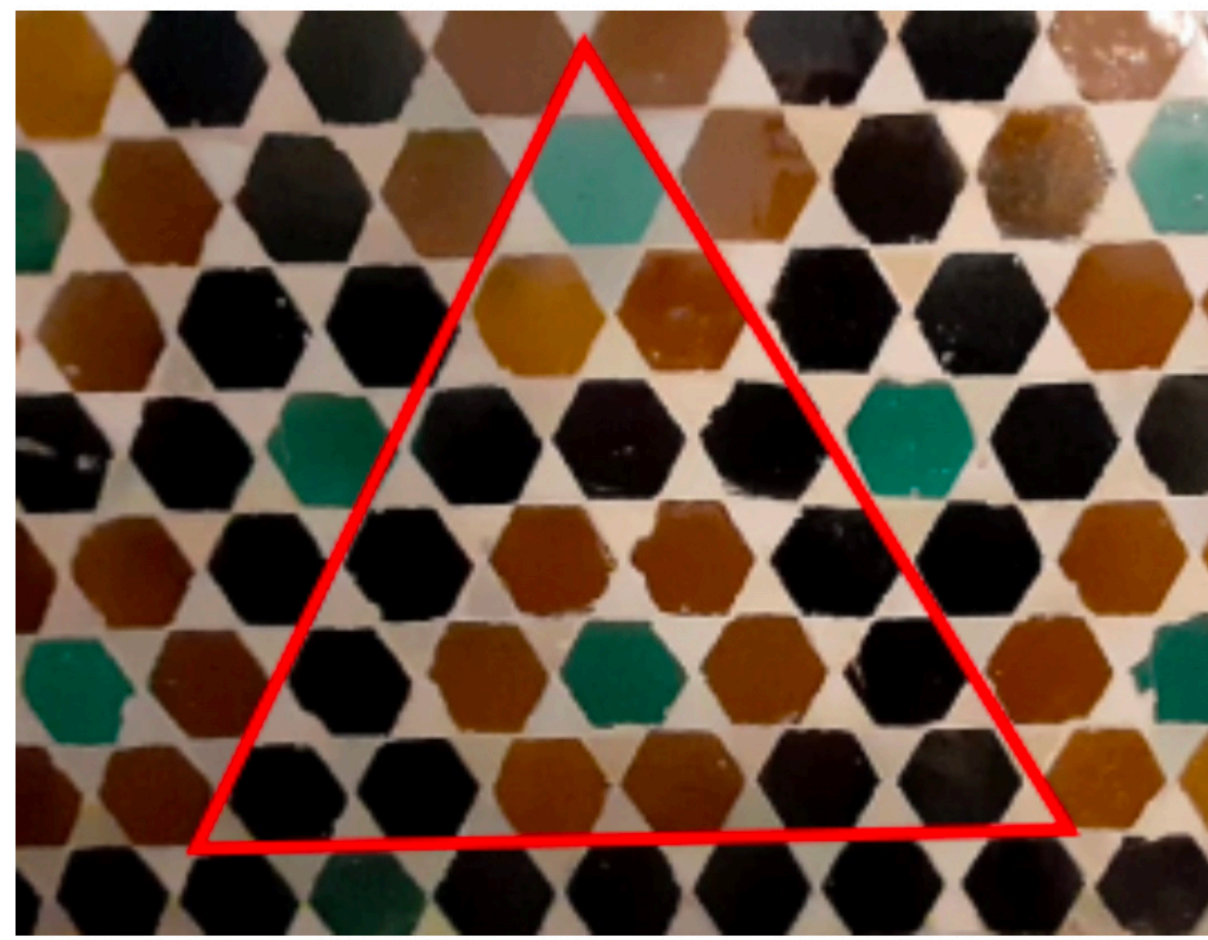
Department of Mathematics, Northeastern University, Boston, MA 02115.

Received May 23, 1991, Revised October 10, 1991



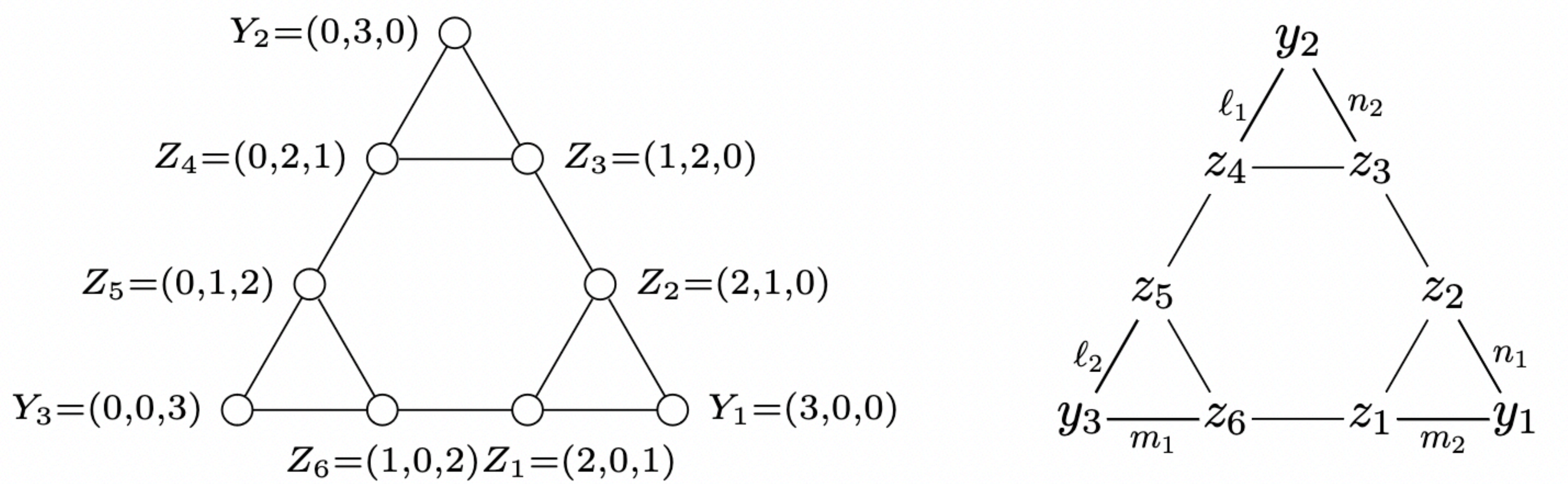
We fix a natural number r and put $T = T_r = \{(i,j,k) \in \mathbf{Z}_+^3 : i+j+k = 2r-1\}$. Put also $H = H_r = \{(i,j,k) \in T_r$: all i,j,k are odd$\}$ and $G = G_r = T_r - H_r$. Thus T_r is the set of vertices of a regular triangular lattice filling the regular triangle with vertices $(2r-1,0,0)$, $(0,2r-1,0)$, and $(0,0,2r-1)$; this triangle is decomposed into the union of elementary triangles having all three vertices in G_r and of elementary hexagons centered at points of H_r (see Figure 1).

Alcázar de Sevilla, 1090



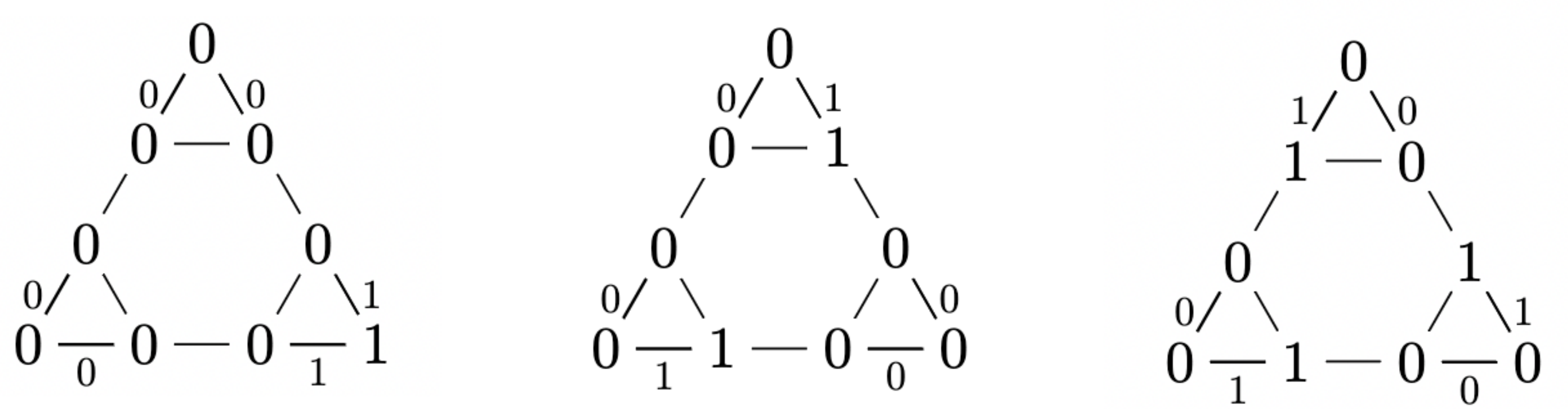
Join work with Emmanuel Briand and Stefan Trandafir

Berenstein-Zelevinski triangles



Sides of the hexagon sum as much as the opposite sides

$$z_1 - z_4 = z_5 - z_2 = z_3 - z_6.$$



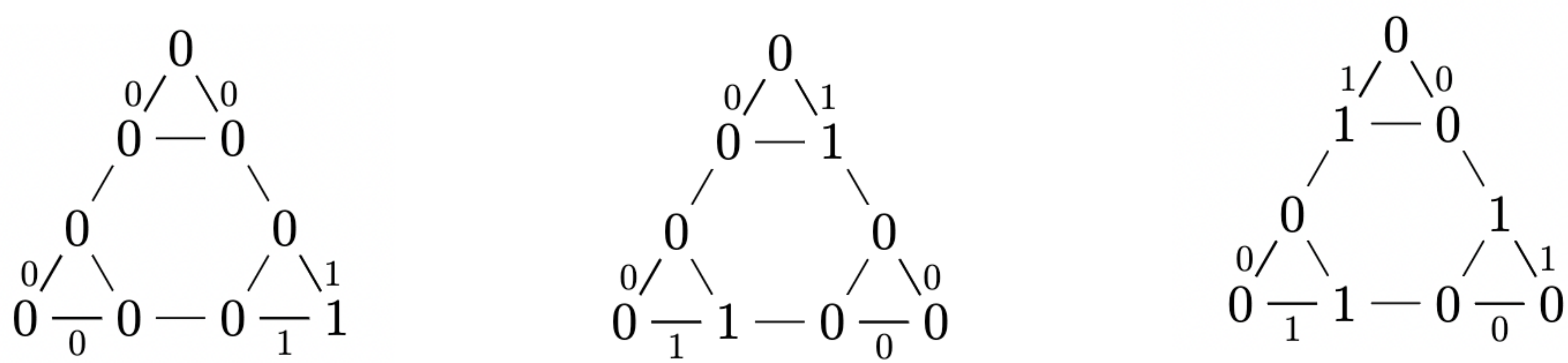
VECTOR SPACE $\mathcal{L}_{\mathrm{BZ}}$ REAL LABELLING OF BZ TRIANGLES

BZ CONE

THE CONE OF ALL POINTS WITH
NON-NEGATIVE LABELINGS

BZ TRIANGLE

AN ELEMENT OF
LATTICE OF INTEGRAL POINTS
IN THE BZ CONE

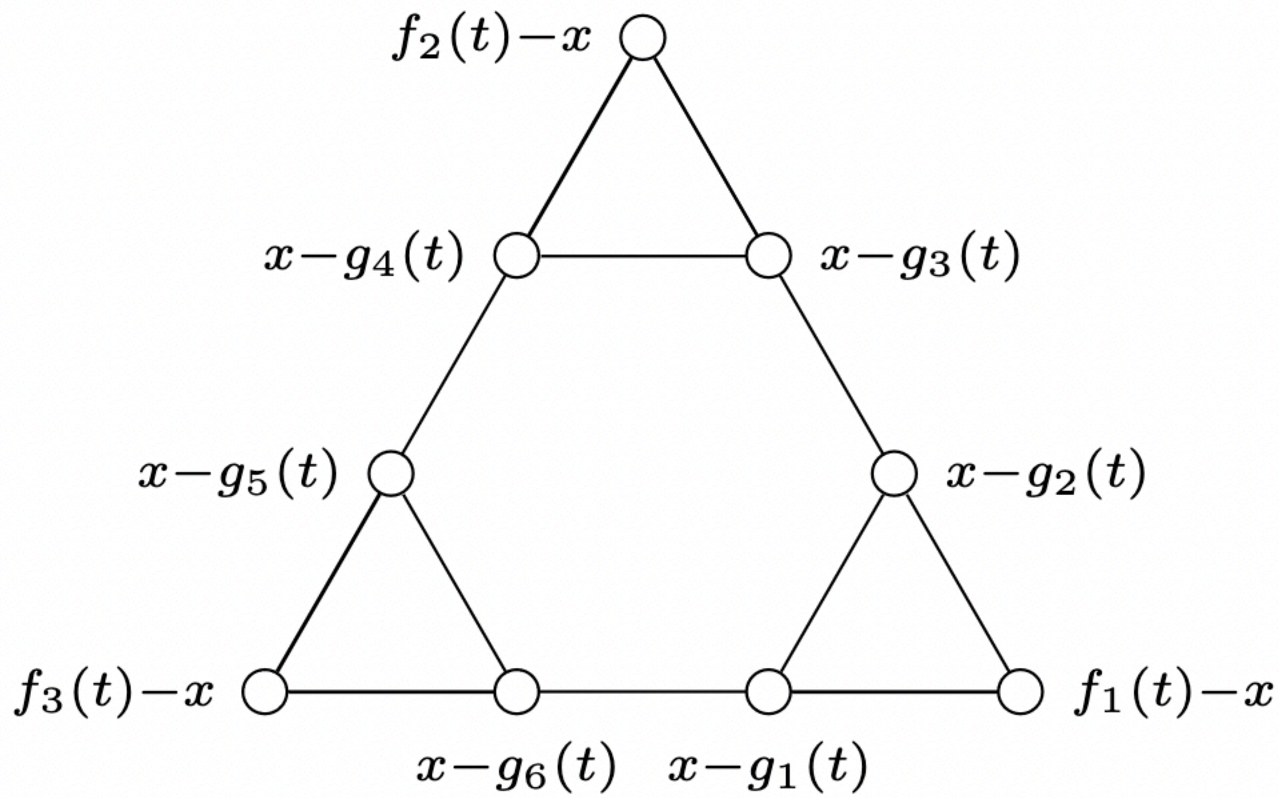


$$
\begin{array}{lll}
f_1(t) = 0, & f_2(t) = \ell_1 - m_2 - \omega(t), & f_3(t) = \ell_2 - n_1 + \omega(t), \\
g_1(t) = -m_2, & g_3(t) = \ell_1 - m_2 - n_2 - \omega(t), & g_5(t) = -n_1 + \omega(t), \\
g_2(t) = -n_1, & g_4(t) = -m_2 - \omega(t), & g_6(t) = \ell_2 - m_1 - n_1 + \omega(t)
\end{array}
$$

$$
\text{with } \omega(t) = \tfrac{1}{3}\left(\ell_1 + m_1 + n_1 - \ell_2 - m_2 - n_2\right).
$$

A parametrization of the space of BZ-triangles

$$
\begin{cases}
\forall i, x \leq f_i(t), \\
\forall j, x \geq g_j(t).
\end{cases}
$$



The rays of the BZ cone

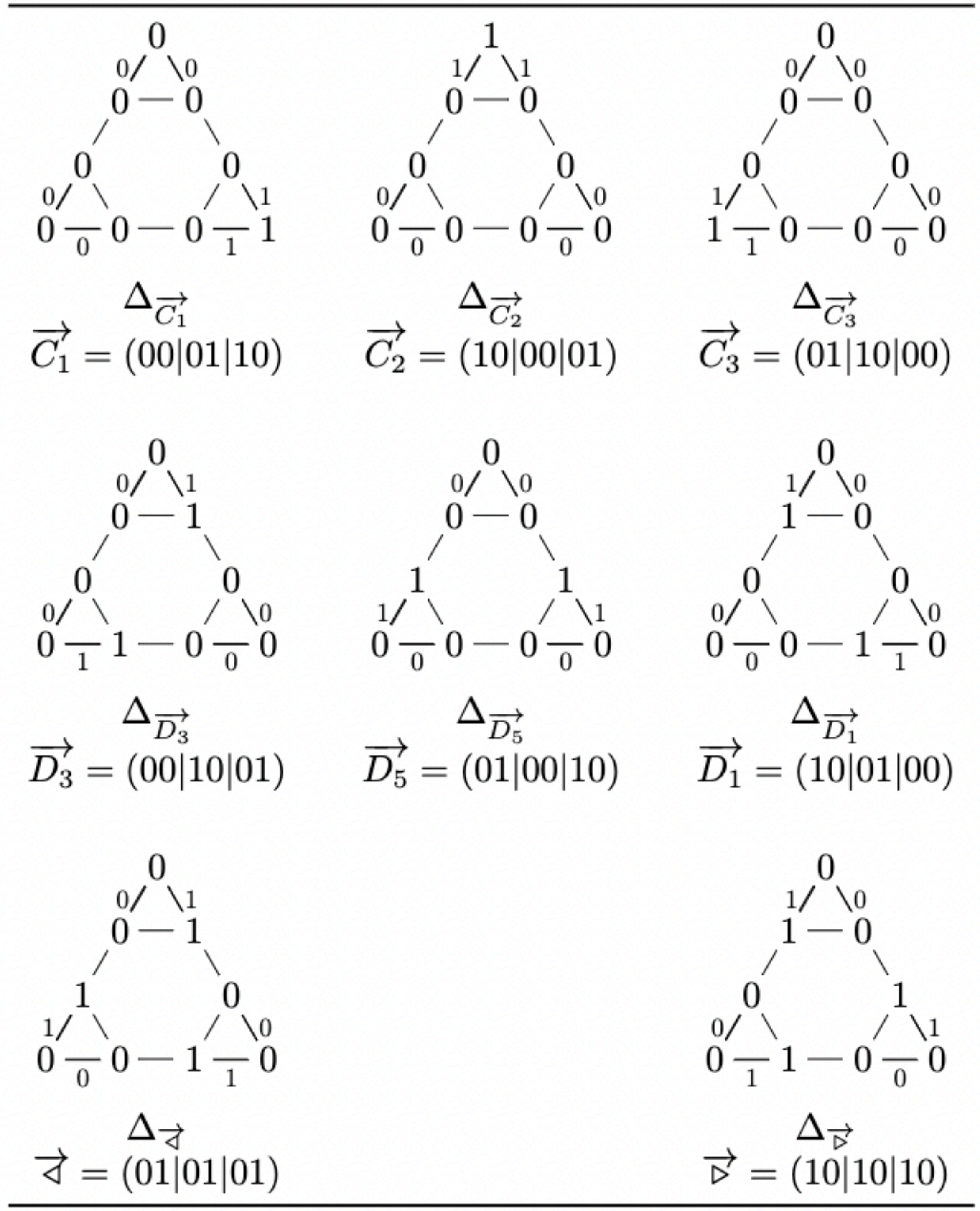
Relation

$$\Delta_{\vec{D_1}} + \Delta_{\vec{D_3}} + \Delta_{\vec{D_5}} = \Delta_{\vec{\triangleleft}} + \Delta_{\vec{\triangleright}}$$

Fundamental BZ triangles

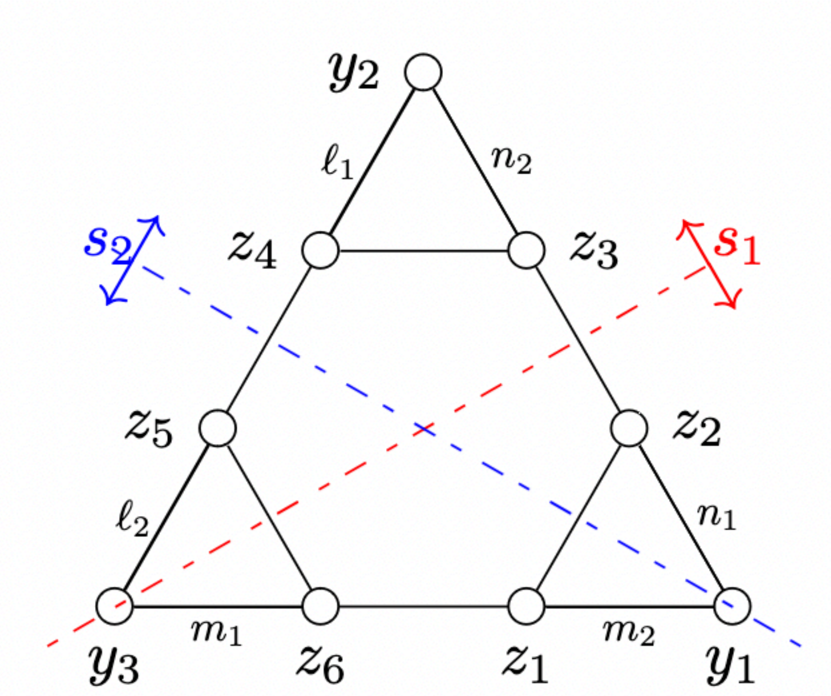
Generate as vector space
and as lattice

Fundamental BZ triangles



Berenstein-Zelevinsky

Symmetries of the BZ triangles



Generated by

$$(\ell; m; n) \leftrightarrow (m^*; \ell^*; n^*)$$

$$(\ell; m; n) \leftrightarrow (\ell^*; n^*; m^*).$$

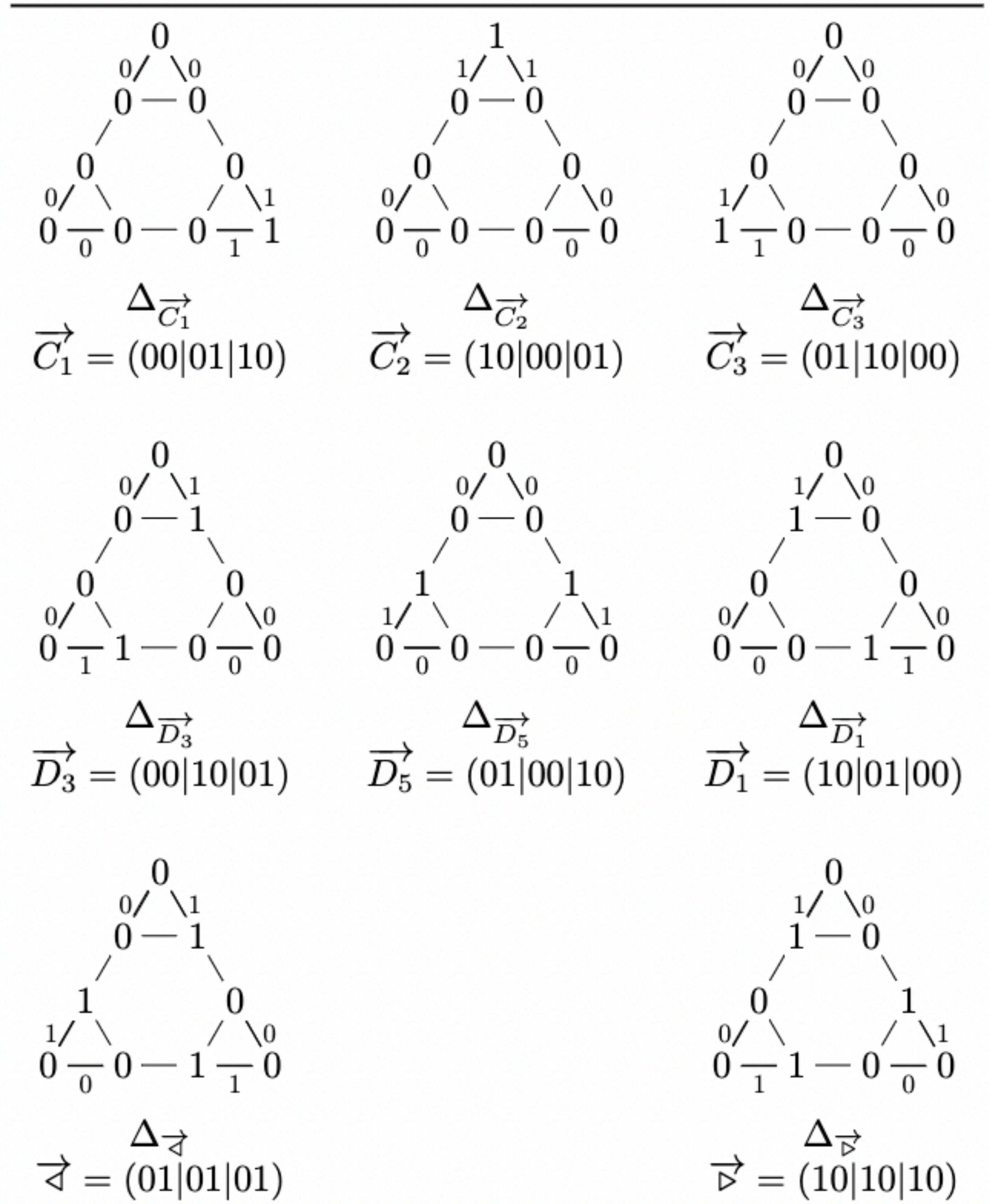
A not the group of permutations of the Dynkin labels
(it is isomorphic to it)

Does not contain the duality symmetry

FUNDAMENTAL BZ TRIANGLES

A LINEAR SYMMETRY OF THE SPACE OF BZ TRIANGLES PERMUTES

MINIMAL RAY GENERATORS FOR THE CONE BZ



ANY LINEAR SYMMETRY SHOULD STABILIZE

$$\{\Delta_{\vec{D_3}}, \Delta_{\vec{D_5}}, \Delta_{\vec{D_1}}\}$$

AND $\{\Delta_{\vec{\triangleleft}}, \Delta_{\vec{\triangleright}}\}$

$$\Delta_{\vec{D_1}} + \Delta_{\vec{D_3}} + \Delta_{\vec{D_5}} = \Delta_{\vec{\triangleleft}} + \Delta_{\vec{\triangleright}}$$

THUS, IT SHOULD ALSO STABILIZE

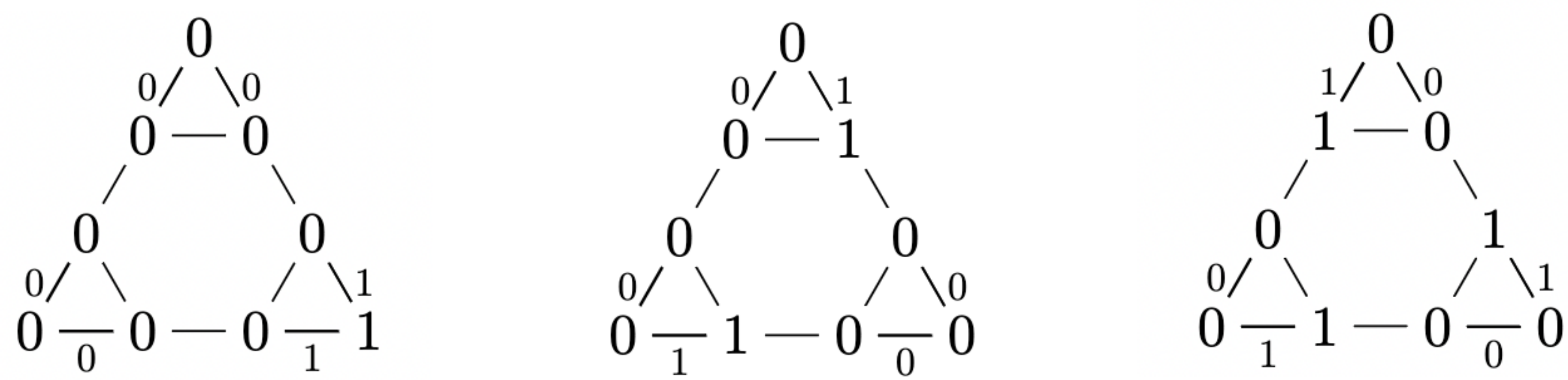
$$\{\Delta_{\vec{C_1}}, \Delta_{\vec{C_2}}, \Delta_{\vec{C_3}}\}$$

GROUP OF LINEAR SYMMETRIES OF ORDER 72

$$\mathfrak{S}_{\{\Delta_{\vec{C_1}},\Delta_{\vec{C_2}},\Delta_{\vec{C_3}}\}} \times \mathfrak{S}_{\{\Delta_{\vec{D_3}},\Delta_{\vec{D_5}},\Delta_{\vec{D_1}}\}} \times \mathfrak{S}_{\{\Delta_{\vec{\triangleleft}},\Delta_{\vec{\triangleright}}\}}.$$

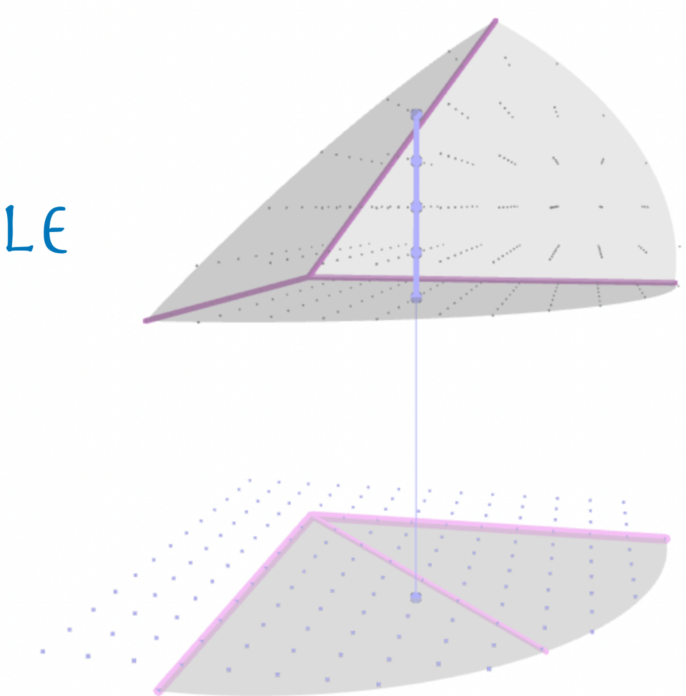
A linear map $\quad pr : \mathcal{L}_{\mathrm{BZ}} \to \mathbb{R}^6$

$$\ell_1 = y_2 + z_4, \quad m_1 = y_3 + z_6, \quad n_1 = y_1 + z_2$$
$$\ell_2 = y_3 + z_5, \quad m_2 = y_1 + z_1, \quad n_2 = y_2 + z_3$$



The lattice of integral points of the BZ cone is sent onto Λ_{TM}

It sends the BZ cone to the cone of the triple multiplicities



$$c(\ell; m; n) = \#\left(pr^{-1}(\ell; m; n) \cap \mathrm{lat}(\mathsf{BZ})\right)$$

Berenstein-Zelevinski

A linear symmetry of the triple multiplicities

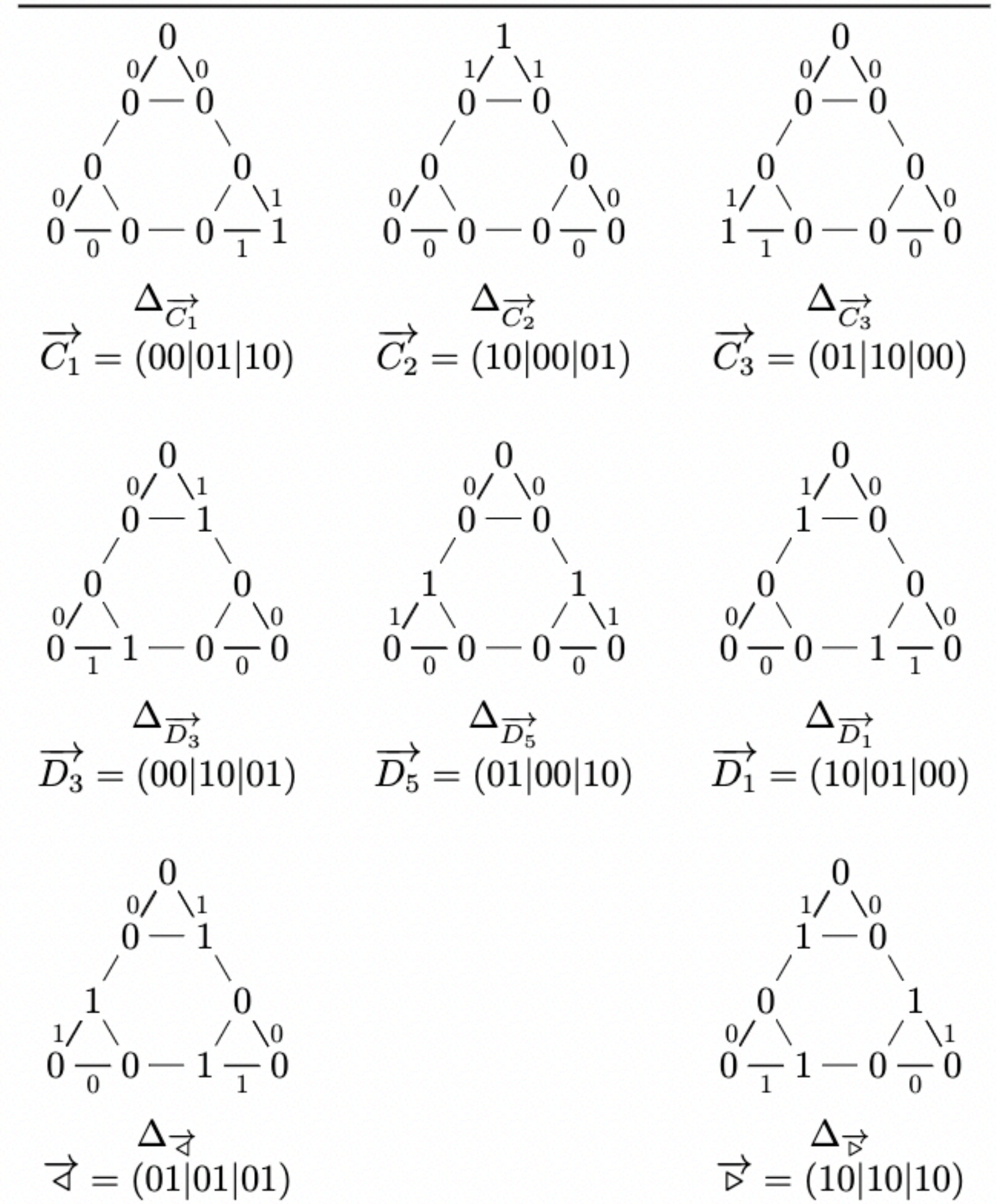
$$\overrightarrow{C_1} + \overrightarrow{C_2} + \overrightarrow{C_3} = \overrightarrow{D_1} + \overrightarrow{D_3} + \overrightarrow{D_5} = \overrightarrow{\triangleleft} + \overrightarrow{\triangleright}.$$

↑ NEW

There are no other relations with all coefficients positive

Triple multiplicities

PROJECTIONS



Any linear symmetry of the triple multiplicities stabilizes the TM cone

Thus, permutes its rays.

$$\{\overrightarrow{C_1}, \overrightarrow{C_2}, \overrightarrow{C_3}, \overrightarrow{D_1}, \overrightarrow{D_3}, \overrightarrow{D_5}, \overrightarrow{\triangleleft}, \overrightarrow{\triangleright}\}.$$

Minimal ray generators

A linear symmetry of the triple multiplicities

$$\overrightarrow{C_1} + \overrightarrow{C_2} + \overrightarrow{C_3} = \overrightarrow{D_1} + \overrightarrow{D_3} + \overrightarrow{D_5} = \overrightarrow{\triangleleft} + \overrightarrow{\triangleright}.$$

Stabilizes $\{\overrightarrow{\triangleleft}, \overrightarrow{\triangleright}\}$

Stabilizes —-or swaps them!

$$\{\overrightarrow{C_1}, \overrightarrow{C_2}, \overrightarrow{C_3}\} \text{ and } \{\overrightarrow{D_1}, \overrightarrow{D_3}, \overrightarrow{D_5}\}$$

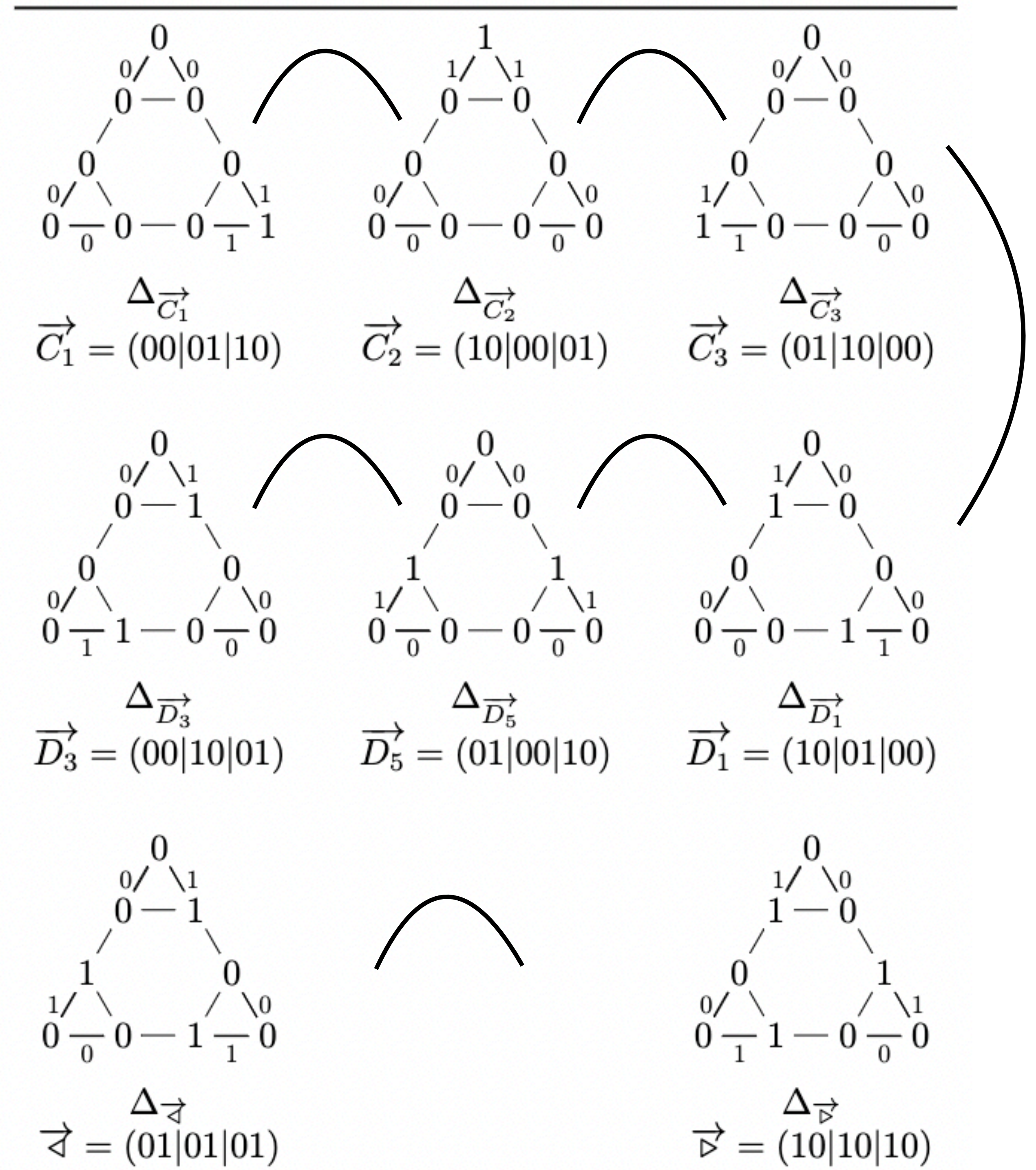
Thus stabilizes

$$\{\overrightarrow{\triangleleft}, \overrightarrow{\triangleright}\} \text{ and } \{\{\overrightarrow{C_1}, \overrightarrow{C_2}, \overrightarrow{C_3}\}, \{\overrightarrow{D_1}, \overrightarrow{D_3}, \overrightarrow{D_5}\}\}$$

Isomorphic $\mathfrak{S}_2 \times (\mathfrak{S}_3 \wr \mathfrak{S}_2)$

Order $2 \times (2 \times (3!)^2)$

Triple multiplicities



Symmetries of the outer triangle

Symmetries of the inner hexagon

The support of the triple multiplicities is a cone.

$$\begin{cases} \forall i \in \{1,2,3\}, & x \leq f_i(t); \\ \forall j \in \{1,2,3,4,5,6\}, & x \geq g_j(t) \end{cases}$$

$$\max_q g_q(t) \leq x \leq \min_p f_p(t)$$

System of 18 inequalities:

$$\forall i \in \{1,2,3\},\ \forall j \in \{1,2,3,4,5,6\},\ g_j(t) \leq f_i(t).$$

The quasi polynomial:

$$c(t) = 1 + \max(0, \min_p f_p(t) - \max_q g_q(t)).$$

THE 18 CHAMBERS ARE FULL DIMENSIONAL

THE GROUP OF SYMMETRIES OF THE BZ TRIANGLES ACTS TRANSITIVELY ON THE CHAMBER COMPLEX

CHAMBERS ARE SIMPLICIAL,

HAVE 6 RAYS, 5 EXTERNAL $t_1, \ldots, t_5$

INTERNAL RAY $(11|11|11)$

$$c(t) = 1 + \mathrm{Vol}_{\Lambda_{\mathrm{TM}}}(\Pi(t_1, \ldots, t_5, t))$$

fundamental domains of the lattice have volume 1

RANK GENERATING FUNCTION

$$(1 + 3q + 3q^2)^2(1 + 2q)(1 + q)^3.$$

AD UTRUMQUE

Grant PID2020-117843GB-I00 funded by MICIU/AEI/10.13039/501100011033.

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