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Quasi-toposes as elementary quotient completions

In [11] the notion of *elementary quotient completion* of an elementary doctrine[1] is introduced as a generalization of the notion of the exact completion of a category with finite products and weak equalizers, presented in [4], see also [2, 10, 13] for other examples.

Such a completion is the free elementary doctrine with stable effective quotients of equivalence relations (in the sense of the doctrine). In general the base category of the completion need not be exact though the exact completion of a category with finite limits turns out to be an instance of this construction.

In this talk we focus on the special class of Lawvere's elementary doctrines called *triposes*, introduced in [7], to build elementary toposes by means of what is now known as the tripos-to-topos construction, see [5]. We characterize those triposes whose elementary quotient completion is an arithmetic quasi-topos—*i.e.* a quasi-topos equipped with a natural number object—as base category.

To obtain the characterization, we extend some known results about exact completions such as Carboni and Vitale's characterization of exact completions in terms of its projective objects in [4], Menni's characterization of the exact completions which are toposes in [12] and Carboni and Rosolini's characterization of the locally cartesian closed exact completions [3]. In particular, we show that

- an elementary doctrine $P : \mathbb{C}^{op} \longrightarrow \mathbf{InfSL}$ closed under effective quotients is the elementary quotient completion of the doctrine determined by the restriction of P to the full subcategory of $\mathbb{C}$ on its projective objects;

- the base category of the elementary quotient completion of P turns weak universal properties of $\mathbb{C}$ into (strong) universal properties of the base of the elementary quotient completion. Those include binary co-products, a natural number object, a parametrized list object, a subobject classifier, a cartesian closed structure, a locally cartesian structure.

We conclude by pointing out some relevant examples of arithmetic quasi-toposes arising as non-exact elementary quotient completions. Most notably they include the category of equilogical spaces of [14, 15, 1], that of assemblies over a partial combinatory algebra (see [6, 16]), and the category of total setoids, in the style of E. Bishop, over Coquand and Paulin's Calculus of Inductive Constructions which is the theory at the base of the proof-assistant Coq.

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[1] Following Lawvere [8, 9], an elementary doctrine is a functor $P : \mathbb{C}^{op} \longrightarrow \mathbf{InfSL}$ from a category $\mathbb{C}$ with finite products to the category of inf-semilattices such that maps of the form $P(< id_A, id_A >)$ have a left adjoint satisfying Beck-Chevalley condition.

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