Twisted Gelfand-Tsetlin modules

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Simple Lie algebra

Let $\mathfrak{g}$ be a simple complex finite dimensional Lie algebra, $\mathfrak{h}$ a fixed Cartan subalgebra, $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$, and W the Weyl group of $\mathfrak{g}$. By Δ we denote the root system of $\mathfrak{g}$ and π a fixed basis of Δ. For a subset Σ of π denote by Δ_Σ the root subsystem in $\mathfrak{h}^*$ generated by Σ. Then the standard parabolic subalgebra $\mathfrak{p}_\Sigma$ of $\mathfrak{g}$ associated to Σ is defined as $\mathfrak{p}_\Sigma = \mathfrak{l}_\Sigma \oplus \mathfrak{u}_\Sigma^+$ with nilradical $\mathfrak{u}_\Sigma^+ := \bigoplus_{\alpha\in\Delta_+\setminus\Delta_\Sigma} \mathfrak{g}_\alpha$ and Levi subalgebra $\mathfrak{l}_\Sigma$ defined by $\mathfrak{l}_\Sigma := \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta_\Sigma} \mathfrak{g}_\alpha$.
Let V be a simple weight $\mathfrak{l}_\Sigma$-module. Set $\mathfrak{p} := \mathfrak{p}_\Sigma$ and consider V as a $\mathfrak{p}$-module with trivial action of the nilradical $\mathfrak{u}_\Sigma^+$. The *generalized Verma* $\mathfrak{g}$-module $M_\mathfrak{p}^\mathfrak{g}(\Sigma, V)$ is the induced module

$$M_\mathfrak{p}^\mathfrak{g}(\Sigma, V) = \mathrm{Ind}_\mathfrak{p}^\mathfrak{g} V = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V.$$

Weight modules of infinite dimension

Recall that a $\mathfrak{g}$-module M is called *weight* if $\mathfrak{h}$ is diagonalizable on M. For $\lambda \in \mathfrak{h}^*$ the subspace M_λ of those $v \in V$ such that $hv = \lambda(h)v$ is the *weight subspace* of weight λ. The dimension of M_λ is the *multiplicity* of weight λ. We say that a weight module is

- *Bounded* if all weight multiplicities are uniformly bounded;
- *Unbounded* is not bounded and all weight multiplicities are finite;
- *Dense* if one weight multiplicity is infinite. In this case, all weight multiplicities are infinity.

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Admissible weight

Let κ be a $\mathfrak{g}$-invariant symmetric bilinear form on $\mathfrak{g}$ and $\widehat{\mathfrak{g}}_\kappa$ the affine Kac–Moody algebra of level κ associated to $\mathfrak{g}$.
For $\lambda \in \widehat{\mathfrak{h}}^*$, we define its integral root system $\widehat{\Delta}(\lambda)$ by

$$\widehat{\Delta}(\lambda) = \{\alpha \in \widehat{\Delta}^{\mathrm{re}};\ \langle \lambda + \widehat{\rho}, \alpha^\vee \rangle \in \mathbb{Z}\},$$

where $\widehat{\rho} = \rho + h^\vee \Lambda_0$.

Definition (Kac-Wakimoto, 1989)

A weight $\lambda \in \widehat{\mathfrak{h}}^*$ is *admissible* provided

i) λ is *regular dominant*, that is $\langle \lambda + \widehat{\rho}, \alpha^\vee \rangle \notin -\mathbb{N}_0$ for all $\alpha \in \widehat{\Delta}^{\mathrm{re}}_+$;

ii) the $\mathbb{Q}$-span of $\widehat{\Delta}(\lambda)$ contains $\widehat{\Delta}^{\mathrm{re}}$.

Admissible module

Let $k \in \mathbb{Q}$ be an admissible number for $\mathfrak{g}$ and let $\kappa = k\kappa_0$.

Definition (Universal affine vertex algebra)

For a $\mathfrak{g}$-module E, let us consider the induced $\widehat{\mathfrak{g}}_\kappa$-module

$$\mathbb{M}_{\kappa,\mathfrak{g}}(E) = U(\widehat{\mathfrak{g}}_\kappa)_{U(\mathfrak{g}[[t]]\oplus\mathbb{C}c)}E,$$

where E is considered as the $\mathfrak{g}[[t]] \oplus \mathbb{C}c$-module on which $\mathfrak{g} \otimes_\mathbb{C} t\mathbb{C}[[t]]$ acts trivially and c acts as the identity. We denote by $\mathbb{L}_{\kappa,\mathfrak{g}}(E)$ the corresponding quotient.

If $E = L(0)$ then $\mathbb{M}_{\kappa,\mathfrak{g}}(E) = \mathcal{V}_\kappa(\mathfrak{g})$ is the *universal affine vertex algebra* associated to the affine Kac–Moody algebra $\widehat{\mathfrak{g}}_\kappa$ and $\mathbb{L}_{\kappa,\mathfrak{g}}(E) = \mathcal{L}_\kappa(\mathfrak{g})$.

Definition (Admissible module)

We say that a $\mathfrak{g}$-module E is *admissible of level k* if $\mathbb{L}_{\kappa,\mathfrak{g}}(E)$ is an $\mathcal{L}_\kappa(\mathfrak{g})$-module.

Open question and principal results

Open problem

Let $\mathfrak{p}$ a subalgebra of $\mathfrak{g}$. Classify all $\mathfrak{g}$-modules induced from simple $\mathfrak{p}$-modules admissible in the Universal Affine Vertex Algebra.

Open question and principal results

Open problem

Let $\mathfrak{p}$ a subalgebra of $\mathfrak{g}$. Classify all $\mathfrak{g}$-modules induced from simple $\mathfrak{p}$-modules admissible in the Universal Affine Vertex Algebra.

Let $Z_k := \mathrm{Zhu}(\mathcal{L}_\kappa(\mathfrak{g}))$ denoted the *Zhu's algebra* of $\mathcal{L}_\kappa(\mathfrak{g})$.

Theorem (Kawasetsu and Ridout, 2021)

A simple weight $\mathfrak{g}$-module $\mathcal{M}$, with finite-dimensional weight spaces, is a Z_k-module if and only if either of the following statements hold:

- *$\mathcal{M}$ is a highest-weight Z_k-module, with respect to some Borel subalgebra of $\mathfrak{g}$.*
- *There is a parabolic subalgebra $\mathfrak{p} \subseteq \mathfrak{g}$, with non-abelian Levi factor $\mathfrak{l}$ of AC-type, and a corresponding irreducible semisimple parabolic family $\mathcal{P}$ of $\mathfrak{g}$-modules such that $\mathcal{M}$ is isomorphic to a submodule of $\mathcal{P}$ and some submodule of $\mathcal{P}$ is an $\mathfrak{l}$-bounded highest-weight Z_k-module.*

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Γ-Gelfand–Tsetlin module

Let us consider the simple Lie algebra $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{C})$ with the set of simple roots $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. Further, let us denote by $\mathfrak{g}_k$ for $k = 1, 2, \ldots, n$ the Lie subalgebra of $\mathfrak{g}$ generated by the root subspaces $\mathfrak{g}_{\alpha_1}, \ldots, \mathfrak{g}_{\alpha_k}$ and $\mathfrak{g}_{-\alpha_1}, \ldots, \mathfrak{g}_{-\alpha_k}$.
Let us denote by $\mathfrak{z}_{\mathfrak{g}_k}$ the center of $U(\mathfrak{g}_k)$ for $k = 1, 2, \ldots, n$. Then the Gelfand–Tsetlin subalgebra Γ of $U(\mathfrak{g})$ to respect $\mathcal{F}$ is generated by $\mathfrak{z}_{\mathfrak{g}_k}$ for $k = 1, 2, \ldots, n$ and by the Cartan subalgebra $\mathfrak{h}$.

Definition (Gelfand-Tsetlin module)

A Gelfand-Tsetlin module (with respect to Γ) M can be decomposed as $M = \bigoplus_{\chi \in \Gamma^*} M(\chi)$, where

$$M(\chi) = \{v \in M \mid \text{ for each } \gamma \in \Gamma \text{ , } \exists k \in \mathbb{Z}_{\geq 0} \text{ such that } (\gamma - \chi(\gamma))^k v = 0\}.$$

Tame Γ-Gelfand–Tsetlin module

Definition

We say that a Γ-Gelfand–Tsetlin $\mathfrak{g}$-module M is *tame* if Γ has a simple spectrum on M, i.e. all Γ-multiplicities are equal to 1.

Remark

If M is tame Γ-Gelfand–Tsetlin $\mathfrak{g}$-module then Γ-weights of M parameterize a basis of M.

Finite-dimensional $\mathfrak{g}$-modules are examples of tame Γ-Gelfand–Tsetlin $\mathfrak{g}$-modules. For infinite-dimensional $\mathfrak{g}$-modules the situation is very more complicated, for example the Verma Γ-Gelfand–Tsetlin $\mathfrak{g}$-module $M(-\rho)$ is not strongly tame (Futorny-Grantcharov-Ramirez-Zadunaisky, 2020). On the other hand, *generic* modules, or more generally, *relation* modules are tame Γ-Gelfand–Tsetlin $\mathfrak{g}$-modules (Futorny-Ramirez-Zhang, 2019).

Twisted Gelfand-Tsetlin module

Definition

Given $m \in \mathbb{N}$, and $\sigma \in S_m$, a weight $\mathfrak{gl}_m$-module M will be called σ-admissible if M is isomorphic to N^σ for some Γ-relation module N. A weight λ will be σ-admissible if the simple highest weight $\mathfrak{gl}_m$-module $L_{\mathfrak{h}}(\lambda)$ is σ-admissible.

Remark

If M is σ-admissible, all Γ_σ-multiplicities are equal to one (Futorny-Ramirez-Zhang, 2019). The next example shows that the converse is not necessarily true.

Example (Arias, H. M. and Ramirez, 2024)

Let λ be the $\mathfrak{gl}(3)$-weight $\left(-\frac{1}{6}, -\frac{2}{3}, \frac{5}{6}\right)$. The tableaux $T_{id}(\lambda + \bar{\rho})$ and, $T_{s_2}(s_2(\lambda + \bar{\rho}))$ are, respectively

$$
\begin{array}{ccccc}
-\frac{1}{6} & & -\frac{5}{3} & & -\frac{7}{6} \\
& -\frac{1}{6} & & -\frac{5}{3} & \\
& & -\frac{1}{6} & &
\end{array}
\qquad\qquad
\begin{array}{ccccc}
-\frac{1}{6} & & -\frac{7}{6} & & -\frac{5}{3} \\
& -\frac{1}{6} & & -\frac{1}{6} & \\
& & -\frac{1}{6} & &
\end{array}
$$

As $T_{id}(\lambda + \bar{\rho})$ is generic, the module $L(\lambda)$ is admissible. Moreover, all its weight multiplicities are 1, and consequently Γ_σ is diagonalizable for any $\sigma \in S_3$. However, $L(\lambda)$ is not s_2-admissible since the associated tableau $T_{s_2}(s_2(\lambda + \bar{\rho}))$ is critical.

Gelfand-Tsetlin admissible modules

Fixed $\mathfrak{g} = \mathfrak{sl}_3$. Let k be an admissible number, so that $k + 3 = p/q$, $p \geq 3$, $q \geq 1$, p and q are coprime integers. All simple Gelfand-Tsetlin admissible $\mathfrak{sl}_3$-module of an admissible level k (Arakawa, Futorny and Ramirez, 2017). These modules have the same annihilator as $L(\lambda)$, where the $\mathfrak{sl}_3$-weight λ corresponds to an admissible number with denominator q. Moreover, explicit basis and the action of the generators of $\mathfrak{sl}_3$ are given. In this case, exist bounded, unbounded and dense $\mathfrak{sl}_3$-modules.

Open problem

Classify all simple Gelfand-Tsetlin admissible $\mathfrak{sl}_n$-module of an admissible level k for any $n > 3$.

Realization of admissible highest weight modules

Lemma (Futorny, H. M. and Křižka, 2023)

If $\lambda \in \mathfrak{h}^*$ dominant integral or dominant regular, then $L_{\mathfrak{b}}^{\mathfrak{g}}(\lambda)$ is a strongly tame Γ-Gelfand–Tsetlin $\mathfrak{g}$-module.

Let k be admissible, i.e., $k + n + 1 = \frac{p}{q}$ with $p, q \in \mathbb{Z}_{>0}$, $(p, q) = 1$, $p > n$ and $\overline{\mathrm{Pr}}_k$ the set of admissible weight of $\mathfrak{g}$ (Arakawa, 2015). Given that $\lambda \in \overline{\mathrm{Pr}}_k$ implies that λ is dominant regular, we have that

Theorem (Futorny, H. M. and Křižka, 2023)

For $\lambda \in \overline{\mathrm{Pr}}_k$, the simple $\mathfrak{g}$-module $L_{\mathfrak{b}}^{\mathfrak{g}}(\lambda)$ is a strongly tame Γ-Gelfand–Tsetlin $\mathfrak{g}$-module.

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Minimal Orbit

The orbit $\mathbb{O}_{min}$ is the unique minimal non-trivial nilpotent orbit of $\mathfrak{g}$ with $\dim \mathbb{O}_{min} = 2n$. We have the following description of $[\overline{Pr}_k^{\mathbb{O}_{min}}]$:

$$\bar{\lambda} - \frac{ap}{q}\varpi_1 = \left(\lambda_1 - \frac{ap}{q}, \lambda_2, \lambda_3, \ldots, \lambda_{n-1}, \lambda_n\right),$$

where $\lambda_i \in \mathbb{Z}_{\geq 0}$, for all $i = 1, \ldots, n$ are such that $\lambda_1 + \ldots + \lambda_n < p - n$ and $a \in \{1, 2, \ldots, q-1\}$.

Theorem (Futorny, H. M. and Ramirez, 2021)

Any simple admissible highest weight module in the minimal nilpotent orbit is a bounded stronly tame Γ-Gelfand–Tsetlin module.

Example for $\mathfrak{sl}_4$

For $\lambda \in \Lambda_k(\mathfrak{p}^{\max}_{\alpha_1}) \subset \overline{\mathrm{Pr}}_k^{\mathcal{O}_{\min}}$, we have

$$\lambda = \left(\lambda_1 - \tfrac{p}{q}a, \lambda_2, \lambda_3\right) \text{ with } a \in \mathbb{N},\ \ a \leq q-1.$$

We set $v_1 - v_3 = \langle \lambda + \rho, \alpha_1^\vee \rangle \notin \mathbb{Z}$, $v_3 - v_2 = \langle \lambda + \rho, \alpha_2^\vee \rangle \in \mathbb{N}$, $v_2 - v_4 = \langle \lambda + \rho, \alpha_3^\vee \rangle \in \mathbb{N}$ such that $v_1 + v_2 + v_3 + v_4 = -6$.

$$T(v) = \begin{array}{ccccccc} v_1 & & v_3 & & v_2 & & v_4 \\ & v_1 & & v_3 & & v_2 & \\ & & v_1 & & v_2+1 & & \\ & & & v_1 & & & \end{array}$$

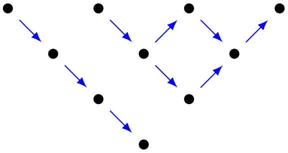
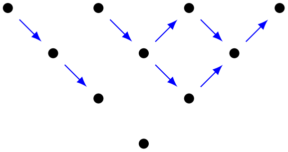
$V_{\mathcal{C}}(T(w))$	$T(w)$	$\mathcal{C}$	$\mathcal{B}_{\mathcal{C}}(T(w))$
$M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$	$T(v)$		$\left\{\begin{matrix} \ell \leq m \leq r \leq 0 \\ -\lambda_2 \leq t \leq 0 \\ -\lambda_3 \leq s \leq 0 \\ s \leq n \leq t - \lambda_2 \end{matrix}\right\}$
$D_f M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$	$T(v)$		$\left\{\begin{matrix} m \leq r \leq 0 \\ -\lambda_2 \leq t \leq 0 \\ -\lambda_3 \leq s \leq 0 \\ s \leq n \leq t - \lambda_2 \end{matrix}\right\}$

Table: Minimal nilpotent orbit

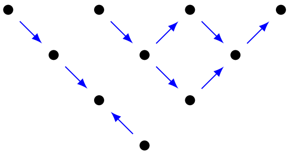
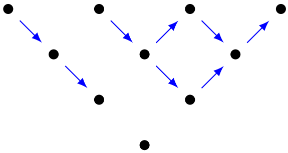
$V_{\mathcal{C}}(T(w))$	$T(w)$	$\mathcal{C}$	$\mathcal{B}_{\mathcal{C}}(T(w))$
$T_f M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$	$T(v+\delta^{1,1})$		$\left\{\begin{array}{c} m \leq r \leq 0 \\ -\lambda_2 \leq t \leq 0 \\ -\lambda_3 \leq s \leq 0 \\ s \leq n \leq t-\lambda_2 \\ m \leq \ell \end{array}\right\}$
$D_f^{\nu} M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$	$T(v+\nu\delta^{1,1})$		$\left\{\begin{array}{c} m \leq r \leq 0 \\ -\lambda_2 \leq t \leq 0 \\ -\lambda_3 \leq s \leq 0 \\ s \leq n \leq t-\lambda_2 \end{array}\right\}$

Table: Minimal nilpotent orbit

Main Theorem: Minimal Orbit

Theorem (Futorny, H. M. and Ramirez, 2021)

Let β be a root of $\mathfrak{g}$, $\mathfrak{b}$ a Borel subalgebra of $\mathfrak{g}$ for which β is a positive root, ρ_β the half-sum of positive (with respect to $\mathfrak{b}$) roots. Let $L_{\mathfrak{b}}(\lambda)$ be an admissible simple $\mathfrak{b}$-highest weight $\mathfrak{g}$-module in the minimal orbit, such that $\langle \lambda, \beta^\vee \rangle \notin \mathbb{Z}$, and $f = f_\beta$. Denote by $A_{\mathfrak{b},\beta}$ the set of all $x \in \mathbb{C} \setminus \mathbb{Z}$ such that $x + \langle \lambda + \rho_\beta, \beta^\vee \rangle \notin \mathbb{Z}$.

a. *The $\mathfrak{g}$-module $D^x_{f_\beta} L_{\mathfrak{b}}(\lambda)$ is admissible in the minimal orbit for any $x \in A_{\mathfrak{b},\beta}$;*
b. *Modules $D^x_{f_\beta} L_{\mathfrak{b}}(\lambda)$, where $\mathfrak{g}_\beta \subset \mathfrak{b}$, $x \in A_{\mathfrak{b},\beta}$, $\langle \lambda, \beta^\vee \rangle \notin \mathbb{Z}$ and $L_{\mathfrak{b}}(\lambda)$ is admissible in the minimal orbit, exhaust all simple $\mathfrak{sl}_2$-induced admissible modules in the minimal orbit. All such modules have bounded weight multiplicities;*
c. *There exists a flag $\mathcal{F}$ such that $D^x_f L_{\mathfrak{b}}(\lambda)$ is $\Gamma_{\mathcal{F}}$-relation Gelfand–Tsetlin $\mathfrak{g}$-module.*

Principal orbit

An element of $[\overline{\mathrm{Pr}}_k^{\mathcal{O}_{\mathrm{prin}}}]$ is represented by the set $\Lambda_k(\mathfrak{b})$, where

$$\Lambda_k(\mathfrak{b}) = \left\{ \mu - \frac{p}{q} \sum_{i=1}^{n} a_i \omega_i;\ \mu \in \overline{\mathrm{Pr}}_{k,\mathbb{Z}},\ a_1, a_2, \ldots, a_n \in \mathbb{N}, \sum_{i=1}^{n} a_i \leq q-1 \right\}.$$

Further, since for $\lambda \in \Lambda_k(\mathfrak{b})$, $\langle \lambda + \rho_{\mathfrak{b}}, \gamma^\vee \rangle \notin \mathbb{Z}$ for all $\gamma \in \Delta_+^{\mathfrak{b}}$, we obtain immediately that a weight $\lambda \in \overline{\mathrm{Pr}}_k^{\mathcal{O}_{\mathrm{prin}}}$ is not only regular dominant but also antidominant.

Theorem (Futorny, H. M. and Křižka, 2023)

If $\lambda \in \overline{\mathrm{Pr}}_k^{\mathcal{O}_{\mathrm{prin}}}$, then $L_{\mathfrak{b}}^{\mathfrak{g}}(\lambda)$ is a strongly tame $\Gamma_{\mathrm{st}}(w(\Pi))$-Gelfand–Tsetlin $\mathfrak{g}$-module for any $w \in W$.

Main Theorem: Principal orbit

Theorem (Futorny, H. M. and Křižka, 2023)

Let $n > 1$, $\lambda \in \overline{\mathrm{Pr}}_k^{\mathcal{O}_{\mathrm{prin}}}$, $\gamma \in \Delta_+^{\mathfrak{b}}$, $\nu \in \mathbb{C} \setminus \mathbb{Z}$ *and* $\nu + \langle \lambda + \rho_{\mathfrak{b}}, \gamma^\vee \rangle \notin \mathbb{Z}$. *Further, let us assume that* $w \in \widetilde{W}$ *satisfies* $\gamma = w(\gamma_1)$. *Then*

- a. $D_{f_\gamma}^{\nu}(L_{\mathfrak{b}}^{\mathfrak{g}}(\lambda))$ *is a simple admissible strongly tame* $\Gamma_{\mathrm{st}}(w(\Pi))$*-Gelfand–Tsetlin* $\mathfrak{g}$*-module which belongs to the principal nilpotent orbit;*
- b. $\mathfrak{g}$*-modules* $D_{f_\gamma}^{\nu}(L_{\mathfrak{b}}^{\mathfrak{g}}(\lambda))$ *for* $\gamma \in \Pi$ *exhaust all simple admissible* $\mathfrak{sl}_2$*-induced* $\mathfrak{g}$*-modules which belongs to the principal nilpotent orbit. All such* $\mathfrak{g}$*-modules have unbounded finite weight multiplicities.*

For $\lambda \in \Lambda_k(\mathfrak{b}) \subset \overline{\mathrm{Pr}}_k^{\mathcal{O}_{\mathrm{prin}}}$, we have

$$\lambda = \left(\lambda_1 - \tfrac{p}{q}a, \lambda_2 - \tfrac{p}{q}b, \lambda_3 - \tfrac{p}{q}a\right) \text{ with } a, b, c \in \mathbb{N} a + b + c \leq q - 1.$$

We set $v_1 - v_3 = \langle \lambda + \rho, \alpha_1^\vee \rangle \notin \mathbb{Z}$, $v_3 - v_2 = \langle \lambda + \rho, \alpha_2^\vee \rangle \notin \mathbb{Z}$, $v_2 - v_4 = \langle \lambda + \rho, \alpha_3^\vee \rangle \notin \mathbb{Z}$ such that $v_1 + v_2 + v_3 + v_4 = -6$.

$$T(v) = \begin{array}{cccccccc} & v_1 & & v_2 & & v_3 & & v_4 \\ & & v_1 & & v_2 & & v_3 & \\ & & & v_1 & & v_2 + 1 & & \\ & & & & v_1 & & & \end{array}$$

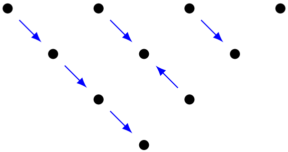
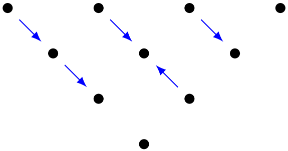
$V_{\mathcal{C}}(T(w))$	$T(w)$	$\mathcal{C}$	$\mathcal{B}_{\mathcal{C}}(T(w))$
$M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$	$T(v)$		$\left\{\begin{matrix} \ell \leq m \leq r \leq 0 \\ s \leq 0 \\ t \leq 0 \\ s \leq n \end{matrix}\right\}$
$D_f M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$	$T(v)$		$\left\{\begin{matrix} m \leq r \leq 0 \\ s \leq 0 \\ t \leq 0 \\ s \leq n \end{matrix}\right\}$

Table: Principal nilpotent orbit

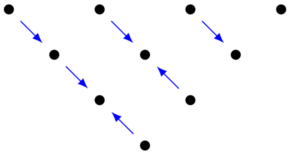
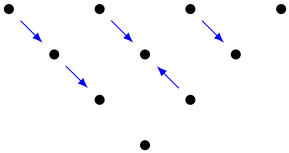
$V_{\mathcal{C}}(T(w))$	$T(w)$	$\mathcal{C}$	$\mathcal{B}_{\mathcal{C}}(T(w))$
$T_f M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$	$T(v+\delta^{1,1})$		$\left\{\begin{matrix} m \leq r \leq 0 \\ s \leq 0 \\ t \leq 0 \\ s \leq n \\ m \leq \ell \end{matrix}\right\}$
$D_f^{\nu} M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$	$T(v+\nu\delta^{1,1})$		$\left\{\begin{matrix} m \leq r \leq 0 \\ s \leq 0 \\ t \leq 0 \\ s \leq n \end{matrix}\right\}$

Table: Principal nilpotent orbit

New results in the Minimal orbit

Theorem (Futorny, H. M. and Křižka, 2024)

Let $L(\lambda)$ be an admissible simple highest weight bounded $\mathfrak{g}$-module, such that $\langle \lambda, \alpha_1^\vee \rangle \notin \mathbb{Z}$, $F = \{f_{\alpha_{1j}} \mid j = 1, 2, \ldots, n\}$ and $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ for some set of complex numbers $\{x_i \mid i = 1, 2, \ldots, n\}$, then

- (a) *$D_F^{\mathbf{x}} L(\lambda)$ is strongly tame Γ_{st}-Gelfand–Tsetlin $\mathfrak{g}$-module if and only if*
$$\sum_{j=i}^{n} x_j + \langle \lambda, \alpha_1^\vee \rangle \notin \mathbb{Z}, \text{ for all } i = 2, \ldots, n;$$
- (b) *The $\mathfrak{g}$-module $D_F^{\mathbf{x}} L(\lambda)$ is simple strongly tame Γ_{st}-Gelfand–Tsetlin $\mathfrak{g}$-module if and only if $x_i \notin \mathbb{Z}$ for all $i = 1, 2, \ldots, n$ and*
$$\sum_{j=i}^{n} x_j + \langle \lambda, \alpha_1^\vee \rangle \notin \mathbb{Z}, \text{ for all } i = 1, 2, \ldots, n.$$

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