Ring theoretical properties of Affine Cellular Algebras.

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jointly with

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Affine Cellular Algebras

D. Kazhdan and G. Lusztig, Inv. Math. 1979
$\longrightarrow$ bases for Hecke algebras, indexed by a poset (Weyl group)
$\longrightarrow$ term "cells" and "cell representations"

J.J. Graham and G.I. Lehrer, "Cellular Algebras", Inv. Math. 1996
$\longrightarrow$ Hecke (type A and B), Brauer, Temperley-Lieb

S. Koenig, C. Xi, J. LMS 1999
$\longrightarrow$ Defn. through ideal structure (cf. quasi-hereditary algebras)

S. Koenig, C. Xi, "Affine Cellular Algebras", Adv. Math 2012
$\longrightarrow$ include affine Hecke algebras and affine Temperley-Lieb algebras

Examples: Kleshchev's graded quasi-hereditary algebras (2015), Khovanov-Lauda-Roquier (2013), Birman-Murakami-Wenzl, affine Brauer,...

Definition

Let R be an associative ring and $\psi \in R$. Then $\tilde{R} = (R, \psi)$ is an associative ring with multiplication

$$a * b = a\psi b, \qquad \forall a, b \in R.$$

$$\varphi : \tilde{R} \to R, \qquad a \mapsto a\psi$$

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Affine Cell ideal

Definition (Koenig-Xi)

An ideal J of a an k-algebra A with k-involution i is called an *affine cell ideal* if $J = i(J)$ and

$$J \simeq (M_n(B), \psi), \qquad \text{as } A\text{-bimodule}$$

for some affine k-algebra B (with k-involution σ & comp. cond.).

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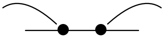
Affine Temperley-Lieb algebras

Let $A = TL_2^a(q)$ be the affine temperley-Lieb algebras on 2 strings over a field K.
The subspace J spanned by all diagrams without through arcs is an ideal of A that is stable under the canonical involution.
There are 2 half diagrams:

$D_1 =$  $D_2 =$ 

Each diagram without through arcs consists of a half diagram on the top and on the bottom and possible a finite number of circles.

Set $V = \mathrm{span}(D_1, D_2)$ and identify J with $V \otimes K[x] \otimes V$.

$V \otimes K[x] \otimes V$ can be also identified with $M_2([x])$ via:

$$D_i \otimes x^n \otimes D_j \mapsto x^n E_{ij}.$$

As algebras and A-bimodule: $J \simeq (M_2(K[x]), \psi)$, where

$$\psi = \begin{bmatrix} q & x \\ x & q \end{bmatrix}$$

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Affine Cellular Algebras

Definition (Koenig-Xi)

An algebra A (with the involution i) is called *affine cellular* if there exists a chain of ideals:

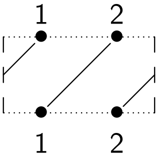
$$0 = J_{-1} \subseteq J_0 \subset J_1 \subset J_2 \subset \cdots \subset J_n = A$$

and $J_j/J_{j-1} \simeq (M_{m_j}(B_j), \psi_j)$ is an affine cell ideal of A/J_{j-1} .

$M_{m_1}(B_1) \times \cdots \times M_{m_n}(B_n)$ is called the *asymptotic algebra* of A.

Affine Temperley-Lieb algebra

Let $A = TL_2^a(q)$ and let $J \simeq (M_2(K[x]), \psi)$ be the ideal spanned by all diagrams without through arcs. Let τ be the diagram:



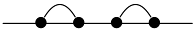
Then any diagram with two through arcs is of the form τ^n for some $n \in \mathbb{Z}$ and

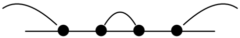
$$A/J \simeq K[\tau, \tau^{-1}].$$

Moreover $0 \subseteq J \subseteq A$ is an affine cellular structure of A.

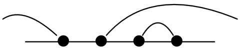
Affine Temperley-Lieb algebras

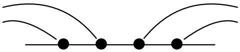
Let $A = TL_4^a(q)$ and J_0 the ideal spanned by the diagrams without through arcs. There are 6 half diagrams:

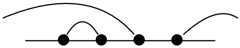
$D_1 =$ 

$D_2 =$ 

$D_3 =$ 

$D_4 =$ 

$D_5 =$ 

$D_6 =$ 

Then $J_0 \simeq (M_6(K[x]), \psi_0)$ with

$$\psi_0 = \begin{bmatrix} q^2 & q & x & qx & qx & q \\ q & q^2 & qx & x & x & x^2 \\ x & qx & q^2 & q & q & qx \\ qx & x & q & q^2 & x^2 & q \\ qx & x & q & x^2 & q^2 & x \\ q & x^2 & qx & q & x & q^2 \end{bmatrix}$$

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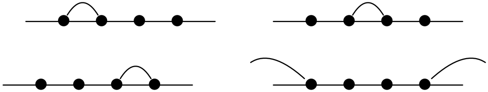
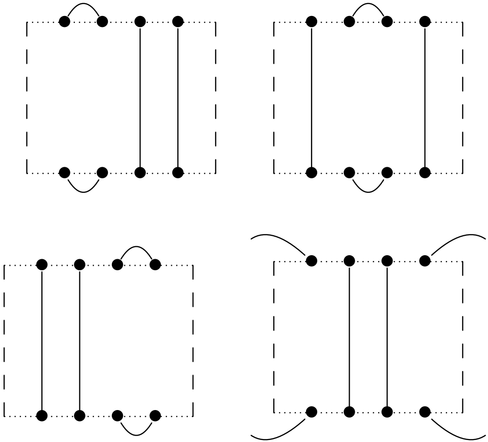
Let J_1 be ideal of all diagrams with at most two through arcs. There are 4 half diagrams of diagrams with exactly two through arcs:



We choose the following diagrams, which will be our matrix units $E_{11}, E_{22}, E_{33}, E_{44}$ in $J_1/J_0 \simeq (M_4(k[y^{\pm 1}]), \psi_1)$:



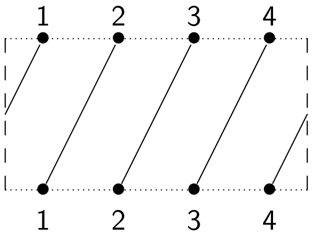
The matrix ψ_1 encoding the multiplication in J_1/J_0 is given as

$$\psi_1 = \begin{pmatrix} q & 1 & 0 & y^{-1} \\ 1 & q & 1 & 0 \\ 0 & 1 & q & y \\ y & 0 & y^{-1} & q \end{pmatrix}$$

where an entry 0 means, that the result leads leads to a diagram without through arcs and hence to an element in J_0.
A power of y means that the result is a "twisted" version of a "standard" diagram.

$$J_1/J_0 \simeq (M_4(K[y, y^{-1}], \psi_1).$$

Note that $A/J_1 \simeq B = K[\tau, \tau^{-1}]$, where τ is the diagram:



In particular $0 \subset J_0 \subseteq J_1 \subseteq A$ is an affine cell structure.

Polynomial identities

Lemma

Any affine cellular algebra satisfies a polynomial identity.

If R has a PI f, then $\tilde{R}$ satisfies f^2.
If J and R/J satisfy a PI, then so does R.

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Affine Cell ideals

Lemma

Let J be an affine cell ideal of A with $J \simeq (M_n(B), \psi)$.

$$I := \langle \psi_{ij} : i, j \rangle \subseteq B.$$

1. *If B/I is f.g. k-module, then J is f.g. as left A-module.*
2. *J is an idempotent ideal if and only if $B = I$.*
3. *J is principal if and only if $\det(\psi)^{-1} \in B$.*

In case (3): $J = Ae$, $e^2 = e$ central and $A \simeq A/J \times M_n(B)$.

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Embeddings

Let $J \simeq (M_n(B), \psi)$ be an affine cell ideal of A.
For any $a \in A$ let $\rho_a : J \to J$ be the right multiplication of a on J.
Define the ring homomorphism

$$\Phi : A \to A/J \times \mathrm{End}({}_A J), \qquad a \mapsto (a + J, \rho_a).$$

Theorem

Φ is injective if and only if $\det(\psi)$ is not a zero divisor in B.

When is $\mathrm{End}({}_A J) \simeq M_n(B)$?

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Semiprime algebras

Theorem

Let A be an affine cellular algebra. The following statements are equivalent:

(a) A/J_j *is semiprime for all* j;

(b) B_j *is reduced and* $\det(\psi_j)$ *is not a zero divisor in* B_j *for all* j;

(c) $\Phi : A \to \mathrm{End}\left({}_{A/J_{m-1}}J_m/J_{m-1}\right) \times \cdots \times \mathrm{End}({}_A J_0)$ *is an embedding and* B_j *is reduced for all* j.

In any of these cases $\mathrm{End}({}_{A/J_{j-1}}J_j/J_{j-1}) \simeq M_{n_j}(B_j)$, *for all* j *and*

$$c(A) = \Phi^{-1}\left(B_m \times \cdots \times B_0\right).$$

Here $m = \min\{l \mid \text{r.ann}_{A/J_{l-1}}(J_l/J_{l-1}) = 0\}$.

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Affine Temperley-Lieb algebras

$A = TL_2^a(q)$: $\det(\psi) = q^2 - x^2 \neq 0$, leads to an embedding

$$\Phi : A \to K[\tau, \tau^{-1}] \times M_2\left(K[x]\right).$$

$A = TL_4^a(q)$: the determinants are non-zero

$$\det(\psi_0) = -(x-q)^4(x+q)^2(x^2 - f(q)x - g(q))(x^2 + f(-q)x + h(q))$$
$$\det(\psi_1) = -y^{-2} - y^2 + q^4 - 4q^2 + 2$$

$$\Phi : A \to K[\tau, \tau^{-1}] \times M_4\left(K[y, y^{-1}]\right) \times M_6\left(K[x]\right).$$

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"Semisimple" case

If $\det(\psi_j)$ *is invertible in* B_j *for all* j*, then*

$$A \simeq M_{m_1}(B_1) \times \cdots \times M_{m_n}(B_n).$$

Moreover the Gelfand-Kirillov dimension of A *is*

$$GKdim(A) = \max\left(Kdim(B_1), \ldots, Kdim(B_n)\right).$$

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Gelfand-Kirillov dimension

Corollary

Let k be a field and A an affine cellular k-algebra with cell chain

$$0 = J_{-1} \subset J_0 \subset \cdots \subset J_n = A,$$

such that $J_j/J_{j-1} \simeq (M_{m_j}(B_j), \psi_j)$ for $1 \leq j \leq n$. Suppose B_j is reduced and $\det(\psi_j)$ is not a zero divisor in B_j for all j. Then

$$GKdim(A) \leq \max\left(Kdim(B_1), \ldots, Kdim(B_m)\right),$$

where $m = \min\{l \mid r.ann_{A/J_{l-1}}(J_l/J_{l-1}) = 0\}$.

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Corollary

The affine Temperley-Lieb algebra $A = TL_n^a(q)$ *is a semiprime Noetherian PI-algebra with* $GKdim(A) = 1$ *for all but finitely many specialisations of the parameter* q*. Moreover, its centre* $c(A)$ *is an affine* k*-algebra of Krull dimension* 1*,* A *is finitely generated over* $c(A)$ *and embeds into its asymptotic algebra.*

Open Problem

Question

When are affine cellular algebras Noetherian?

Theorem (Posner)

A semiprime PI-ring R is right Noetherian and finitely generated over its centre $c(R)$ if and only if $c(R)$ is a Noetherian ring.

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Example (Schelter, 1976)

Extensions: $K \leq K_1, K_2 \leq L$ with $K = K_1 \cap K_2$ and $[L : K_i] < \infty$.

$$A = \begin{pmatrix} K_1 + xL[x] & xL[x] \\ xL[x] & K_2 + xL[x] \end{pmatrix} = \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix} \oplus xM_2\left(L[x]\right).$$

$$J_0 = xM_2(L[x]) \simeq (M_2(B_0), \psi) \text{ with } B_0 = L[x]; \ \psi = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}.$$

$A/J_0 \simeq K_1 \times K_2 = B_1$.

$$c(A) = \left\{ \begin{pmatrix} k + xf & 0 \\ 0 & k + xf \end{pmatrix} \mid k \in K, f \in L[x] \right\} \simeq K \oplus xL[x].$$

A is Noetherian PI and $c(A)$ is non-Noetherian if $[L : K] = \infty$.

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Question

Is the centre of an affine cellular algebra affine cellular?

Thank you for your attention!

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