VECTOR PARTITION FUNCTIONS AND KRONECKER COEFFICIENTS

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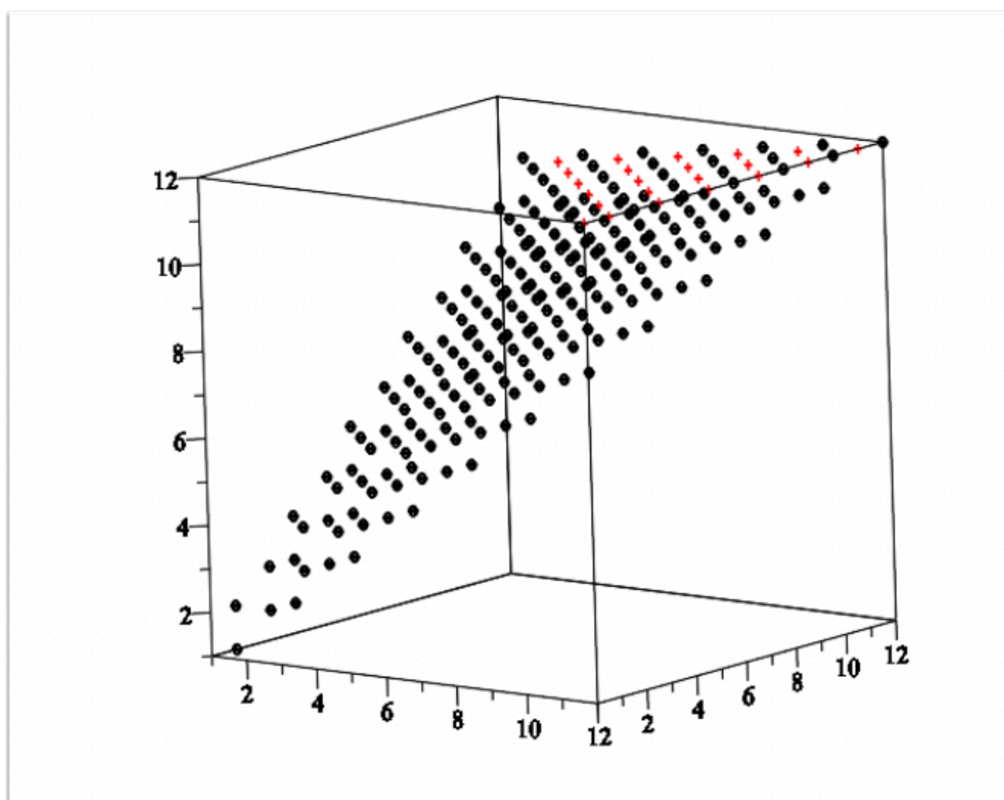
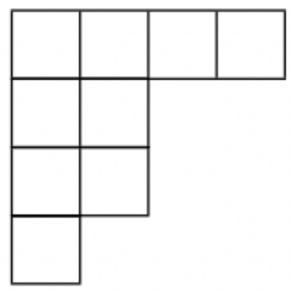
https://arxiv.org/abs/1811.10015

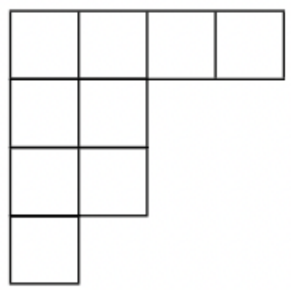


STRUCTURAL COEFFICIENTS

	The symmetric group $\mathbb{S}_n$.	The complex general lineal group $GL(n, \mathbb{C}) = GL(V)$
	Indexed by partitions λ of n, S^λ	Indexed by partitions λ of length n, W^λ

	The trivial representation	The symmetric Powers of V
	The sign Representation	The exterior Powers of V

STRUCTURAL COEFFICIENTS

	The symmetric group $\mathbb{S}_n$.	The complex general lineal group $GL(n,\mathbb{C}) = GL(V)$
Irreducible Representations	**Indexed by partitions λ of n, S^λ**	**Indexed by partitions λ of length n, W^λ**
Littlewood-Richardson coefficients	The solution of A branching problem	$W^\mu \otimes W^\nu = \bigoplus_{\substack{\lambda \vdash \lvert\nu\rvert + \lvert\mu\rvert \\ \ell(\lambda) \leq n}} c^\lambda_{\mu,\nu} W^\lambda$
Kronecker coefficients	$S^\mu \otimes S^\nu = \bigoplus_{\lambda \vdash n} c^\lambda_{\mu,\nu} S^\lambda$	The solution of a branching problem.

Littlewood-Richardson coefficients

$$s_\lambda[X+Y] = \sum_{\mu,\nu} c^\lambda_{\mu,\nu} s_\mu[X] s_\nu[Y]$$

Kronecker coefficients

$$s_\lambda[XY] = s_\lambda(x_1y_1, x_1y_2 \cdots , x_ny_m) = \sum_{\mu,\nu} g_{\mu,\nu,\lambda}\, s_\mu[X] s_\nu[Y]$$

THE KRONECKER FUNCTION

$$\kappa_{n,m,l}(\mu,\nu,\lambda) = \kappa_{n,m,l}(\mu_1,\ldots,\mu_n,\nu_1,\ldots,\nu_m,\lambda_1,\ldots,\lambda_l) := g_{\mu,\nu,\lambda}.$$

What kind of function is the Kronecker function?

Can we compute it?

What is its asymptotical behavior?

Can we bound the Kronecker coefficients?

POLYTOPES AND QUASIPOLYNOMIALS

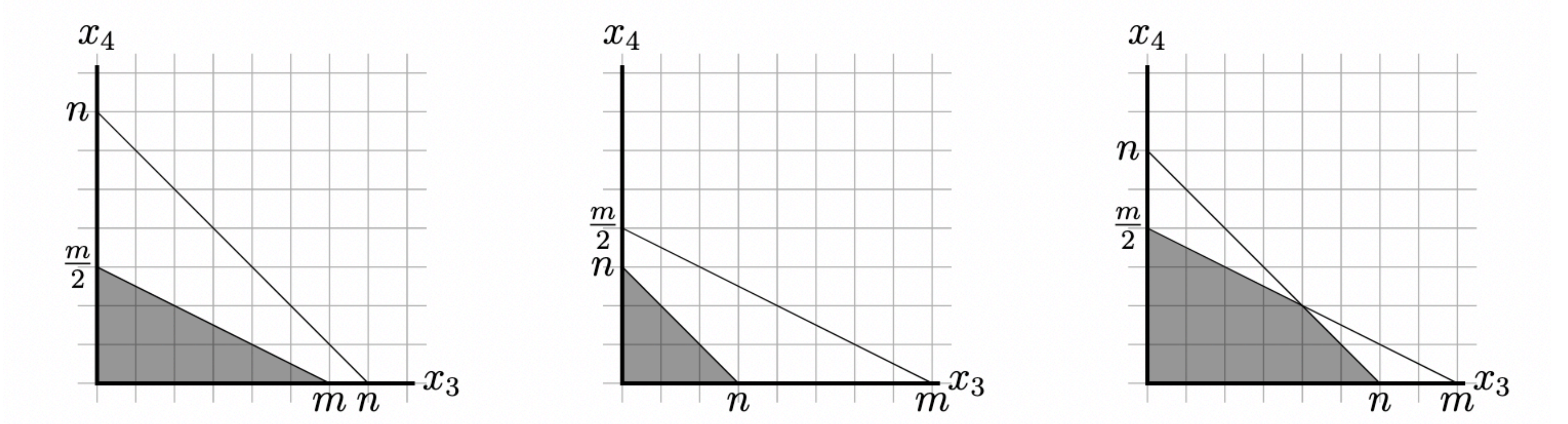
A *polyhedron* $\mathcal{P}$ is the set of solutions of a (finite) system and inequalities:

$$\mathcal{P} = \{\mathbf{x} \in \mathbb{R}^d : A\mathbf{x} \leq \mathbf{b}\},$$

for a fixed matrix A and vector $\mathbf{b}$, where the "$\leq$" sign is to be understood componentwise.

Non-negative solutions

$$\begin{cases} x_3 + x_4 \leq n \\ x_3 + 2x_4 \leq m \end{cases}$$



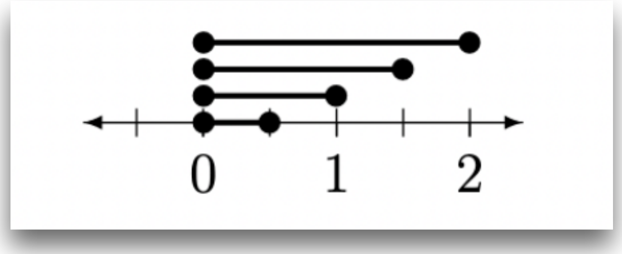
rational

bounded (Polytope)

dimension

POLYTOPES AND QUASIPOLYNOMIALS

The one dimensional polytope [0,1/2] and its first four dilations.



The "volumen" of the k-th dilation is a quasipolynomial in k

$$\phi_{\mathcal{P}}(k) = \left\lfloor \frac{k}{2} \right\rfloor + 1 = \begin{cases} \frac{k+2}{2} & \text{if } k \equiv 0 \mod 2 \\ \frac{k+1}{2} & \text{if } k \equiv 1 \mod 2 \end{cases}$$

A function $\phi : \mathbb{N} \to \mathbb{Q}$ is a *(one-variable) quasipolynomial* if there exist polynomials $p_0, p_1, \ldots, p_{k-1}$ in $\mathbb{Q}[t]$ and a natural number $m > 0$, a *period* of ϕ, such that

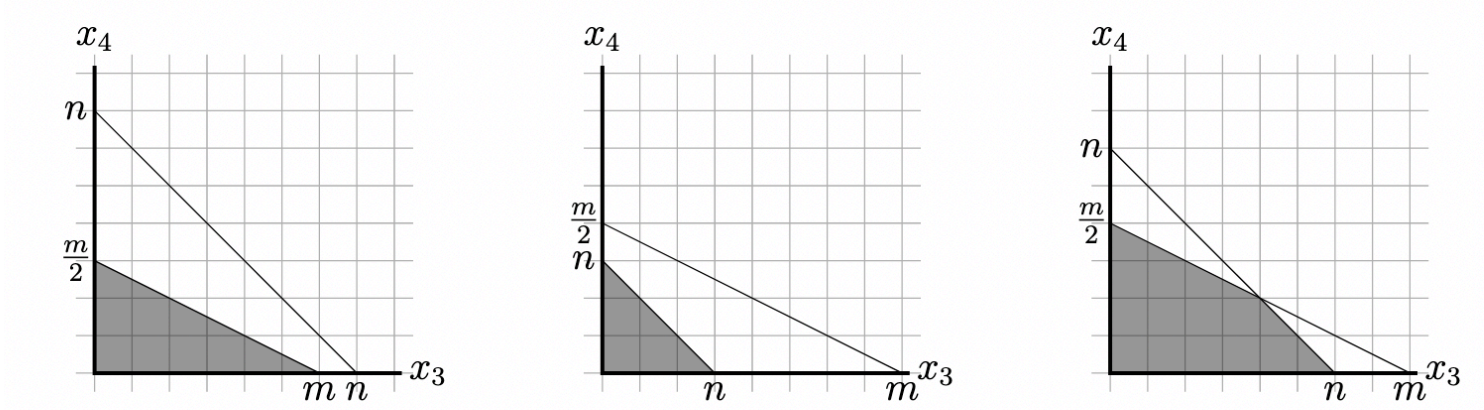
$$\phi(t) = p_i(t), \text{ for } t \equiv i \mod m.$$

Example 2. Let $p_S(n,m)$ count the number of vector partitions of $\mathbf{b} = (n,m)$ with parts in $S = \{(1,0),(0,1),(1,1),(1,2)\}$. Equivalently, this is the number of nonnegative integer solutions $\mathbf{x}$ to the system $A\mathbf{x} = \mathbf{b}$, where

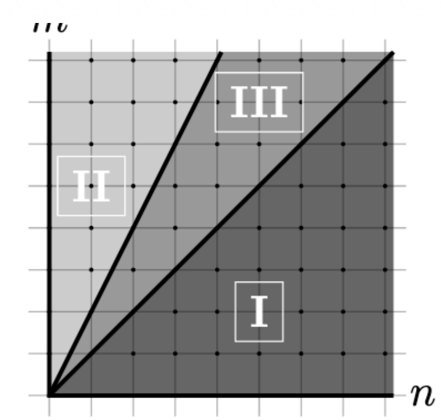
$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} n \\ m \end{bmatrix}$$

The linear system is equivalent to the system of inequalities

$$\begin{cases} x_3 + x_4 \leq n \\ x_3 + 2x_4 \leq m \end{cases}$$



chamber complex



The resulting piecewise quasipolynomial is then:

Region		$p_S(n,m)$
I	$m \leq n$	$\frac{m^2}{4} + m + \frac{7}{8} + \frac{(-1)^m}{8}$
II	$2n \leq m$	$\frac{n^2}{2} + \frac{3n}{2} + 1$
III	$n \leq m \leq 2n$	$nm - \frac{n^2}{2} - \frac{m^2}{4} + \frac{n+m}{2} + \frac{7}{8} + \frac{(-1)^m}{8}$

A vector partition is a way of decomposing
as a sum of nonzero vectors with nonnegative coordinates.

$$(1,2,1) = (1,1,0) + (0,1,0) + (0,0,1)$$

Where the parts belong to a fixed multiset S

The vector partition partition function

$$p_S : \mathbb{N}^d \to \mathbb{N}$$

is the function that evaluated at v gives the
number of partitions of v with parts in S.

Cauchy's definition of a Schur function / Weyl character formula:

$$s_\lambda[X] = \frac{a_{\lambda+\delta}(x_1, x_2, \ldots, x_n)}{a_\delta(x_1, x_2, \ldots, x_n)} = \frac{\det(x_i^{\lambda_j+n-j})_{i,j=1}^n}{\prod_{1\le i<j\le n}(x_i - x_j)}$$

The comultiplication formula for the Kronecker coefficients:

$$s_\lambda[XY] = \sum_{\mu,\nu} g_{\mu,\nu,\lambda}\, s_\mu[X] s_\nu[Y]$$

$$\frac{a_{\delta_n}[X] a_{\delta_m}[Y]}{a_{\delta_{nm}}[XY]}\, a_{\lambda+\delta_{nm}}[XY] = \sum_{\mu,\nu} g_{\mu,\nu,\lambda}\, a_{\mu+\delta_n}[X] a_{\nu+\delta_m}[Y]$$

Rational Function

polynomial

polynomial

$$\frac{a_{\delta_n}[X]a_{\delta_m}[Y]}{a_{\delta_{nm}}[XY]}\, a_{\lambda+\delta_{nm}}[XY] = \sum_{\mu,\nu} g_{\mu,\nu,\lambda}\, a_{\mu+\delta_n}[X]a_{\nu+\delta_m}[Y]$$

The quotient simplifies to

$$\frac{a_{\delta_2}[X]a_{\delta_2}[Y]}{a_{\delta_4}[XY]} = \frac{1}{x^2y\ (1-y/x)(1-xy)(1-x)(1-y)}$$

$$\bar{F}_{2,2}(x,y) := \frac{1}{(1-y/x)(1-xy)(1-x)(1-y)},$$

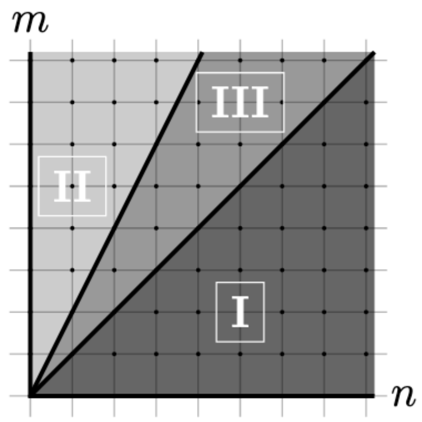
After a change of basis we obtain the vector partition function

$$\begin{aligned} F_{2,2}(s_0,s_1) &= \sum \tilde{g}_{\mu,\nu,\lambda} s_0^{\nu_2-\lambda_3-\lambda_4} s_1^{\mu_2+\nu_2-\lambda_2-\lambda_3-2\lambda_4} \\ &= \frac{1}{(1-s_0)(1-s_1)(1-s_0s_1)(1-s_0s_1^2)}. \end{aligned}$$

This is the vector partition function of our example.

$$F_{2,2}(s_0, s_1) = \sum \tilde{g}_{\mu,\nu,\lambda} s_0^{\nu_2-\lambda_3-\lambda_4} s_1^{\mu_2+\nu_2-\lambda_2-\lambda_3-2\lambda_4}$$
$$= \frac{1}{(1-s_0)(1-s_1)(1-s_0 s_1)(1-s_0 s_1^2)}.$$

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} n \\ m \end{bmatrix}$$



$$\frac{a_{\delta_n}[X] a_{\delta_m}[Y]}{a_{\delta_{nm}}[XY]} a_{\lambda+\delta_{nm}}[XY] = \sum_{\mu,\nu} g_{\mu,\nu,\lambda}\, a_{\mu+\delta_n}[X] a_{\nu+\delta_m}[Y]$$

$$\frac{a_{\delta_n}[X]a_{\delta_m}[Y]}{a_{\delta_{nm}}[XY]}\ a_{\lambda+\delta_{nm}}[XY] = \sum_{\mu,\nu} g_{\mu,\nu,\lambda}\, a_{\mu+\delta_n}[X]a_{\nu+\delta_m}[Y]$$

the following 7-term linear combination of vector partition functions $p_S(n,m)$:

$$\begin{aligned} g_{\mu,\nu,\lambda} = p_S(\nu_2-(\lambda_3+\lambda_4), \nu_2-(\lambda_3+\lambda_4)+\mu_2-(\lambda_2+\lambda_4)) \\ -p_S(\nu_2-(\lambda_3+\lambda_4), \nu_2-(\lambda_3+\lambda_4)+\mu_2-(\lambda_2+\lambda_3+1)) \\ -p_S(\nu_2-(\lambda_3+\lambda_4), \nu_2-(\lambda_3+\lambda_4)+\mu_2-(\lambda_1+\lambda_4+1)) \\ +p_S(\nu_2-(\lambda_3+\lambda_4), \nu_2-(\lambda_3+\lambda_4)+\mu_2-(\lambda_1+\lambda_3+2)) \\ -p_S(\nu_2-(\lambda_2+\lambda_4+1), \nu_2-(\lambda_2+\lambda_4+1)+\mu_2-(\lambda_3+\lambda_4-1)) \\ +p_S(\nu_2-(\lambda_2+\lambda_4+1), \nu_2-(\lambda_2+\lambda_4+1)+\mu_2-(\lambda_2+\lambda_3+1)) \\ +p_S(\nu_2-(\lambda_2+\lambda_4+1), \nu_2-(\lambda_2+\lambda_4+1)+\mu_2-(\lambda_1+\lambda_4+1)) \end{aligned} \quad (13)$$

A TYPICAL EXAMPLE OF A DILATION OF THE KRONECKER FUNCTION

$\lambda = (132, 38, 19, 11)$, $\mu = (110, 90)$, and $\nu = (120, 80)$.

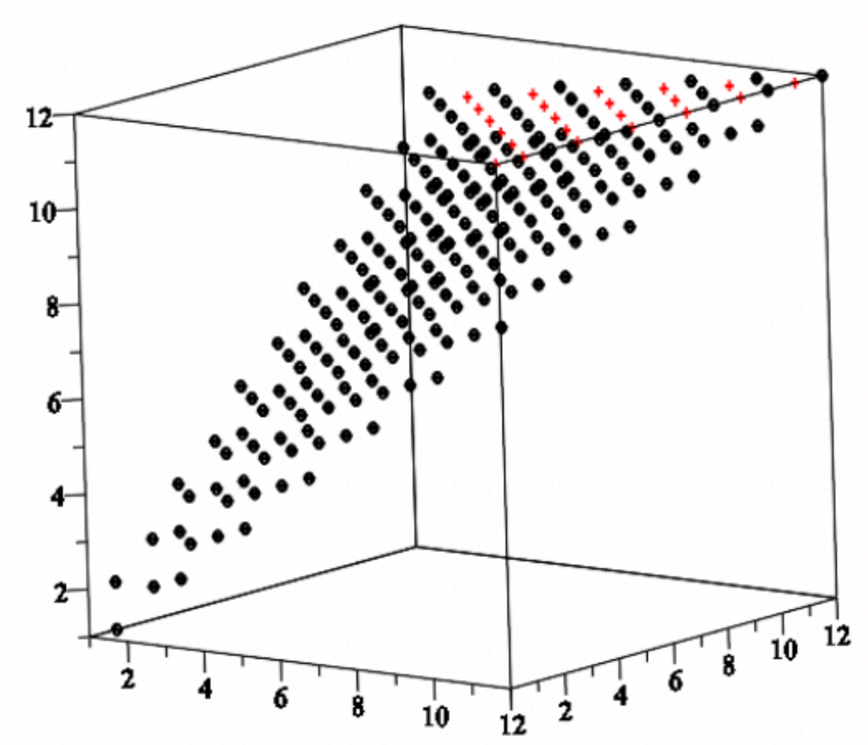
$$p_S(50k, 91k) - p_S(50k, 83k - 1) - p_S(31k - 1, 91k - 2) + p_S(31k - 1, 64k - 2).$$

$$g_{k\mu,k\nu,k\lambda} = 52k^2 + \frac{25}{2}k + \frac{3}{4} + \frac{(-1)^k}{4}$$

KRONECKER CONE

The cone generated by the nonzero Kronecker coefficients

What we want to know about the Kronecker cone?
Saturated?
Dimension
Equations of the walls
Chamber complex
Quasipolynomial in each chamber

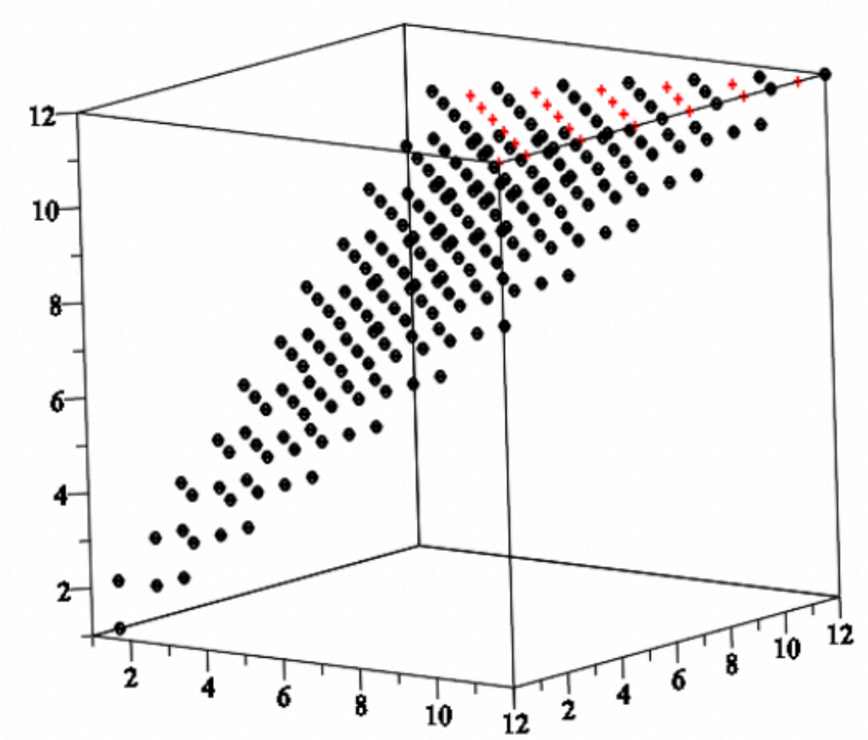


(i, j, k) *is black if* $g_{(24-i,i)(24-j,j)(24-k,k)}$ *is nonzero (assuming,* $j \leq i \leq k \leq 24/2$*).*

There are holes in the Kronecker cone

$$
\begin{aligned}
g_{(k,k),(k,k),(k,k)} &= [x^k y^k](x^k - 2x^{k+1} + x^{k+2})\bar{F}_{2,2}(x,y) \\
&= [s_0^k s_1^k](1 - 2s_1 + s_1^2)F_{2,2}(s_0, s_1) \\
&= p_S(k,k) - 2p_S(k,k-1) + p_S(k,k-2) \\
&= \begin{cases} 1, & k \text{ even} \\ 0, & k \text{ odd}, \end{cases}
\end{aligned}
$$

The Kronecker coefficients
do not count
integer points in a polytope.



(i,j,k) *is black if* $g_{(24-i,i)(24-j,j)(24-k,k)}$ *is nonzero (assuming,* $j \le i \le k \le 24/2$*).*

What we want to know about the Kronecker cone?
Dimension

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} n \\ m \end{bmatrix}$$

Quadratic !!

The degree of the Kronecker vector partition function equals the dimension of the null space.

The degree of the Kronecker function can be smaller, due to cancelations in the signed sums.

What we want to know about the Kronecker cone?

Equations of the walls

Proposition 5. *The atomic Kronecker coefficient* $\tilde{g}_{\mu,\nu,\lambda}$ *is nonzero if and only if*

$$\begin{cases} \lambda_2 + \lambda_3 + 2\lambda_4 \leq \mu_2 + \nu_2 \\ \lambda_3 + \lambda_4 \leq \nu_2. \end{cases}$$

Moreover, the value of $\tilde{g}_{\mu,\nu,\lambda}$ *is given by a quadratic quasipolynomial:*

$$\tilde{g}_{\mu,\nu,\lambda} = p_S(\nu_2 - (\lambda_3 + \lambda_4), \mu_2 + \nu_2 - (\lambda_2 + \lambda_4) - (\lambda_3 + \lambda_4)),$$

where p_S *is the vector partition function of Example 2.*

This problem has been considered by Bravyi in the 2-2-4 case where he computed a list of 3 inequalities known as Bravyi's inequalities.

What we want to know about the Kronecker cone?

Quasipolynomial in each chamber

Meinrenken and Sjamaar,
Singular reduction and quantization.
Topology, 1999.

the following 7-term linear combination of vector partition functions $p_S(n,m)$*:*

$$\begin{aligned} g_{\mu,\nu,\lambda} = p_S(\nu_2 - (\lambda_3+\lambda_4), \nu_2 - (\lambda_3+\lambda_4) + \mu_2 - (\lambda_2+\lambda_4)) \\ - p_S(\nu_2 - (\lambda_3+\lambda_4), \nu_2 - (\lambda_3+\lambda_4) + \mu_2 - (\lambda_2+\lambda_3+1)) \\ - p_S(\nu_2 - (\lambda_3+\lambda_4), \nu_2 - (\lambda_3+\lambda_4) + \mu_2 - (\lambda_1+\lambda_4+1)) \\ + p_S(\nu_2 - (\lambda_3+\lambda_4), \nu_2 - (\lambda_3+\lambda_4) + \mu_2 - (\lambda_1+\lambda_3+2)) \\ - p_S(\nu_2 - (\lambda_2+\lambda_4+1), \nu_2 - (\lambda_2+\lambda_4+1) + \mu_2 - (\lambda_3+\lambda_4-1)) \\ + p_S(\nu_2 - (\lambda_2+\lambda_4+1), \nu_2 - (\lambda_2+\lambda_4+1) + \mu_2 - (\lambda_2+\lambda_3+1)) \\ + p_S(\nu_2 - (\lambda_2+\lambda_4+1), \nu_2 - (\lambda_2+\lambda_4+1) + \mu_2 - (\lambda_1+\lambda_4+1)) \end{aligned} \tag{13}$$

What we want to know about the Kronecker cone?

Chamber complex

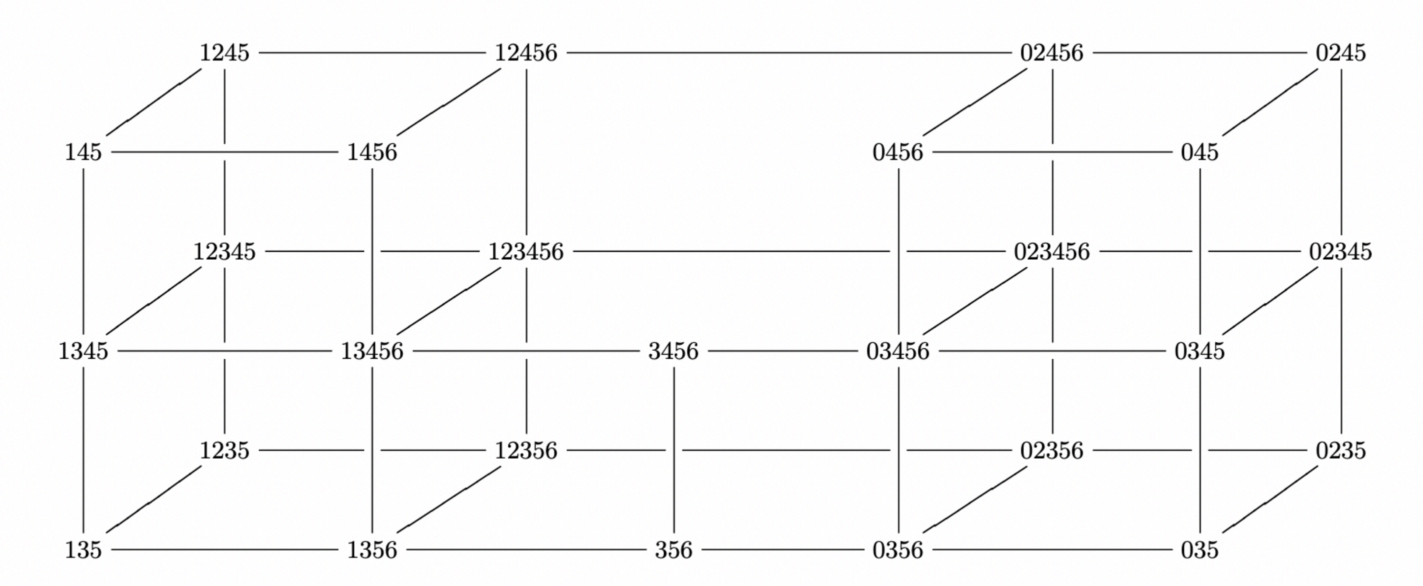
Briand-R-Orellana

https://arxiv.org/pdf/0812.0861.pdf

177 maximal cell

74 quasipolynomial formulas Pσ of the form:

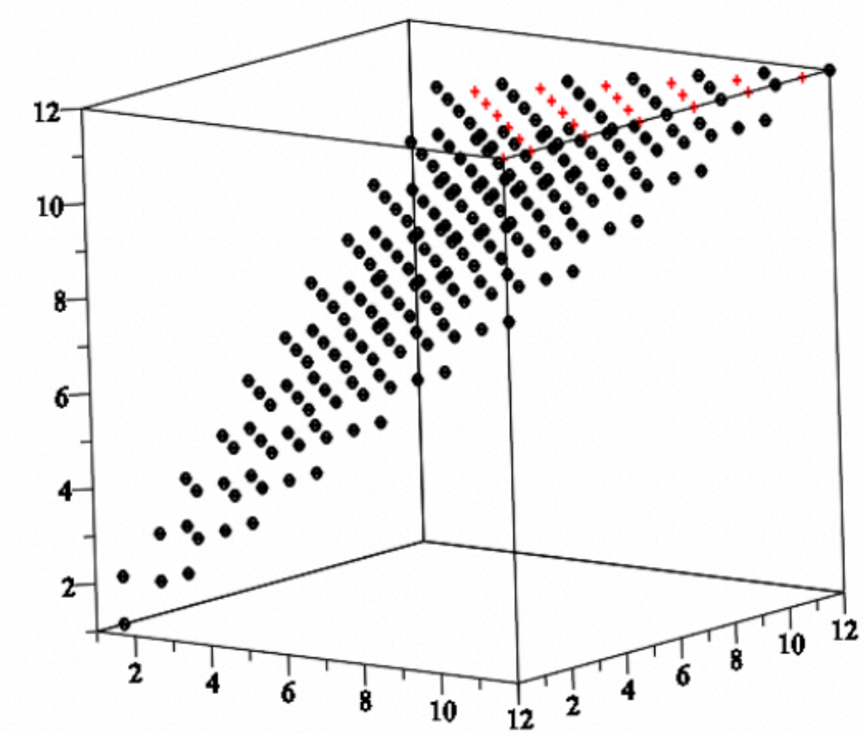
$$P_\sigma = 1/4\, Q_\sigma + 1/2\, L_\sigma + M_\sigma/4$$



Chamber complex for the reduced case.

What we want to know about the Kronecker cone?

Location of the zeroes



Walls (Manivel)
+ monotonicity

What we want to know about the Kronecker cone?

An upper bound for the Kronecker Coefficients?

The atomic Kronecker coefficient is
an upper bound for the case 2-2-4

THE REDUCED KRONECKER COEFFICIENTS

$$\bar{g}^{\lambda}_{\mu,\nu}$$

$$s_{2,2} * s_{2,2} = s_4 + s_{1,1,1,1} + s_{2,2}$$
$$s_{3,2} * s_{3,2} = s_5 + s_{2,1,1,1} + s_{3,2} + s_{4,1} + s_{3,1,1} + s_{2,2,1}$$
$$s_{4,2} * s_{4,2} = s_6 + s_{3,1,1,1} + 2s_{4,2} + s_{5,1} + s_{4,1,1} + 2s_{3,2,1} + s_{2,2,2}$$
$$s_{5,2} * s_{5,2} = s_7 + s_{4,1,1,1} + 2s_{5,2} + s_{6,1} + s_{5,1,1} + 2s_{4,2,1} + s_{3,2,2} + s_{4,3} + s_{3,3,1}$$
$$s_{6,2} * s_{6,2} = s_8 + s_{5,1,1,1} + 2s_{6,2} + s_{7,1} + s_{6,1,1} + 2s_{5,2,1} + s_{4,2,2} + s_{5,3} + s_{4,3,1} + s_{4,4}$$
$$s_{7,2} * s_{7,2} = s_9 + s_{6,1,1,1} + 2s_{7,2} + s_{8,1} + s_{7,1,1} + 2s_{6,2,1} + s_{5,2,2} + s_{6,3} + s_{5,3,1} + s_{5,4}$$
$$s_{\bullet,2} * s_{\bullet,2} = s_{\bullet} + s_{\bullet,1,1,1} + 2s_{\bullet,2} + s_{\bullet,1} + s_{\bullet,1,1} + 2s_{\bullet,2,1} + s_{\bullet,2,2} + s_{\bullet,3} + s_{\bullet,3,1} + s_{\bullet,4}$$

Briand-R-Orellana

https://arxiv.org/pdf/0907.4652.pdf

$$\mathrm{stab}(\alpha,\beta) = |\alpha| + |\beta| + \alpha_1 + \beta_1$$

Murnaghan If $|\mu| + |\nu| = |\lambda|$

Then the reduced Kronecker coefficients are equal to the Littlewood-Richardson coefficients.

Some Kronecker coefficients are Littlewood-Richardson coefficients. Where do they sit in the Kronecker cone?

Theorem 22. *The Littlewood–Richardson cone coincides with the intersection of the hyperplane* $\lambda_4 = 0$ *and the face of the Kronecker cone defined by the first Bravyi inequality,* $\lambda_2 + \lambda_3 + 2\lambda_4 = \mu_2 + \nu_2$.

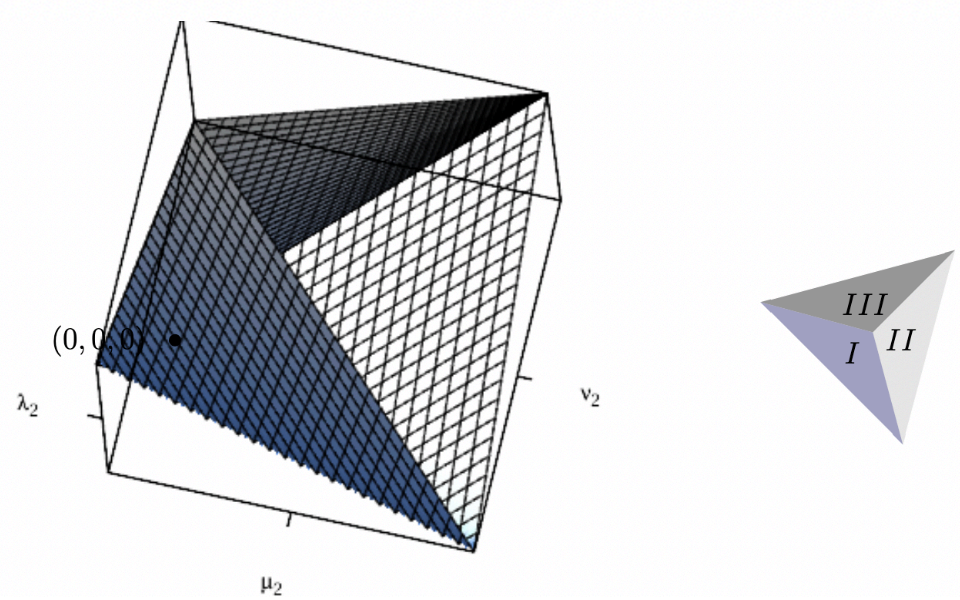
2-2-4: These are very easy instances of LR coefficients, they are described by Pieri's rule. Indeed they are all atomic Kronecker coefficients.

Quasipolynomial formulas for computing the reduced Kronecker coefficients.

Corollary 20. *Fix λ_2, μ_2, and ν_2. Let $a = \max(\lambda_2, \mu_2, \nu_2) \geq b \geq c = \min(\lambda_2, \mu_2, \nu_2)$ be a total ordering of λ_2, μ_2, ν_2. Set $\ell = b + c - a$. Then*

$$\bar{g}_{(\lambda_2),(\mu_2),(\nu_2)} = \bar{g}_{(a),(b),(c)} = \begin{cases} \frac{\ell}{2} + \frac{3+(-1)^\ell}{4} = \lfloor \frac{\ell}{2} \rfloor + 1 & \text{if } \ell \geq 0 \\ 0 & \text{if } \ell < 0. \end{cases} \tag{25}$$

The chamber complex for this quasipolynomial is illustrated in Figure 6. The walls are the hyperplanes $I) : \mu_2 + \nu_2 = \lambda_2$, $II) : \mu_2 + \lambda_2 = \nu_2$, $III) : \lambda_2 + \nu_2 = \mu_2$. The reduced Kronecker coefficient indexed by points on any of these walls always has value equal to one.

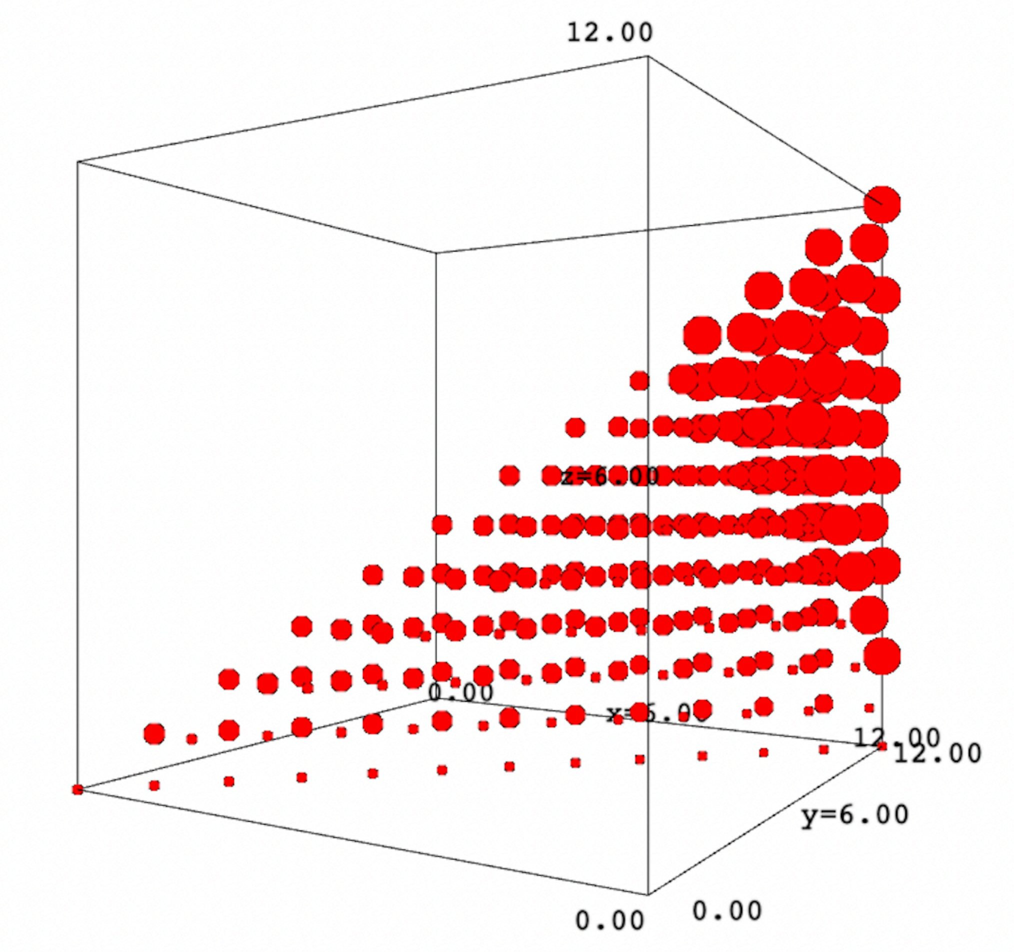


Linear growth

The chamber complex for the reduced Kronecker coefficients indexed by three one-row partitions.

Some Kronecker coefficients are atomic Kronecker coefficients. Where do they sit in the Kronecker cone?

By Stefan Trandafir (now at Sevilla)



Final comments

Many of the results of this paper have been extended
to larger dimensional situations by
Stefan Trandafir (SFU PhD 2024).

Stefan has computed that the number of chambers for the Kronecker vector partition indexed by 2-n-2n grows quickly:

3, 34, 4328, ...

M. Christandl, B. Doran, M. Walter https://arxiv.org/pdf/1204.4379.pdf
V. Baldoni, M. Vergne, https://arxiv.org/abs/1601.04325