Weighted (Co)Limits

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- Thanks to the organizers of CT2017!
- Thanks to Emily, Alexander, and Brendan for running the Kan Extension Seminar!
- And thanks to all my fellow students in the seminar for their conversation and insight!

Goal: We want to develop a good theory of limits and colimits for enriched categories.

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And give a few nice examples.

Review of (Co)Limits

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• For colimits, switch the direction of the cone around.

$$\mathcal{E}(\operatorname{colim} D, E) \cong \mathbf{Set}^{\mathcal{D}^{\operatorname{op}}}(1, \mathcal{E}(D(-), E)).$$

Thickening the Cones

- When sets are our base, we can analyze any set of morphisms in a category one at a time. That is, it suffices to look at points 1 → E(E, D_i) in our cones.
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Thickening the Cones

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- We will let the shape of the legs of our cones vary with the objects of the diagrams. These shapes are called *weights*.

Definition

Let $D : \mathcal{D} \to \mathcal{E}$ be a diagram in a category \mathcal{E} enriched in \mathcal{V} . Given a *functor of weights* $W : \mathcal{D} \to \mathcal{V}$, the **weighted limit** $\lim_{W} D$, if it exists, satisfies the following universal property:

$$\mathcal{E}(E, \lim_W D) \cong \mathcal{V}^{\mathcal{D}}(W(-), \mathcal{E}(E, D(-))).$$

- Powers (over any base).
- Ø Kernel Pairs (over sets).
- Solution Limits of Cauchy Sequences (over positive real numbers).
- In Homotopy Pushouts (over topological spaces).

- Let $\mathcal{D} = 1$ be the walking object.
- A diagram $D: \mathcal{D} \to \mathcal{E}$ is just an object of \mathcal{E} .
- A weight $W : \mathcal{D} \to \mathcal{V}$ is just an object of the base.
- The weighted limit $\lim_{W} D$ is given by

$$\mathcal{E}(E, \lim_{W} D) \cong \mathcal{V}(W, \mathcal{E}(E, D)),$$

showing that maps into $\lim_{W} D$ are W-tuples of maps into D. Therefore, $\lim_{W} D = D^{W}$, the W-power of D.

Kernel Pairs

- Let $\mathcal{D} = \bullet \to \bullet$ be the walking arrow.
- A diagram $D: \mathcal{D} \to \mathcal{E}$ is an arrow $A \xrightarrow{f} B$ in \mathcal{E} .
- Take $W : \mathcal{D} \to \mathbf{Set}$ to be $2 \xrightarrow{!} 1$.
- The weighted limit is then given by

$$\mathcal{E}(E, \lim_W D) \cong \mathbf{Set}^{\bullet \to \bullet}(2 \to 1, \mathcal{E}(E, A) \xrightarrow{f_*} \mathcal{E}(E, B)).$$

• Substituting in $\lim_{W} D$ for E and pushing $\mathbf{id}_{\lim_{W} D}$ through the isomorphism gives us

$$\begin{array}{ccc} 2 \longrightarrow \mathcal{E}(E,A) \\ \downarrow & & \downarrow_{f_*} \\ 1 \longrightarrow \mathcal{E}(E,B) \end{array}$$

or, in \mathcal{E} ,

$$\lim_{W} D \rightrightarrows A \to B,$$

showing that $\lim_{W} D$ is the kernel pair.

Limits of Cauchy Sequences

- Let $\mathcal{D} = \{0, 1, 2, \dots, \infty\}$ be the natural numbers, with $\mathcal{D}(i, j) = \infty$.
- A diagram $D: \mathcal{D} \to \mathcal{E}$ is a sequence in \mathcal{E} .
- Let $W : \mathcal{D} \to [0,\infty]$ be $i \mapsto \frac{1}{2^i}$, with $\infty \mapsto 0$.
- The weighted limit is then given by

$$\mathcal{E}(E, \lim_{W} D) = [0, \infty]^{\mathcal{D}}(W(-), \mathcal{E}(E, D(-))),$$

which means

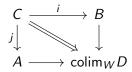
$$0 = \mathcal{E}(\lim_{W} D, \lim_{W} D) = \sup_{i \in \mathcal{D}} \left(\mathcal{E}(E, D_i) - \frac{1}{2^i} \right).$$

so that $D_i \rightarrow \lim_W D$ as a sequence.

- Let $\mathcal{D} = \bullet \leftarrow \bullet \rightarrow \bullet$, so that a diagram is a span in \mathcal{E} .
- Let $W : \mathcal{D}^{op} \to \mathbf{Top}$ be the cospan $* \xrightarrow{0} [0, 1] \xleftarrow{1} *$.
- The weighted colimit is given by

$$\mathcal{E}(\operatorname{colim}_W D, E) \cong \operatorname{Top}^{\mathcal{D}^{\operatorname{op}}}(W(-), \mathcal{E}(D(-), E)),$$

which makes it the initial such cocone:



Thanks for Listening!