Relating Chomsky Normal Form and Greibach Normal Form by Exponential Transposition

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http://www.iti.cs.tu-bs.de/~koslowj/RESEARCH

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- At issue is the use of node-labeled trees in the theory of formal languages.

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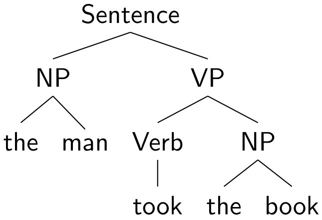

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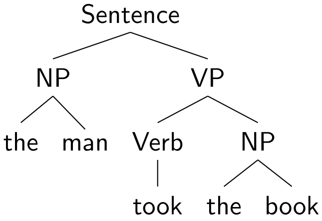

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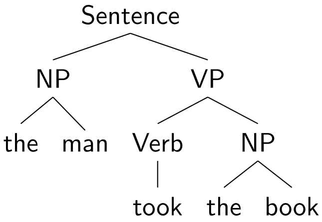

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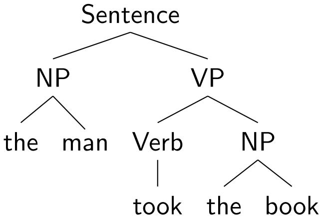

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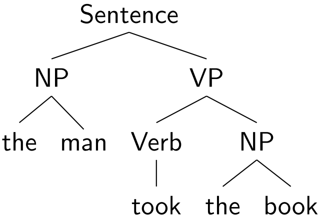

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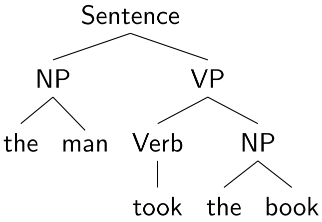

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Note the distinction between leaves and (capitalized) inner nodes.

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Node-labeled trees in the 1960's also formed the basis for the new field of tree grammars/automata/languages, see Thatcher's survey of 1973.

Background on grammars and normal forms

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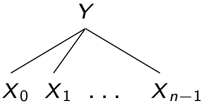
The words $w \in \mathcal{T}^*$ with $S \Rightarrow^* w$ constitute the language generated by G, where $\Rightarrow^*$ is the reflexive transitive hull of $\Rightarrow$.

Traditional tree descriptions of normal-form productions:



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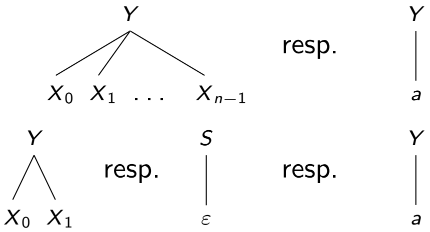

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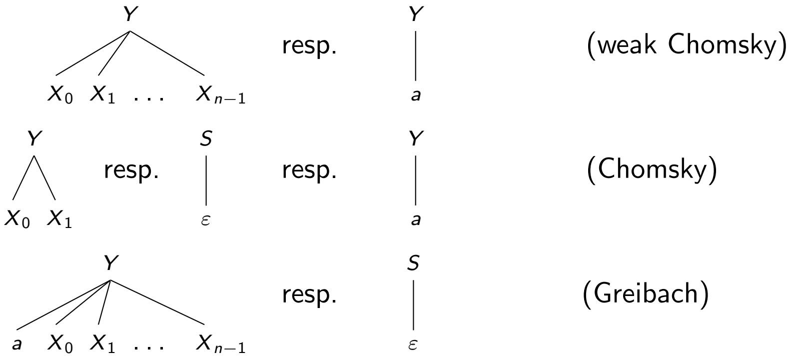

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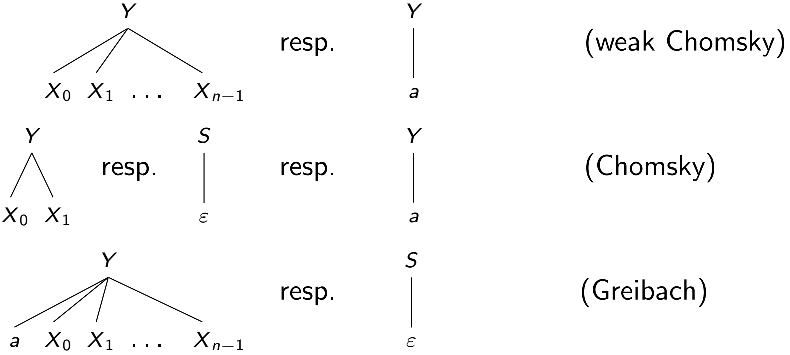

Strictly speaking, the tree for $S \longrightarrow \varepsilon$ is not correct; it should be just a leaf with nonterminal S. However, this is hard to distinguish from cases, where the derivation is not yet finished.

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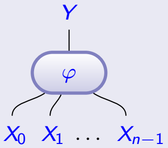

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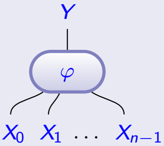

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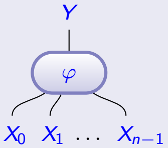

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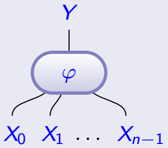

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A CFG à la Walters over $\mathcal{T}$ is a faithful multigraph morphism $\boldsymbol{G} \xrightarrow{\gamma} \mathcal{T}_{\langle 0 \rangle}$ with $\boldsymbol{G}$ finite,

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To describe the language generated by γ as directly as possible, we take a different approach from that of Walters.

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The intention is to have the default multiarrows obey certain identifications in the free multicategory $\mathcal{T}^*_{\langle 0 \rangle}$; hence its construction needs to be revised:

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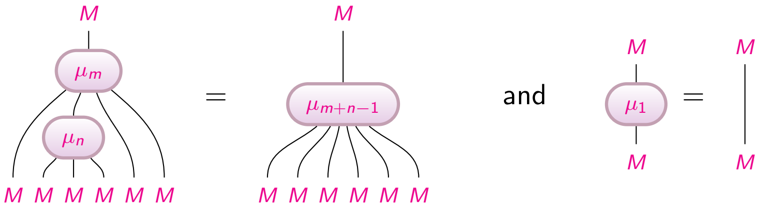
Identifications in $\mathcal{T}^*_{\langle 0 \rangle}$



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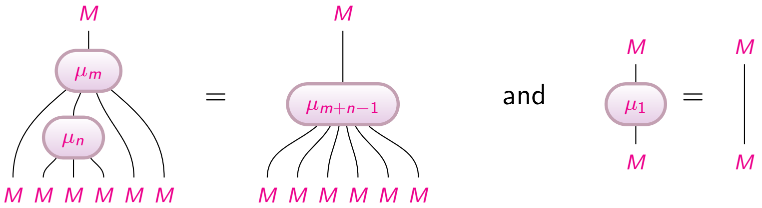

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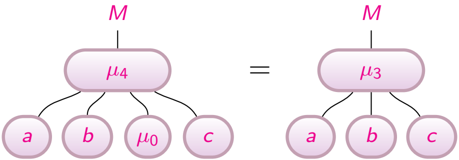

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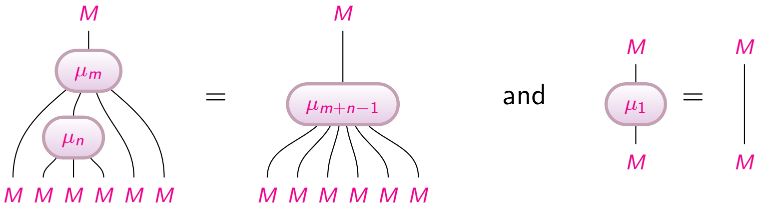

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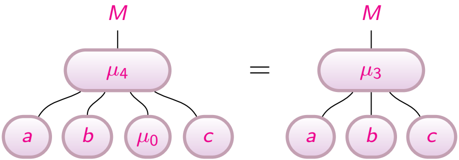

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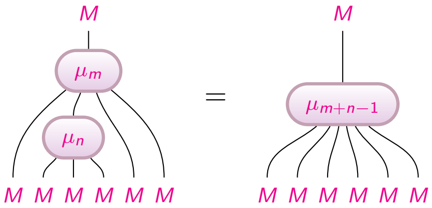

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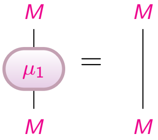

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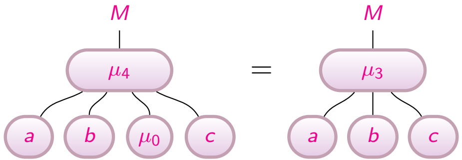

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This motivates us to write ε not only for μ_0, but also for μ_n, $n \in \mathbb{N}$.

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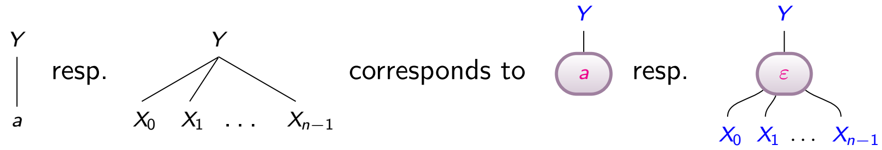

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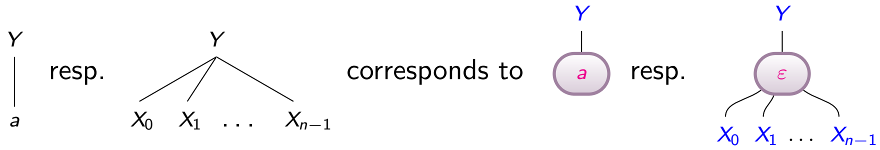

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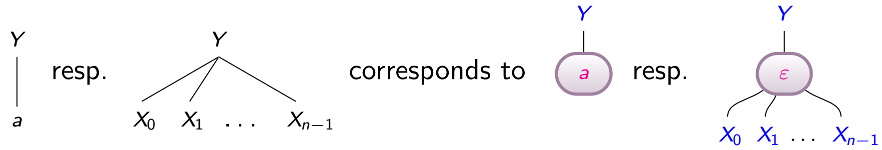

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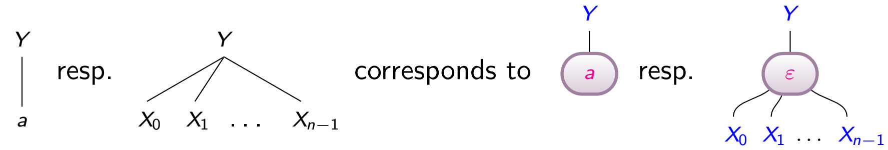

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Besides a certain elegance of the new approach and the better handling of ε-productions ("peanuts"), how do we "sell" this to computer scientists or the tree-people (Ents?), who seem to be perfectly happy with the traditional approach?

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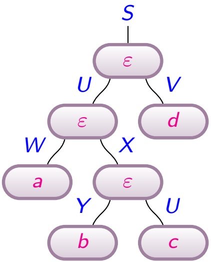

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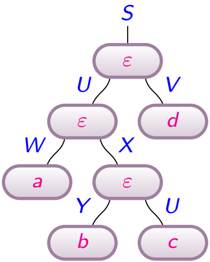

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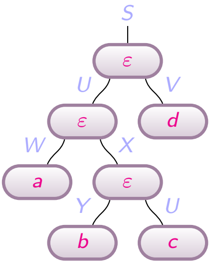

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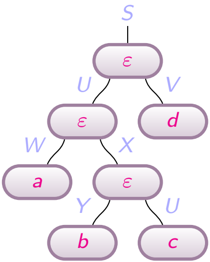

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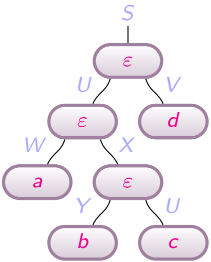

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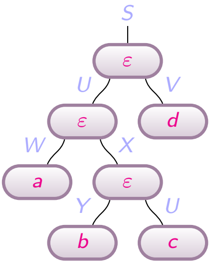

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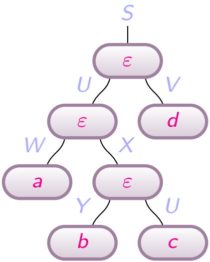

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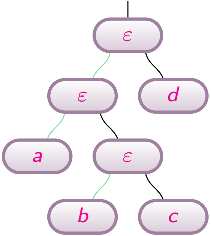

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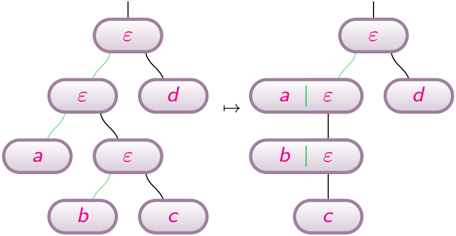

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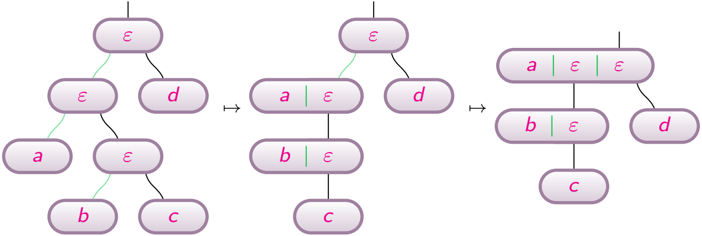

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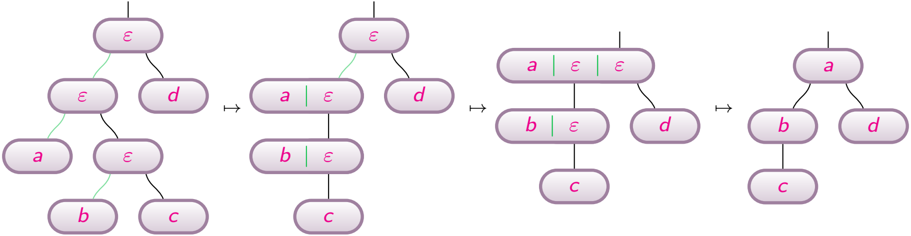

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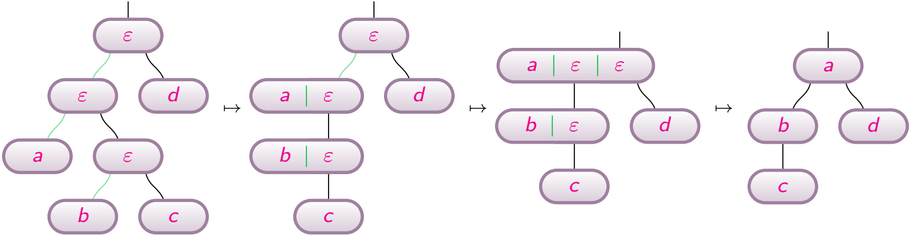

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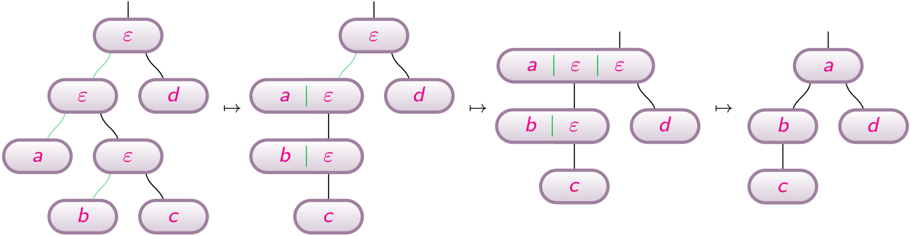

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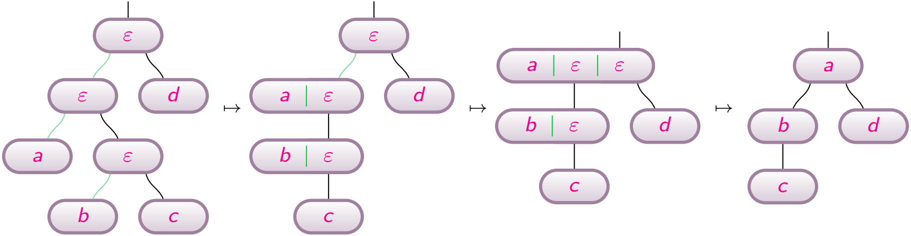

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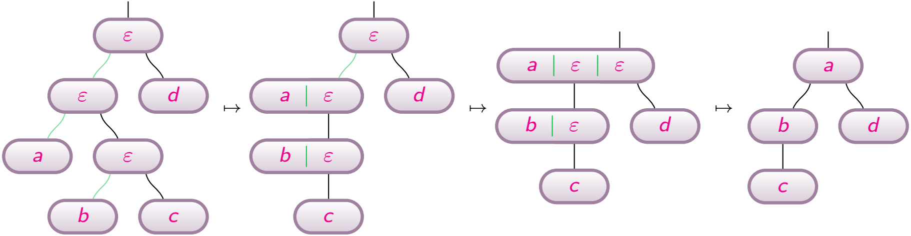

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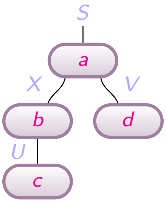
(There is a second (better?) isomorphism utilizing reverse Polish notation.)



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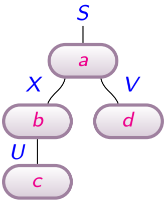

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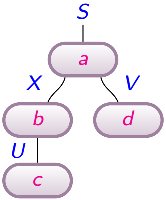

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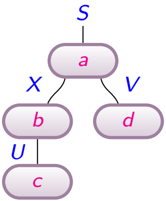

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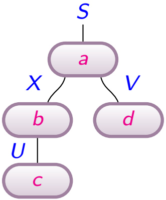

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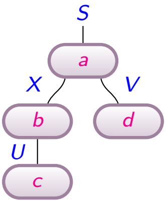

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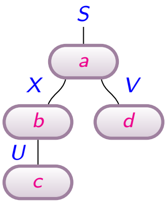

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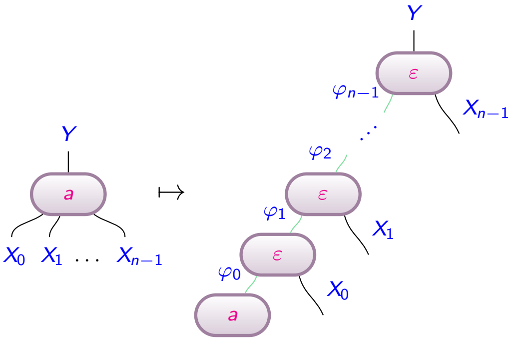

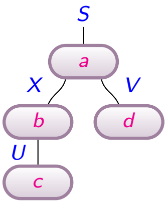

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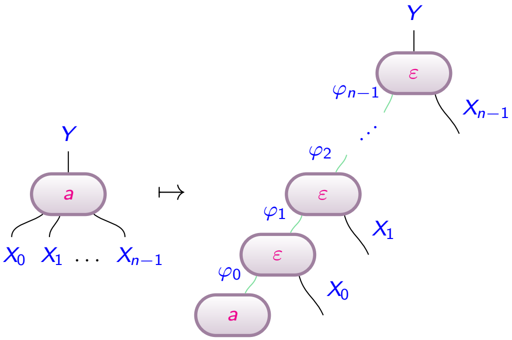

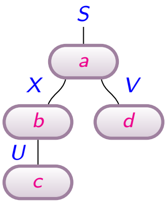

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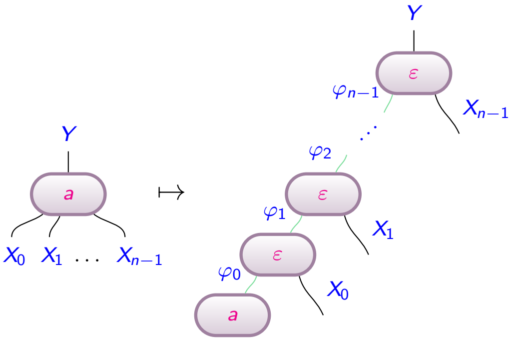

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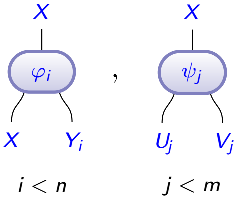

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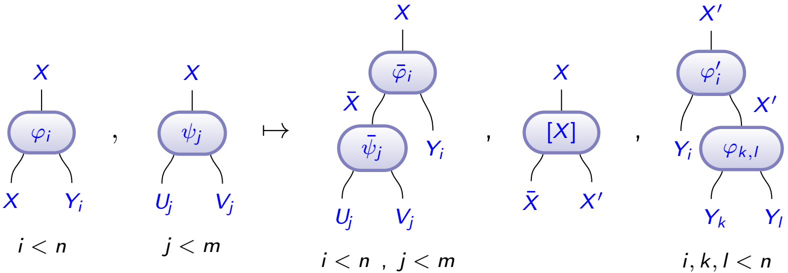

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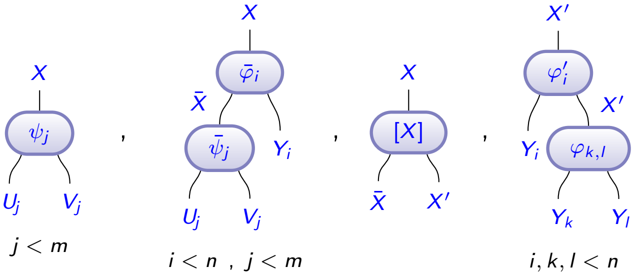

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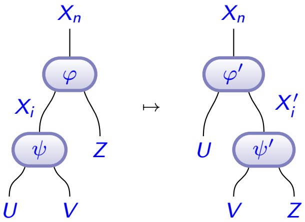

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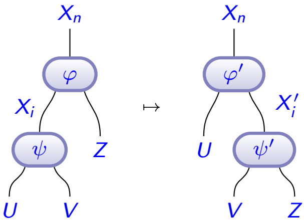

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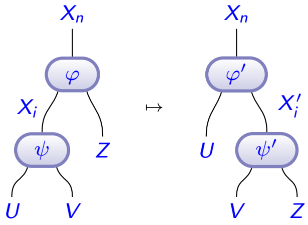

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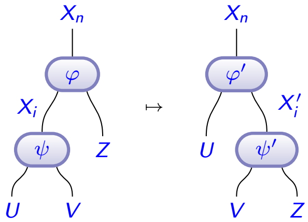

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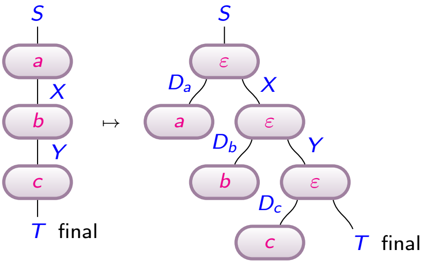

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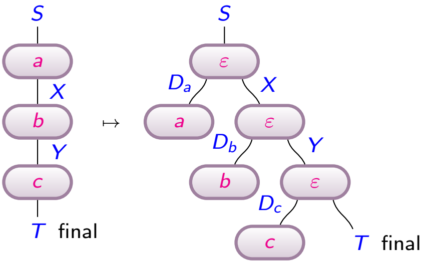

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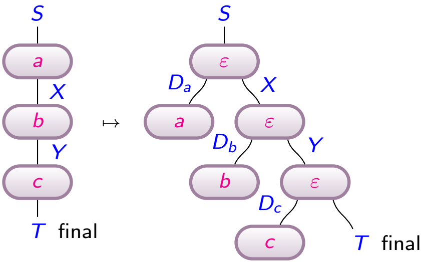

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Final states provide an externally imposed mechanism for termination, as $\mathcal{T}_{\langle 1 \rangle}$ has no default for this.

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The familiar bijection for labeled transition systems

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Closing up under "vertical" composition, we obtain bijections

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Oplax (monoidal) natural transformations turn out to be the appropriate type of morphism in the "denominators" to encode simulations.

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$$\frac{\boldsymbol{G} \xrightarrow{\gamma} \mathcal{T}^\star_{\langle \mathbb{N} \rangle}}{\mathcal{T}^\star_{\langle \mathbb{N} \rangle} \xrightarrow{\Gamma} \boldsymbol{rel}} \qquad \begin{array}{l}\text{(faithful strict monoidal functor)} \\ \text{(strict monoidal lax functor)}\end{array}$$

where $\boldsymbol{G}$ now is a (strict monoidal) category. Again, general (strict monoidal) categories can serve as controls instead of $\mathcal{T}^*_{\langle 1 \rangle}$, resp., $\mathcal{T}^\star_{\langle \mathbb{N} \rangle}$.

Oplax (monoidal) natural transformations turn out to be the appropriate type of morphism in the "denominators" to encode simulations.

Instead of $\boldsymbol{rel}$, matrix categories over other rigs yield further instances of this phenomenon, like probabilistic or weighted transition systems.

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How can we extend pure PDA's as above in a finite fashion in order to handle general grammars?

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A combination of the last two ideas indeed will do the trick.

Polygraphs with linear adjoints

Recall that planar poly-bicategories as well as polycategories admit a very nice notion of so-called linear adjoints, together with a mate calculus.

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2-PDA's with external states are well-known to be equivalent to Turing machines.

To do

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- Work out the details for polygraph comprehension.

Thank you!