A recursive presentation of switching on ballot tableau pairs
or
the many faces of the involutive nature of Littlewood-Richardson coefficient commutativity

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Workshop on Representation Theory and Related areas

Lisboa, December 17, 2016

Plan

1. Motivation: LR coefficients as structure constants *versus* combinatorial numbers
2. LR tableaux, Gelfand-Tsetlin patterns (and LR hives)
3. Involution commutators of LR tableaux (and LR hives)
    - based on the Schützenberger involution/jeu de taquin
    - our involution commutator based on internal Schensted insertion

Littlewood-Richardson coefficients as structure constants

- *The ring of symmetric polynomials: the product of Schur polynomials*. Let $x = (x_1, \dots, x_d)$ be a sequence of indeterminates. Schur polynomials $s_\lambda(x)$ for all partitions λ with $\ell(\lambda) \le d$, form a $\mathbb{Z}$-linear basis for the ring $\Lambda_d := \mathbb{Z}[x]^{\mathfrak{S}_d}$ of symmetric polynomials in x,

$$s_\mu s_\nu = \sum_{\substack{\lambda \\ \ell(\lambda)\le d}} c^{\lambda}_{\mu\,\nu} s_\lambda, \quad c^{\lambda}_{\mu\,\nu} \in \mathbb{Z}^{+}_{\ge 0}.$$

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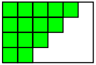
- *Schubert calculus of Grassmannians: the product in the cohomology ring of Grassmannians*. Schur polynomials $s_\lambda(x)$ with λ inside a rectangle $d \times (n-d)$ $(0 < d < n)$ may be interpreted as representatives of Schubert classes σ_λ



Schubert classes $\{\sigma_\lambda\}_{\lambda \subseteq n \times (n-d)}$ form a $\mathbb{Z}$-linear basis for the cohomology ring $H^*(G(d,n))$ of the Grassmannian $G(d,n)$ (the set of all complex d-dimensional linear subspaces of $\mathbb{C}^n$), and

$$\sigma_\mu \sigma_\nu = \sum_{\lambda \subseteq d \times (n-d)} c^{\lambda}_{\mu\,\nu} \sigma_\lambda.$$

Littlewood-Richardson coefficients as structure constants

- These numbers $c^{\lambda}_{\mu,\nu}$ also arise as *tensor product multiplicities*.
  - general linear group $GL_d(\mathbb{C})$. Schur polynomials $s_\lambda(x)$ may be interpreted as irreducible characters of $GL_d(\mathbb{C})$. The decomposition of the tensor product of two irreducible polynomial representations V^μ and V^ν of $GL_d(\mathbb{C})$ into irreducible representations of $GL_d(\mathbb{C})$, is given by

$$V^\mu \otimes V^\nu = \bigoplus_{\ell(\lambda)\leq d} V^{\lambda^{\oplus c^{\lambda}_{\mu\,\nu}}}.$$

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  - $U_q(\mathfrak{gl}_d)$ the quantum group of $\mathfrak{gl}_d$: $\mathfrak{gl}_d$-crystal bases. Let B_λ denote the crystal basis of the irreducible representation V_λ of $U_q(\mathfrak{gl}_d)$. B_λ can be taken to be the set of all SSYTs of shape λ, in the alphabet $\{1,\dots,d\}$, equipped with crystal operators. The decomposition of the tensor product of $\mathfrak{gl}_d$-crystals is given by

$$B_\mu \otimes B_\nu \cong \bigoplus_{\substack{\lambda \\ T\in\mathcal{LR}(\lambda/\mu,\nu)}} B_\lambda(T),$$

with $B_\lambda(T) \cong B_\lambda$.

- *Positivity of Littlewood-Richardson coefficients in matrix existence problems*. There exist $d \times d$ non singular matrices A, B and $C = AB$, over a *discrete valuation ring*, with Smith invariants μ, ν and λ respectively iff $c_{\mu\,\nu}^{\lambda} > 0$.

- *Positivity of Littlewood-Richardson coefficients in matrix existence problems*. There exist $d \times d$ non singular matrices A, B and $C = AB$, over a *discrete valuation ring*, with Smith invariants μ, ν and λ respectively iff $c^{\lambda}_{\mu\,\nu} > 0$.

- *The commutativity of Littlewood-Richardson coefficients*

$$c^{\lambda}_{\mu\nu} = c^{\lambda}_{\nu\mu}$$

$$c^{\lambda}_{\mu\nu} > 0 \text{ iff } c^{\lambda}_{\nu\mu} > 0.$$

Our structure coefficients are combinatorial numbers

- The structure coefficient $c_{\mu,\nu}^{\lambda}$ is

  the cardinality of an explicit set of combinatorial objects.

- It is possible to determine $c_{\mu,\nu}^{\lambda} > 0$ without determining its exact value.
- We are interested in exploiting symmetry properties of these combinatorial objects.
- In particular, we are interested in exhibiting the commutativity symmetry

$$c_{\mu,\nu}^{\lambda} = c_{\nu,\mu}^{\lambda}.$$

Littlewood-Richardson coefficients as numbers which count

- The Littlewood-Richardson (LR) rule (*D.E. Littlewood and A. Richardson 34; M.-P. Schützenberger 77; G.P. Thomas 74*) states that the coefficients appearing in the expansion of a product of Schur polynomials s_μ and s_ν

$$s_\mu(x)\, s_\nu(x) = \sum_\lambda c^\lambda_{\mu\nu}\, s_\lambda(x)$$

are given by

$$\begin{aligned} c^\lambda_{\mu\nu} &= \#\{T \in \mathcal{LR}(\lambda/\mu,\nu)\} = \#\mathcal{LR}(\lambda/\mu,\nu) \\ c^\lambda_{\nu,\mu} &= \#\{T \in \mathcal{LR}(\lambda/\nu,\mu)\} = \#\mathcal{LR}(\lambda/\nu,\mu). \end{aligned}$$

$\mathcal{LR}(\lambda/\mu,\nu)$ the set of Littlewood-Richardson tableaux of shape λ/μ and weight ν.

- The connection to the cohomology of Grassmannians was made by L. Lesieur (1947).

Littlewood-Richardson tableaux (*D.E. Littlewood and A. Richardson, 1934*) or ballot tableaux

- A Young tableau T of shape λ/μ is said to be a Littlewood-Richardson tableau if
  - it is *semistandard* (SSYT)
    1. the entries in each row of λ/μ are weakly increasing from left to right, and
    2. the entries in each column of λ/μ are strictly increasing from top to bottom,
  - and satisfy the *lattice permutation property or ballot condition*
    1. the content of each initial segment of the reading word, right to left across rows and top to bottom, is a partition.

		1	1	2
	1	2		
1	3			

2112131

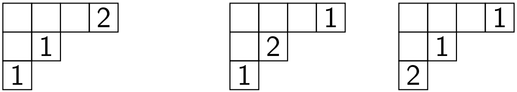
		1	1	1
	2	2		
1	3			

1112231

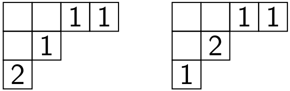
		1	1	1
	1	2		
2	3			

1112132

one has 3 candidates 1,2,3 each receiving 4,2,1 votes respectively. A particular ordering of the votes is then a sequence of length 7 where at any stage candidate 1 has at least many votes as candidate 2, and candidate 2 has at least many votes as candidate 3.



$$c^{421}_{31,\,21} = 2$$



$$c^{421}_{21,\,31} = 2$$

The split of LR tableaux into Gelfand-Tsetlin patterns led to hives

- I.M. Gelfand, A.V. Zelevinsky (1986), A.D. Berenstein, A.V.Zelevinsky (1989)

$T =$

							1	1
					1	1	2	2
			1	1	2			
1	2	2	3					
3								

$\mu = 75300 \quad \nu = 75200 \quad \lambda = 99641$

$$G_\mu = \begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & 0 & & 0 & & & \\ & & 3 & & 1 & & 0 & & \\ & 5 & & 5 & & 3 & & 1 & \\ 7 & & 7 & & 6 & & 4 & & 1 \end{array} \qquad G_\nu = \begin{array}{ccccccccc} & & & & 7 & & & & \\ & & & 7 & & 5 & & & \\ & & 6 & & 5 & & 2 & & \\ & 4 & & 3 & & 1 & & 0 & \\ 2 & & 2 & & 0 & & 0 & & 0 \end{array} \qquad G_\lambda = \begin{array}{ccccccccc} & & & & 7 & & & & \\ & & & 7 & & 5 & & & \\ & & 9 & & 6 & & 2 & & \\ & 9 & & 8 & & 4 & & 1 & \\ 9 & & 9 & & 6 & & 4 & & 1 \end{array}$$

$T_\mu =$

1	2	2	2	3	3	4
2	3	3	4	4		
4	5	5				

$T_\nu =$

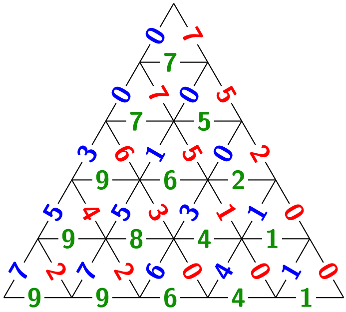
1	1	2	2	3	3	4
2	2	3	4	4		
4	5					

$T_\lambda =$

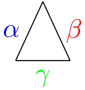
1	1	1	1	1	1	1	3	3
2	2	2	2	2	3	4	4	5
3	3	4	4	5	5			
4	5	5	5					
5								

$2 \subseteq 42 \subseteq 630 \subseteq$

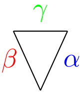
Interlock the three GT patterns



- A *hive in the edge representation form, R.C. King, C. Tollu, F. Toumazet (2006)*, is a labelling of all edges of a planar, equilateral triangular graph satisfying the triangle and the betweeness conditions



$\alpha + \beta = \gamma$



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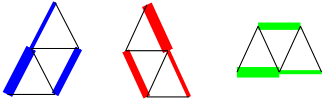
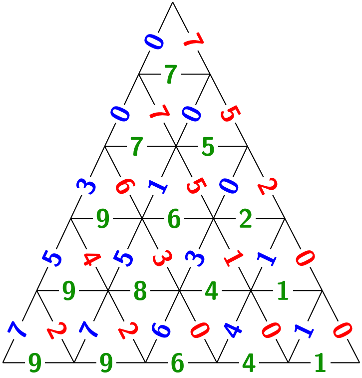
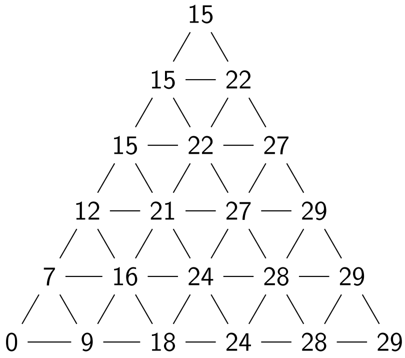
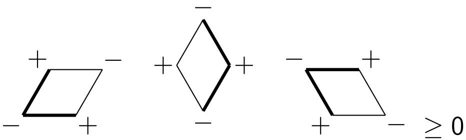


Edge and vertex representation of a hive A. Knutson, T. Tao (1999), and A.S. Buch (2000)

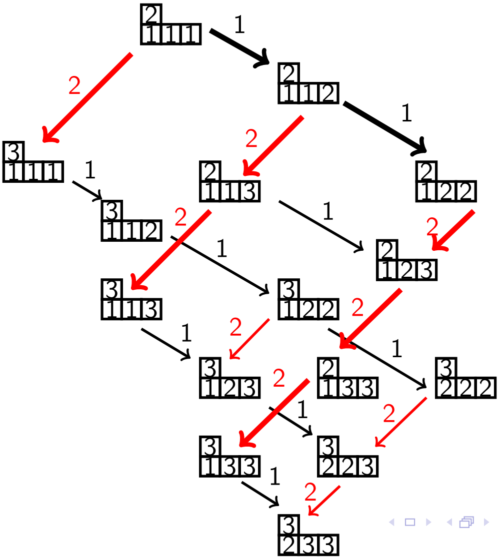






Interlocking GT-pairs and tensor product of $\mathfrak{gl}_n$-crystals

- The crystal basis B_λ of the irreducible representation V_λ of $U_q(\mathfrak{gl}_n)$, can be taken to be the set of all SSYTs of shape λ, in the alphabet $[n]$, equipped with crystal operators.



Example

$$U = \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \qquad V = \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & 4 \\ \hline 4 \\ \cline{1-1}\end{array} \qquad U \otimes V \cong (U \leftarrow V) = \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \leftarrow 24234 =$$

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 2 & 3 \\ \cline{1-2}\end{array} \leftarrow 4234 = \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 2 & 3 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array} \leftarrow 234 = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 2 & 2 \\ \hline 2 & 3 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array} \leftarrow 34 =$$

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 2 & 2 \\ \hline 2 & 3 \\ \cline{1-2} 3 & 4 \\ \cline{1-2}\end{array} \leftarrow 4 = P(U \otimes V) = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 2 & 2 \\ \hline 2 & 3 \\ \cline{1-2} 3 & 4 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array} \qquad Q(U \otimes V) = \begin{array}{|c|c|c|c|}\hline & & & 1 \\ \hline & 1 \\ \cline{1-2} 2 & 2 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}$$

- The map $U \otimes V \to (P(U \otimes V), Q(U \otimes V))$ gives the $\mathfrak{gl}_n$-isomorphism

$$B_\mu \otimes B_\nu \cong \bigoplus_{\substack{\lambda \\ T \in \mathcal{LR}(\lambda/\mu,\nu)}} B_\lambda(T),$$

with $B_\lambda(T) = B_\lambda \times \{T\} \cong B_\lambda$. The multiplicity of B_λ in $B_\mu \otimes B_\nu$ is

$$\#\text{highest (lowest) weight elements of weight } \lambda \ (rev\lambda) \text{ in } B_\mu \otimes B_\nu = |\mathcal{LR}(\lambda/\mu,\nu)| = c^\lambda_{\mu,\nu}$$

- *The lowest and the highest weight elements of B_λ in $B_\mu \otimes B_\nu$.*

  The $c^{\lambda}_{\mu,\nu}$ crystal connected components in $B_\mu \otimes B_\nu$ with highest weight λ are distinguished by

  - highest weight element $Y_\mu \otimes T_\nu$, and
  - lowest weight element $T_\mu \otimes Y_{rev\,\nu}$,
    where (T_μ, T_ν) is the GT-pattern pair of $T \in \mathcal{LR}(\lambda/\mu, \nu)$.

Involution commutators and the Schützenberger involution

- Henriques-Kamnitzer involution LR commutator (arxiv 2004). For each $T \in \mathcal{LR}(\lambda/\mu,\nu)$ there exists $T^* \in \mathcal{LR}(\lambda/\nu,\mu)$ such that the map

$$\begin{array}{ccc} B(\mu)\otimes B(\nu) & \to & B(\nu)\otimes B(\mu) \\ U\otimes V & \mapsto & \xi(V)\otimes\xi(U) \end{array}$$

sends

$$\begin{array}{ccccc} Y_\mu\otimes T_\nu & \to & T^*_\nu\otimes\xi Y_\mu, & & \xi T_\nu = T^*_\nu \\ T_\mu\otimes\xi Y_\nu & \to & Y_\nu\otimes T^*_\mu, & & \xi T_\mu = T^*_\mu. \end{array}$$

For each $T \in \mathcal{LR}(\lambda/\mu,\nu)$ there exists $T^* \in \mathcal{LR}(\lambda/\nu,\mu)$ such that

$$T^*_\nu = \xi T_\nu,\ \ T^*_\mu = \xi T_\mu, \quad \xi \text{ the Schützenberger involution.}$$

$$\begin{array}{rccc} Com_{HK}: \mathcal{LR}(\lambda/\mu,\nu) & \to & \mathcal{LR}(\lambda/\nu,\mu) \\ T & \to & T^* \end{array} \quad \text{is an involution.}$$

- Pak-Vallejo LR commutator bijections (arxiv 2004):

$$\begin{array}{rcll} \rho_2 : \mathcal{LR}(\lambda/\mu,\nu) & \to & \mathcal{LR}(\lambda/\nu,\mu) & \\ T & \to & Q : & Q_\nu = \xi T_\nu \end{array}$$

$$\begin{array}{rcll} \rho_2^{-1} : \mathcal{LR}(\lambda/\mu,\nu) & \to & \mathcal{LR}(\lambda/\nu,\mu) & \\ T & \to & U : & U_\mu = \xi T_\mu \end{array}$$

Conjectured to be an involution $\rho_2 = \rho_2^{-1} \Leftrightarrow Q = U$. It is exactly the H-K commutator.

Example:

- $$T = \begin{array}{cccccccc} \square & \square & \square & \square & \square & \square & 1 & 1 \\ \square & \square & \square & \square & \square & 1 & & \\ \square & \square & 1 & 2 & 2 & & & \\ 1 & 2 & 2 & 3 & & & & \end{array} \longrightarrow T_\nu = \begin{array}{ccccc} 1 & 1 & 2 & 3 & 4 \\ 3 & 3 & 4 & 4 & \\ 4 & & & & \end{array}$$

  $$\longrightarrow \xi T_\nu = \begin{array}{ccccc} 1 & 1 & 1 & 1 & 2 \\ 2 & 2 & 4 & 4 & \\ 3 & & & & \end{array} \longrightarrow Q = \begin{array}{cccccccc} \square & \square & \square & \square & \square & 1 & 1 & 1 \\ \square & \square & \square & \square & 1 & 2 & & \\ \square & 1 & 2 & 2 & 2 & & & \\ 1 & 2 & 3 & 3 & & & & \end{array}$$

Benkart,Sottile-Stroomer switching on ballot tableau pairs: ρ_1

$$\begin{array}{cc}\blacksquare & b\\ a & \end{array} \to \begin{array}{cc} a & b\\ \blacksquare & \end{array}, \quad a\le b \qquad\qquad \begin{array}{cc}\blacksquare & b\\ a & \end{array} \to \begin{array}{cc} b & \blacksquare\\ a & \end{array}, \quad a > b$$

$$Y_\mu \cup T = \begin{array}{cccccc} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & 1 & 1\\ \mathbf{2} & \mathbf{2} & \mathbf{2} & \mathbf{2} & 2 & \\ \mathbf{3} & 1 & 2 & 3 & & \\ \mathbf{4} & 2 & 3 & 4 & & \end{array} \to \begin{array}{cccccc} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & 1 & 1\\ \mathbf{2} & \mathbf{2} & \mathbf{2} & \mathbf{2} & 2 & \\ 1 & 2 & 3 & \mathbf{3} & & \\ 2 & 3 & 4 & \mathbf{4} & & \end{array}$$

$$\begin{array}{cccccc} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & 1 & 1\\ \mathbf{2} & \mathbf{2} & 2 & \mathbf{2} & \mathbf{2} & \\ 1 & 2 & 3 & \mathbf{3} & & \\ 2 & 3 & 4 & \mathbf{4} & & \end{array} \to \begin{array}{cccccc} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & 1 & 1\\ 1 & 2 & 2 & \mathbf{2} & \mathbf{2} & \\ \mathbf{2} & \mathbf{2} & 3 & \mathbf{3} & & \\ 2 & 3 & 4 & \mathbf{4} & & \end{array}$$

Benkart-Sottile-Stroomer switching on ballot tableau pairs: ρ_1

1	**1**	**1**	**1**	1	1
1	2	2	**2**	**2**	
2	3	3	**3**		
2	**2**	4	**4**		

$\rightarrow$

1	**1**	**1**	**1**	1	1
1	2	2	**2**	**2**	
2	3	3	**3**		
4	**2**	**2**	**4**		

1	1	1	**1**	**1**	**1**
1	2	2	**2**	**2**	
2	3	3	**3**		
4	**2**	**2**	**4**		

$\rightarrow$

1	1	1	**1**	**1**	**1**
2	2	2	**2**	**2**	
1	3	3	**3**		
4	**2**	**2**	**4**		

1	1	1	**1**	**1**	**1**
2	2	2	**2**	**2**	
3	3	**1**	**3**		
4	**2**	**2**	**4**		

$= Y_\nu \cup U = \rho_1(Y_\mu \cup T) \quad U \equiv Y_\mu, \ Y_\nu \equiv T.$

Internal and External insertion on skew-tableaux

For skew tableaux there are two types of row insertion: external and internal. External insertion is similar to Schensted's original procedure.

Sagan-Stanley internal insertion operator $\bar{\phi}_i$ on $Y \cup T$, Y Yamanouchi tableau, T a skew-tableau:

$$Y \cup T = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 1 & 2 & 2 \\ \cline{1-5} 3 & \textcircled{2} & 3 & 3 \\ \cline{1-4} 1 & 3 & 4 & 4 \\ \cline{1-4} 5 & 5 \\ \cline{1-2} \end{array} \qquad \bar{\phi}_3(Y \cup T) = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 1 & 2 & 2 \\ \cline{1-5} 3 & \textcircled{3} & 3 & 3 \\ \cline{1-4} 1 & \textcircled{2} & 4 & 4 \\ \cline{1-4} \textcircled{3} & 5 \\ \cline{1-2} \textcircled{5} \\ \cline{1-1} \end{array}$$

- Definition of internal insertion operator $\bar{\phi}_i$ on $Y_\mu \cup T$. Need $(i, \mu_i + 1)$ to be an inner corner of T:
  - $\bar{\phi}_i$ bumps the entry, say x, in the inner corner cell $(i, \mu_i + 1)$ of T, and replaces it with i, and then inserts (externally as in the Schensted procedure) the bumped element x in the sutableau consisting of the last $n - i$ rows of T.

Internal insertion Knuth relations on skew-tableaux

Elementary Knuth transformations on words:

$$kij \equiv kji = \begin{array}{|c|c|}\hline i & j \\ \hline k & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\,, \quad i < k \leq j, \quad ijk \equiv jik = \begin{array}{|c|c|}\hline i & k \\ \hline j & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\,, \quad i \leq k < j.$$

Lemma

(A. 2016) $Y \cup T$ with n rows. Whenever the compositions are defined it holds:

$$\bar{\phi}_k\bar{\phi}_i\bar{\phi}_n(Y \cup T) = \bar{\phi}_k\bar{\phi}_n\bar{\phi}_i(Y \cup T),\ \ 1 \leq i < k \leq n,$$
$$\bar{\phi}_i\bar{\phi}_n\bar{\phi}_k(Y \cup T) = \bar{\phi}_n\bar{\phi}_i\bar{\phi}_k(Y \cup T),\ \ 1 \leq i \leq k < n.$$

Proposition

(A. 2016; Internal insertion Knuth relations on skew-tableaux.) $Y \cup T$ with n rows.

$$\bar{\phi}_k\bar{\phi}_i\bar{\phi}_j(Y \cup T) = \bar{\phi}_k\bar{\phi}_j\bar{\phi}_i(Y \cup T),\ \ 1 \leq i < k \leq j \leq n,$$
$$\bar{\phi}_i\bar{\phi}_j\bar{\phi}_k(Y \cup T) = \bar{\phi}_j\bar{\phi}_i\bar{\phi}_k(Y \cup T),\ \ 1 \leq i \leq k < j \leq n.$$

Proof.

The action of $\bar{\phi}_k\bar{\phi}_i\bar{\phi}_j$ and $\bar{\phi}_k\bar{\phi}_j\bar{\phi}_i$ ($\bar{\phi}_i\bar{\phi}_j\bar{\phi}_k$ and $\bar{\phi}_j\bar{\phi}_i\bar{\phi}_k$) on $Y \cup T$ inserts Knuth equivalent words in the subtableau consisting of the last $n - j$ rows of $Y \cup T$. □

(A. 99; A., King, Terada, 2016) Involution commutator ρ_3

$$Y_\mu \cup T = \begin{array}{|c|c|c|c|c|c|} \hline 1&1&1&1&1&1\\ \hline 2&2&2&2&2& \\ \cline{1-5} 3&1&2&3 \\ \cline{1-4} 4&2&3&4 \\ \cline{1-4} \end{array} \qquad T_\nu = \begin{array}{ccccccc} &&&2&&& \\ &&2&&1&& \\ &3&&2&&1& \\ 3&&3&&2&&1 \end{array}$$

$$\emptyset \cup \emptyset \to \begin{array}{|c|c|c|c|c|c|} \hline 1&1&1&1&1&1 \\ \hline \end{array} \to \begin{array}{|c|c|c|c|c|c|} \hline 1&1&1&1&1&1 \\ \hline 2 \\ \cline{1-1} \end{array} \to \begin{array}{|c|c|c|c|c|c|} \hline 1&1&1&1&1&1 \\ \hline 2&2&2&2&2 \\ \cline{1-5} \end{array}$$

$$\begin{array}{|c|c|c|c|c|c|} \hline 1&1&1&1&1&1 \\ \hline 2&2&2&2&2 \\ \cline{1-5} 3 \\ \cline{1-1} \end{array} \to \begin{array}{|c|c|c|c|c|c|} \hline 1&1&1&1&1&1 \\ \hline 2&2&2&2&2 \\ \cline{1-5} 3&2 \\ \cline{1-2} \end{array} \to \begin{array}{|c|c|c|c|c|c|} \hline 1&1&1&1&1&1 \\ \hline 2&2&1&2&2 \\ \cline{1-5} 3&2&2 \\ \cline{1-3} \end{array} \to \begin{array}{|c|c|c|c|c|c|} \hline 1&1&1&1&1&1 \\ \hline 2&2&1&2&2 \\ \cline{1-5} 3&2&2&3 \\ \cline{1-4} \end{array}$$

$$\begin{array}{|c|c|c|c|c|c|} \hline 1&1&1&1&1&1 \\ \hline 2&2&1&2&2 \\ \cline{1-5} 3&2&2&3 \\ \cline{1-4} 4 \\ \cline{1-1} \end{array} \to \begin{array}{|c|c|c|c|c|c|} \hline 1&1&1&1&1&1 \\ \hline 2&2&1&2&2 \\ \cline{1-5} 3&3&2&3 \\ \cline{1-4} 4&2 \\ \cline{1-2} \end{array} \to \begin{array}{|c|c|c|c|c|c|} \hline 1&1&1&1&1&1 \\ \hline 2&2&2&2&2 \\ \cline{1-5} 3&3&1&3 \\ \cline{1-4} 4&2&2 \\ \cline{1-3} \end{array} \to \begin{array}{|c|c|c|c|c|c|} \hline 1&1&1&1&1&1 \\ \hline 2&2&2&2&2 \\ \cline{1-5} 3&3&1&3 \\ \cline{1-4} 4&2&2&4 \\ \cline{1-4} \end{array} = Y_\nu \cup U$$