# On the local cartesian closure of exact completions

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# Overview

- Background.
- Weak simple products and internal projectives.
- Local cartesian closure from closure under relations.
- Applications: categories of constructive sets and homotopy categories.

# Projective covers

A projective cover of a category $\mathbb{E}$ is a full subcategory $\mathbb{P}$ such that:

1. every $X \in \mathbb{P}$ is (regular) projective,
2. for every $A \in \mathbb{E}$ there is $X \twoheadrightarrow A$ with $X \in \mathbb{P}$.

A category has enough projectives if and only if it has a projective cover.

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Let $X, Y, Z \in \mathbb{P}$,

$$\begin{array}{ccccc} & & P & & \\ & & \downarrow & & \\ X & \longleftarrow & X \times Y & \longrightarrow & Y \end{array}$$

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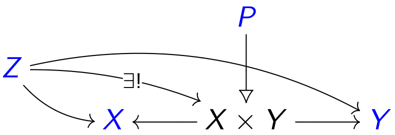

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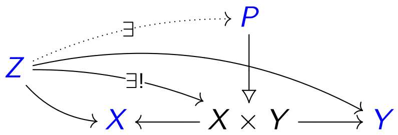


by (1).

$P$ will be denoted as $X \times^w Y$.

# Exact completions

**Theorem (Carboni–Vitale)**

*Exact categories with enough projectives are the exact completion of any of their projective covers.*
*Conversely, any category with weak finite limits is a projective cover of an exact category, namely its exact completion.*

Fix $\mathbb{E}$ exact with enough projectives, and $\mathbb{P}$ a projective cover of it.

**Proposition**

*For* $X_1, \ldots, X_n \in \mathbb{P}$,

$$\mathrm{Sub}_{\mathbb{E}}(X_1 \times \cdots \times X_n) \cong (\mathbb{P}/(X_1, \ldots, X_n))_{\mathrm{po}}.$$

# Cartesian closed exact completions

If $\mathbb{E}$ is cartesian closed then it has exponentials and right adjoints $\forall_A$ to $\times A\colon \mathrm{Sub}_{\mathbb{E}}(I) \to \mathrm{Sub}_{\mathbb{E}}(I \times A)$.

Hence $\mathbb{P}$ has

- weak exponentials,
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## Proposition (Carboni–Rosolini)

*$\mathbb{E}$ is cartesian closed if and only if $\mathbb{P}$ has weak exponentials and $\times^w X$ is left adjoint for every $J, X$.*

# Weak simple products

If $\mathbb{E}$ is cartesian closed, there is a simple product functor $\Pi_A$:

$$\mathbb{E}/I \underset{\Pi_A}{\overset{\times A}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathbb{E}/(I \times A)$$

and both exponentials and $\forall_A$ can be defined from it.

**Definition (Carboni–Rosolini)**

A weak simple product for $J \leftarrow Y \rightarrow X$ is given by

- an object $W \rightarrow J$ in $\mathbb{P}/J$ and
- a weak evaluation $W \times^w X \rightarrow Y$ in $\mathbb{P}/(J, X)$ which factors through $W \times X$.

which is weakly terminal among such pairs.

If $\mathbb{E}$ is cartesian closed, then $\mathbb{P}$ has weak simple products for any span.

# Weak simple products

**Claim (Carboni–Rosolini)**

*If $\mathbb{P}$ has weak simple products for any span, then $\mathbb{E}$ is cartesian closed.*

Weak simple products give rise to:

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**Proposition**

*For any $X \in \mathbb{P}$ the following are equivalent.*

1. $\times^w X \dashv \Pi^w_X$ *for every* $J \in \mathbb{P}$.
2. $J \times X$ *is projective for every* $J \in \mathbb{P}$.
3. $X$ *is internally projective.*

# Weak simple products

**Theorem (Carboni–Rosolini, fixed)**

*If objects in $\mathbb{P}$ are internally projective (i.e. if projectives are closed under binary products), then $\mathbb{E}$ is cartesian closed if and only if $\mathbb{P}$ has weak simple products.*

Applications:

- $\mathbf{Top}_{\mathrm{ex}}$, $(\mathbf{Top}_0)_{\mathrm{ex}}$ and equilogical spaces, the effective topos.
- Every ex/lex completion, in particular M. Menni's characterisation of ex/lex completions that are toposes.

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Non-applications:

- Any topos where not every projective is internally projective, e.g. presheaves on the poset of natural numbers with two distinct infinity points (T. Trimble).
- Setoids as the ex/wlex completion of types in Martin-Löf type theory: types are closed under pullback iff UIP holds.
- The ex/wlex completions of homotopy categories (for l.c.c.).

# Closure under relations

Spans with domain $Z \times^w X$ can be seen as families, indexed by $Z$, of total (pseudo-)relations with domain $X$ and codomain $Y$.


The following are equivalent:

1. $Z \times^w X \to Y$ factors through some $Z \times X \xrightarrow{f} Y$,
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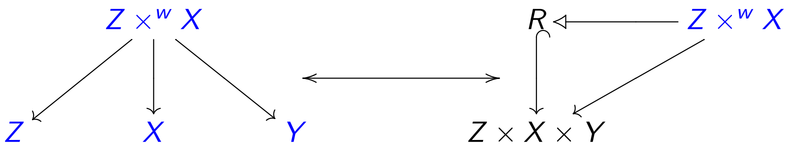


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In a weak simple product for $J \leftarrow Y \to X$, the weak evaluation $W \times^w X \to Y$ factors through $W \times X$.

So it collects codes for just functional relations, whereas spans correspond to arbitrary relations.

# Fullness in Constructive Set Theory (CZF)

Let $\mathrm{Rel}(A, B)$ denote the class of total relations with domain $A$ and codomain $B$.
A set $F \subset \mathrm{Rel}(A, B)$ is full in $\mathrm{Rel}(A, B)$ if

$$\forall R \in \mathrm{Rel}(A, B)\, \exists S \in F \;\; S \subseteq R.$$

## Fullness Axiom (P. Aczel)

For every two sets $A$ and $B$ there is a set $F$ full in $\mathrm{Rel}(A, B)$.

For any $f : A \to B$, $G_f \in \mathrm{Rel}(A, B)$. So there is $S \in F$ such that $S \subseteq G_f$. But then $S = G_f$.

Hence a full set contains all functional relations.

In fact, the Fullness Axiom implies that the class of functions between two sets is itself a set.

# Categorical Fullness

An object $F$ is full for $X$ and $Y$ if there is an arrow $F \times^{w} X \to Y$ such that

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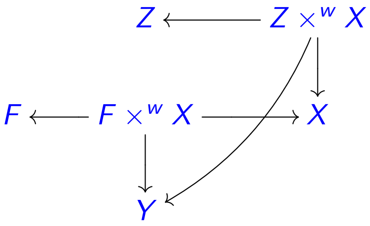

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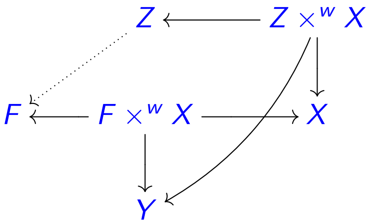

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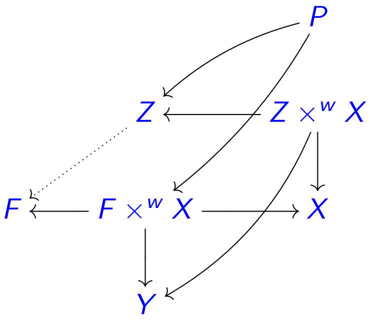

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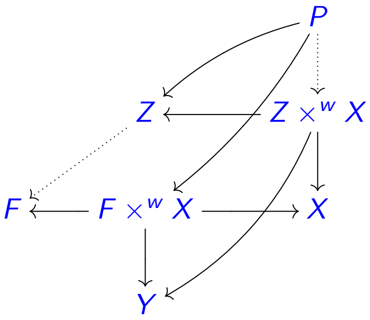

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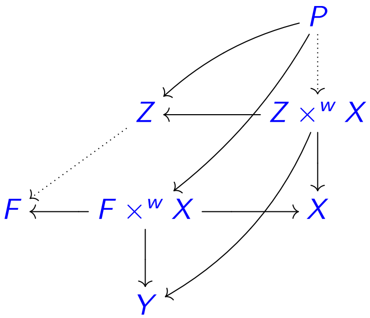


Say that $\mathbb{P}$ is closed for relations, or r-closed, if it has a full object for any pair of objects.

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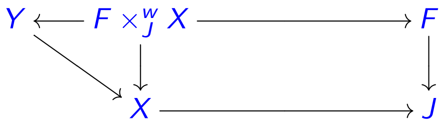
An arrow $F \to J$ is locally full for $Y \to X \to J$ if there is an arrow $F \times^w_J X \to Y$ in $\mathbb{P}/X$ such that,



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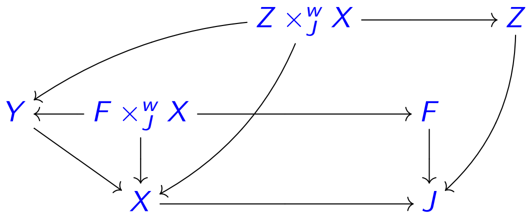

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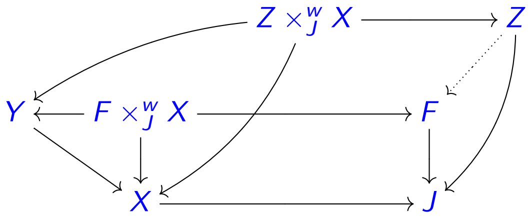

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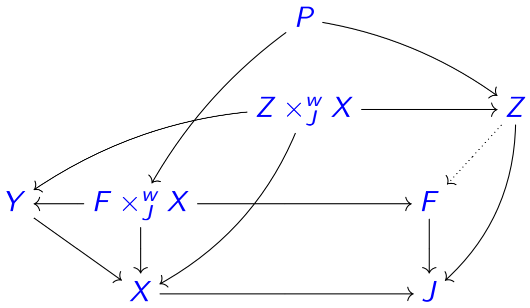

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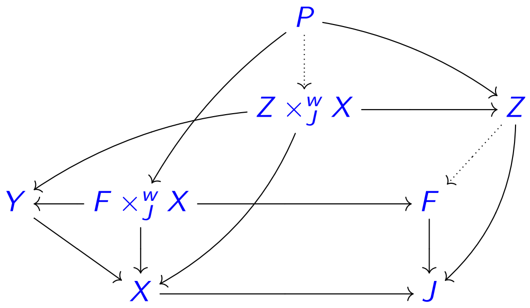

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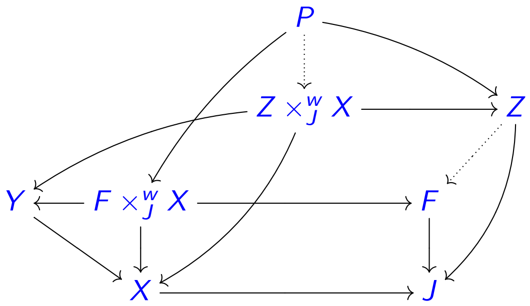


Say that $\mathbb{P}$ is locally closed for relations, or locally r-closed, if it has a locally full arrow for any pair of composable arrows.

# Categorical Fullness

**Lemma**

*If $\mathbb{P}$ is locally r-closed then, for every $f: A \to B$ in $\mathbb{E}$, $f^*\colon \mathrm{Sub}_{\mathbb{E}}(B) \to \mathrm{Sub}_{\mathbb{E}}(A)$ has a right adjoint $\forall_f$ which satisfies Beck-Chevalley.*

**Lemma**

*If $\mathbb{P}$ is locally r-closed, then it has weak exponentials*

**Theorem**

*If $\mathbb{P}$ is locally r-closed, then $\mathbb{E}$ is cartesian closed.*

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**Lemma**

*If $\mathbb{P}$ is locally r-closed, then it has weak exponentials*

If $\mathbb{P}$ is locally r-closed, then every $\mathbb{P}/J$ is also locally r-closed.

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# Categorical Fullness

## Theorem

$\mathbb{P}$ *locally r-closed* $\xrightarrow{(1)}$ $\mathbb{E}$ *locally cartesian closed* $\xrightarrow{(2)}$ *Every* $\mathbb{P}/J$ *has weak simple products.*

*All equivalent if arrows in* $\mathbb{P}$ *are internally projective, i.e. if projectives are closed under pullback.*

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**Theorem**

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Applications of (1):

- Setoids and types and, more generally, models of a categorical constructive set theory and their choice objects.
- The homotopy category of topological spaces and, more generally, homotopy categories of certain model categories.

# Setoids and categories of constructive sets [1]

| **Std** | **Type** |
|---|---|
| extensive | weakly extensive |
| well-pointed | well-pointed & elAC |
| n.n.o. | weak n.n.o. |
| l. cart. closed | l. r-closed |

Well-pointed: $\mathbf{1}$ is strong generator, projective, indecomposable, non-zero.
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| $\mathbb{E}$ | | $\mathbb{P}$ |
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Well-pointed locally cartesian closed pretoposes with an n.n.o. and enough projectives are finitely axiomatised by a theory called CETCS (Palmgren).

[1] joint work with Erik Palmgren

# Models of CETCS from choice objects [1]

**Theorem**

*If $\mathbb{P}$ is weakly lextensive, locally r-closed, well-pointed with elAC and a weak natural numbers object, then $\mathbb{E}$ is a model of CETCS.*

All but local r-closure are also necessary conditions.

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In the presence of small disjoint sums, local r-closure is also a necessary condition.

**Theorem**

*Let $\mathbb{E}$ be well-pointed and with small disjoint sums. Then $\mathbb{E}$ is locally cartesian closed if and only if $\mathbb{P}$ is locally r-closed.*

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**Theorem**

*Let $\mathbb{E}$ be well-pointed and with small disjoint sums. Then $\mathbb{E}$ is locally cartesian closed if and only if $\mathbb{P}$ is locally r-closed.*

**Corollary**

*Suppose $\mathbb{E}$ has small disjoint sums. Then $\mathbb{E}$ is a model of CETCS if and only if $\mathbb{P}$ is weakly lextensive, locally r-closed, well-pointed with elAC and a weak natural numbers object.*

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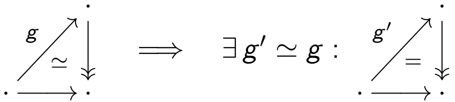
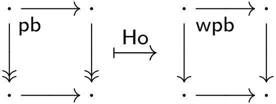
# R-closure in homotopy categories

## Proposition

*Let $\mathbb{M}$ be a right proper model category where every object is cofibrant. If $\mathbb{M}$ is weakly locally cartesian closed, then* $\mathrm{Ho}(\mathbb{M})$ *is locally r-closed.*

If all objects in $\mathbb{M}$ are cofibrant, then its subcategory of fibrant objects $\mathbb{M}_f$ is a path category (van den Berg–Moerdijk).
In particular:

1. 


2. 

# R-closure in homotopy categories

## Proposition (Carboni–Rosolini)

**Top** *is weakly locally cartesian closed.*

- $\mathbf{Top}_{Strøm}$ is a proper model category where every object is fibrant and cofibrant.
- $\mathbf{sSet}_{Quillen}$ is a proper model category where every object is cofibrant.

## Corollary

- $\mathrm{Ho}(\mathbf{Top}_{Strøm})$ *is locally r-closed.*
- $\mathrm{Ho}(\mathbf{Top}_{Quillen}) \sim \mathrm{Ho}(\mathbf{sSet}_{Quillen})$ *is locally r-closed.*

*So their exact completions are locally cartesian closed pretoposes.*

This Corollary also follows from results in van den Berg and Moerdijk, *Exact completion of path categories and Algebraic Set Theory*, 2016.

Thank you!

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