Supercharacter Sheaves for Algebra Groups

João Dias

Universidade de Lisboa

Seminar of Representation Theory and Related Areas

17/12/2016

Supercharacter Theory

Definition of a Supercharacter Theory for a finite group G

A pair $(\mathcal{X}, \mathcal{K})$, with $\mathcal{X}$ a collection of characters of G and $\mathcal{K}$ a partition of G is a Supercharacter Theory if:

1. Each irreducible character of G is constituent of an unique character of $\mathcal{X}$,
2. $|\mathcal{X}| = |\mathcal{K}|$,
3. The characters $\chi \in \mathcal{X}$ are constant on the elements of $\mathcal{K}$,
4. $\{1\} \in \mathcal{K}$.

Definition of a Supercharacter Theory for a finite group G

A pair $(\mathcal{X}, \mathcal{K})$, with $\mathcal{X}$ a collection of characters of G and $\mathcal{K}$ a partition of G is a Supercharacter Theory if:

1. Each irreducible character of G is constituent of an unique character of $\mathcal{X}$,
2. $|\mathcal{X}| = |\mathcal{K}|$,
3. The characters $\chi \in \mathcal{X}$ are constant on the elements of $\mathcal{K}$,
4. $\{1\} \in \mathcal{K}$.

Idempotents and Supercharacters

We have a bijection between:

- $\{\{e_i\}_{i \in I}\}$, with $\{e_i\}_{i \in I}$ a decomposition of the unit in $\mathbb{C}G$ in central ortogonal idempotents, with e_1 the trivial character,
- $\{(\mathcal{X}, \mathcal{K}) \mid (\mathcal{X}, \mathcal{K})$ a supercharacter theory for $G\}/\sim$

Some facts about Supercharacters

Given a supercharacter theory on G we have:

- The superclasses are union of conjugacy classes to each other,
- The supercharacters are ortogonal,
- For each supercharacter $\chi \in \mathcal{X}$ and an element $k \in K$ in a superclass we have that $\chi(k)\frac{|K|}{\chi(1)}$ is an algebraic integer.
- The supercharacters are multiples of $\sum_{\chi \in X} \chi(1)\chi$,
- The supercharacters determine uniquely the superclasses.

Algebra groups

Consider $G = 1 + J$ with J a finite dimensional $\mathbb{F}_q$ nilpotent algebra without unity.

Algebra groups

Consider $G = 1 + J$ with J a finite dimensional $\mathbb{F}_q$ nilpotent algebra without unity.

Action on the Algebra groups

We have the following actions of $G \times G$ in G, J, and $Irr(J, +)$

- $(g, h).(x) = gxh^{-1}$,
- $(g, h).(1 + x) = 1 + gxh^{-1}$,
- $((g, h).\lambda)(x) = \lambda(g^{-1}xh)$.

Supercharacter theory for Algebra groups

We define a supercharacter theory as follows:

- $\mathcal{K} = \{1 + GxG \mid x \in J\}$,
- $\mathcal{X} = \{\frac{1}{n_\lambda} \sum_{\mu \in G\lambda G} \hat{\mu} \mid \lambda \in Irr(J, +)\}$,

with $n_\lambda = \frac{|G\lambda G|}{|\lambda G|}$ and $\hat{\mu}(1 + x) = \mu(x)$.

Supercharacter theory for Algebra groups

We define a supercharacter theory as follows:

- $\mathcal{K} = \{1 + GxG \mid x \in J\}$,
- $\mathcal{X} = \{\frac{1}{n_\lambda} \sum_{\mu \in G\lambda G} \hat{\mu} \mid \lambda \in Irr(J, +)\}$,

with $n_\lambda = \frac{|G\lambda G|}{|\lambda G|}$ and $\hat{\mu}(1 + x) = \mu(x)$.

An equivalent way to define the supercharacters

We also have that the set:

$$\mathcal{X}' = \{Ind_{R_\lambda}^{G}(\widehat{\lambda_{|R_\lambda}}) \mid \lambda \in Irr(J, +)\},$$

where $R_\lambda = \{g \in G | g\lambda = \lambda\}$, defines the same supercharacter theory.

Supercharacter Sheaves

Algebra groups in algebraic geometry

Consider $G_0 = 1 + J_0$ with J_0 a finite dimensional $\mathbb{F}_q$ nilpotent algebra without unity. And let:

- $G = G_0 \otimes \mathbb{F}$ a connected $\mathbb{F}$ algebraic group,
- $J = J_0 \otimes \mathbb{F}$ an abelian connected $\mathbb{F}$ algebraic group,
- J^* the Serre Dual, it's an abelian connected $\mathbb{F}$ algebraic group.

We have that $(J^*)^{F^n} = Irr(J_0 \otimes \mathbb{F}_{q^n}, +)$.

Algebra groups in algebraic geometry

Consider $G_0 = 1 + J_0$ with J_0 a finite dimensional $\mathbb{F}_q$ nilpotent algebra without unity. And let:

- $G = G_0 \otimes \mathbb{F}$ a connected $\mathbb{F}$ algebraic group,
- $J = J_0 \otimes \mathbb{F}$ an abelian connected $\mathbb{F}$ algebraic group,
- J^* the Serre Dual, it's an abelian connected $\mathbb{F}$ algebraic group.

We have that $(J^*)^{F^n} = Irr(J_0 \otimes \mathbb{F}_{q^n}, +)$.

Derived category of constructible complexes of $\ell-$adic sheaves

Consider the category $\mathcal{D}(G) = D_c^b(G, \mathbb{Q}_\ell)$ and the bifunctor:

$$M * N := \mu_!(\pi_1^*(M) \otimes \pi_2^*(N)).$$

And if H is acting in G denote by $\mathcal{D}_H(G)$ the full subcategory of $\mathcal{D}(G)$ with elements $a^* M \simeq \pi^* M$.

Minimal idempotents in $\mathcal{D}_H(G)$

An element $e \in \mathcal{D}_H(G)$ is called idempotent if $e * e \simeq e$, and it's a minimal idempotent if for all idempotent $f \in \mathcal{D}_H$, we have $e * f \simeq e$ or $e * f \simeq 0$.

Minimal idempotents in $\mathcal{D}_H(G)$

An element $e \in \mathcal{D}_H(G)$ is called idempotent if $e * e \simeq e$, and it's a minimal idempotent if for all idempotent $f \in \mathcal{D}_H$, we have $e * f \simeq e$ or $e * f \simeq 0$.

Definition of Supercharacter Sheaves

Let H be a group action on G, such that the conjugacy action is a subaction.
Then a supercharacter sheaf $\mathcal{L}$ (for the supercharacter theory defined by H) is an indecomposable perverse sheaf such that $\mathcal{L} * e \simeq \mathcal{L}$, for some $e \in \mathcal{D}_H(G)$ minimal idempotent.

Action on the algebra group G

We extend the actions of $G_0 \times G_0$ on G_0 and J_0 to action on $G \times G$ on G and J. So we have:

- A compatible action on J^* of $G \times G$ (taking the fixed points defines the same action as above),
- If Ω is a biorbit, a right orbit or a conjugacy class then Ω^{F^n} is a biorbit, a right orbit or a conjugacy class respectively for G^{F^n}.
- For each n it defines the same supercharacter theory as above for G^{F^n}.

Equivalence Functors

We have the following functors:

$$\mathcal{D}_{G\times G}(G) \xleftarrow{\phi^*} \mathcal{D}_{G\times G}(J) \xleftarrow{\mathcal{F}} \mathcal{D}_{G\times G}(J^*).$$

With $\phi : G \to J$ and $\mathcal{F}$ the Fourier-Deligne transform. That satisfy:

- They are equivalence of categories,
- $\mathcal{F}(M \otimes N) \simeq \mathcal{F}(M) *_J \mathcal{F}(N)$,
- $\phi^*(M *_J N) \simeq \phi^* M *_G \phi^* N$,
- They perserve minimal idempotents,
- They perserve preversety.

Construction of Supercharacter Sheaves

Let $\Omega \subset J^*$ be a biorbit, and $(\bar{\mathbb{Q}}_\ell)_\Omega$ be the (unique) indecomposable complex of $\ell-$adic sheaves on Ω, then $\phi^*(\mathcal{F}(i_\Omega)_!((\bar{\mathbb{Q}}_\ell)_\Omega))$ is a supercharacter sheaf.
Furthermore all supercharacter sheaves are of that form.

Construction of Supercharacter Sheaves

Let $\Omega \subset J^*$ be a biorbit, and $(\bar{\mathbb{Q}}_\ell)_\Omega$ be the (unique) indecomposable complex of $\ell-$adic sheaves on Ω, then $\phi^*(\mathcal{F}(i_\Omega)_!((\bar{\mathbb{Q}}_\ell)_\Omega))$ is a supercharacter sheaf.
Furthermore all supercharacter sheaves are of that form.

Relations of Supercharacter Sheaves and Biorbits

We have the bijection:

$$\mathcal{S} : PSCS(G) \to \{\Omega \subset J^* \mid \Omega \text{ is a sum of biorbits}\}$$

With $PSCS(G)$ the collection of supercharacter sheaves and his (tensor) products, such that $\mathcal{S}(\mathcal{L}_1 \otimes \mathcal{L}_2) = \mathcal{S}(\mathcal{L}_1) + \mathcal{S}(\mathcal{L}_2)$, and $F^n(\mathcal{S}(\mathcal{L})) = \mathcal{S}(\mathcal{L})$ if and only if $(F^n)^*\mathcal{L} \simeq \mathcal{L}$.

Properties of Supercharacter Sheaves

- Exists a (minimal) collection of supercharacter sheaves $\{\mathcal{L}_i\}$, called indecomposable, such that all supercharacter sheaf is a product of some indecomposable.
- A supercharacter sheaf associated to a biorbit Ω is a character sheaf if and only if Ω is a conjugacy orbit.
- The supercharacter sheaves define a partition of the character sheaves $\{\{M \in CS(G) \mid M * \mathcal{L} \simeq M\} \mid \mathcal{L} \in SCS(G)\}$
- We have a ("well behaved") isomorphism between $< \mathcal{L} \in SCS(G) \mid (F^n)^*\mathcal{L} \simeq \mathcal{L} >_{\mathbb{C}}$ and $\mathbf{SC}(G^{F^n})$ (where the first is the free $\mathbb{C}-$algebra generated by the supercharacter sheaves with the (tensor) product and convolution.)

References

- A motivated introduction to character sheaves and the orbit method for unipotent groups in positive characteristic,
  by M. Boyarchenko and V. Drinfeld, in Preprint, September 2006, arXiv:math.RT/0609769.
- Characters of unipotent groups over finite fields,
  by M. Boyarchenko , in Selecta Math. 16 (2010),no. 4, 857–933.
- Character sheaves on unipotent groups in positive characteristic: foundations,
  by M. Boyarchenko and V. Drinfeld, in Preprint, August 2011 arXiv:0810.0794.
- Supercharacters and superclasses for algebra groups,
  by Diaconis, Persi and Isaacs, I, in Transactions of the American Mathematical Society 360.5 (2008): 2359-2392.