An Asymptotic Supercharacter theory for $U_\infty(\mathbb{F}_q)$

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Outline

- The infinite unitriangular group: an overview
- The dual space and Bi-action: Basic pairs
- Super-Representations
- Supercharacters: Asymptotic approximation
- Super-Representations $\leftrightarrow$ Basic Pairs $\leftrightarrow$ Asymptotic-Supercharacters

The infinite unitriangular group

The infinite analogue to $U_n(\mathbb{F}_q)$

- Denote $U_n(\mathbb{F}_q) = G_n = 1 + \mathfrak{u}_n$ the $n \times n$ unitriangular group over $\mathbb{F}_q$
- $G_n \hookrightarrow G_{n+1}$
- $U_\infty(\mathbb{F}_q) = \bigcup_{n \in \mathbb{N}} G_n = \varinjlim G_n = G_\infty$
- Discrete group; Unitary representation theory

The dual space $\mathfrak{u}_\infty^*$

- $\mathfrak{u}_n \simeq \mathfrak{u}_n^*$
- $\mathfrak{u}_\infty^* = \hom_{\mathbb{F}_q}(\mathfrak{u}_\infty, \mathbb{F}_q)$
- $\mathfrak{u}_\infty = \varinjlim \mathfrak{u}_n \;\Rightarrow\; \hom_{\mathbb{F}_q}(\mathfrak{u}_\infty, \mathbb{F}_q) = \varprojlim \hom_{\mathbb{F}_q}(\mathfrak{u}_n, \mathbb{F}_q)$
- We conclude that $\mathfrak{u}_\infty^* \simeq \overline{\mathfrak{u}_\infty}$

Bi-action and Super-representations

- Fix $\theta : \mathbb{F}_q^+ \to \mathbb{C}$ a non-trivial group morphism
- Given $f \in \mathfrak{u}_\infty^*,\ g, h \in G_\infty,\ x \in \mathfrak{u}_\infty$ define

$$gfh(x) := f(hxg)$$

- Let $G_f = \{g \in G_\infty :\ gf = f\}$, and $\lambda_f : G_f \to \mathbb{C}$

$$\lambda_f(1 + x) = \theta(f(x))$$

- $(\pi_f, V_f) = Ind_{G_f}^{G_\infty}(\lambda_f, \mathbb{C})$

Basic pairs and Super-representations

- $\Phi(n) = \{(i,j) : \ 1 \leq i < j \leq n\}$
- $\Phi(\infty) = \{(i,j) : \ 1 \leq i < j \leq \infty\}$
- $D \subseteq \phi(n)$ basic if it has at most one non-zero entry in each line and column
- $D \subseteq \Phi(\infty)$ basic if there is a family of basic sets $D_n \subseteq \Phi(n)$

$$D_n \to D$$

- $\phi : D \to \mathbb{F}_q \setminus 0$, s.t $\phi(\infty) = 1$; (D, ϕ) is said to be a basic pair

Basic pairs and Super-representations

Proposition

There is a bijection between the Bi-orbits of $\mathfrak{u}_\infty^*$ and the basic pairs of $\Phi(\infty)$. Moreover If f_1 and f_2 are in the same Bi-orbit, then

$$\pi_{f_1} \simeq \pi_{f_2}$$

- $f \leftrightarrow (D, \phi)$ we denote $\pi_f = \pi_{D,\phi}$
- There is an explicit model of $(\pi_{D,\phi}, V_{D,\phi})$

Main properties of Super-representations

- The *elementary* Super-representations $\pi^{\alpha}_{i,j}$ are irreducible an the corresponding dimension is $j-i-1$
- for $j<\infty$, $\pi^{\alpha}_{i,j}$ is type I and $\pi_{i,\infty}$ is type II
- $\pi_{D,\phi} \simeq \bigotimes_{(i,j)\in D} \pi_{i,j}^{\phi(i,j)}$
- "$\pi_{D_n,\phi_n} \xrightarrow{n} \pi_{D,\phi}$"

Asymptotic Supercharacters

Supercharacters and Superclasses

- $\chi : G_\infty \to \mathbb{C}$
  - positive definite
  - class function
  - $\chi(1) = 1$
- $g = 1 + x \in G_\infty,\ K_g := 1 + GxG$

Supercharacters

A supercharacter of G_∞ is a character constant in superclasses.

Supercharacters and Super-representations

- Basic characters of G_n are parametrized by basic pairs of $\Phi(n)$:
  - They form a Supercharacter theory
  - $\xi_{D,\phi} = Ind_{G_D}^{G_n} \lambda_{D,\phi}$

Lemma

For all converging sequence $\{(D_n, \phi_n) \subseteq \Phi(n)\}$ such that $(D_n, \phi_n) \to (D, \phi)$, the following limit exists, for all $g \in G_\infty$:

$$\chi_{D,\phi}(g) := \lim_{n \to \infty} \hat{\xi}_{D_n,\phi_n}(g)$$

Moreover, $\chi_{D,\phi}$ is a supercharacter.

Supercharacters and Super-representations

Proposition

For all (D, ϕ) basic pair, the representation $\pi_{D,\phi}$ yields the supercharacter

$\chi_{D,\phi}$

Main properties

- There is an explicit formula for $\chi_{D,\phi}$
- There is a measure ω in $\mathfrak{u}_\infty^*$ such that $\chi_{D,\phi} = \chi_f = \int\limits_{\mu \in GfG} \tilde{\mu} d\omega(\mu)$
- $\chi_{D,\phi} = \prod\limits_{(i,j) \in D} \chi_{i,j}^{\phi(i,j)}$

Extreme Supercharacters and The ergodic method

- The set of all Supercharacters form a Choquet's simplex:
  - All supercharacters have a unique extreme-supercharacter integral decomposition
- The restriction to of basic characters of G_{n+1} to G_n defines a ramification graph Γ
- To Central measures of Γ correspond supercharacters

Extreme Supercharacters and The ergodic method

- Adapting the Birkhoff's ergodic theorem, one has:

Proposition

For all extreme supercharacter ψ, there is a family of basic characters ξ_n such that

$$\psi = \lim_{n \to \infty} \hat{\xi}_n$$

In particular $extSC(G_\infty) \subseteq ASC(G_\infty)$

In progress (Carlos André + Filipe Gomes)

One has the equality

$$extSC(G_\infty) = ASC(G_\infty)$$

Extreme Supercharacters and The ergodic method

- The set of all characters is a Choquet's Simplex

In progress

Let λ be an extreme character and π_λ the corresponding representation.

There is one and only one basic pair $(D, \phi) \subseteq \Phi(\infty)$ such that

$$\pi_\lambda \leq \pi_{D,\phi}$$

Final remarks

Some final remarks

- $ASC(G_\infty)$ depends on the inclusion $G_n \hookrightarrow G_{n+1}$
- G_∞ is a prototype of "infinite algebra groups" and McLain Groups
- G_∞ is a discrete subgroup of $\varprojlim G_n$

Thank you

Questions ?