The canonical intensive quality of a pre-cohesive topos

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Joint work with
Matías Menni

Monday, July 17, 2017

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and help from F.W. Lawvere

Axiomatic Cohesion

I. Categories of space as cohesive backgrounds
II. Cohesion versus non-cohesion; quality types
III. Extensive quality; intensive quality in its rarefied and condensed aspects; the canonical qualities form and substance
IV. Non-cohesion within cohesion via constancy on infinitesimals
V. The example of reflexive graphs and their atomic numbers
VI. Sufficient cohesion and the Grothendieck condition
VII. Weak generation of a subtopos by a quotient topos

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"I look forward to further work on each of these aspects"

Axiomatic Cohesion

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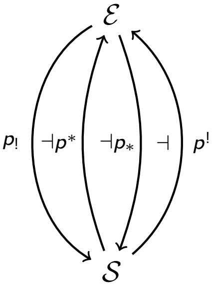

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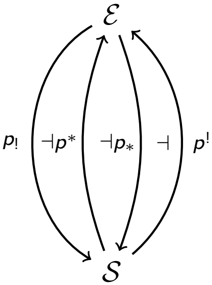

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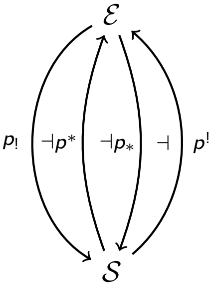

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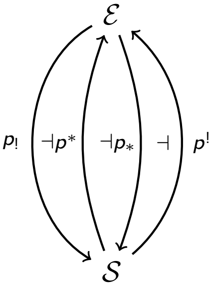

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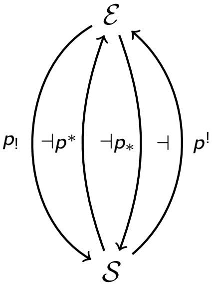

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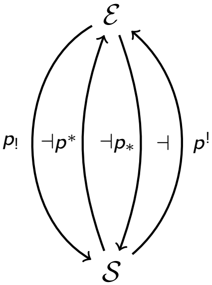

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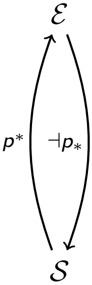
Continuity Axiom: iv) $p_!(E^{p^* S}) \to (p_! E)^S$ iso.

Quality type



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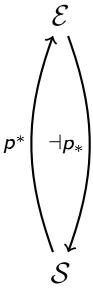

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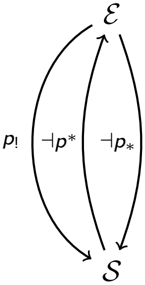

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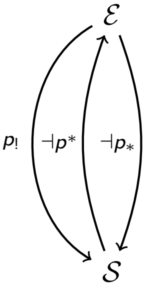

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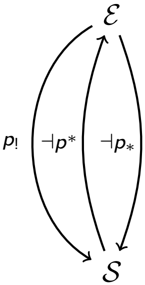

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and $\theta : p_* \to p_!$ is iso.

"A quality type is a category of cohesion in one extreme sense"

Canonical Quality Type

$\mathcal{L}$ the full subcategory of $\mathcal{E}$ of those objects X for which $\theta_X : p_* \to p_!$ is iso

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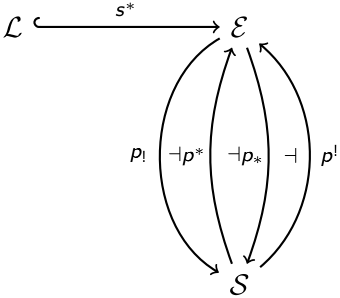


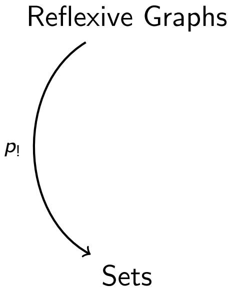
Canonical Quality Type

Reflexive Graphs

Sets

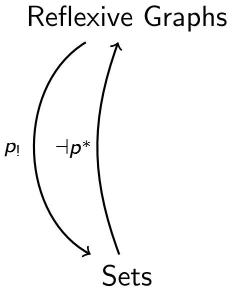
Canonical Quality Type

$p_!$
connected
components



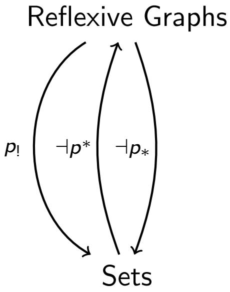
Canonical Quality Type

$p_!$	p^*
connected components	discrete



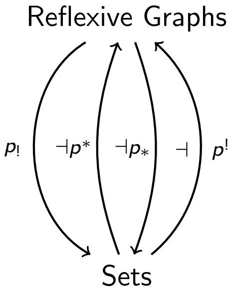
Canonical Quality Type

$p_!$	p^*	p_*
connected components	discrete	points



Canonical Quality Type

$p_!$	p^*	p_*	$p^!$
connected components	discrete	points	codiscrete



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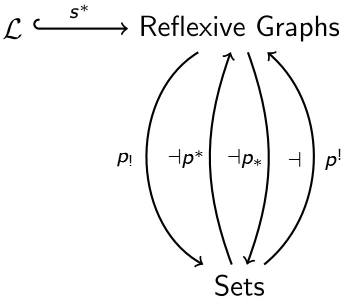

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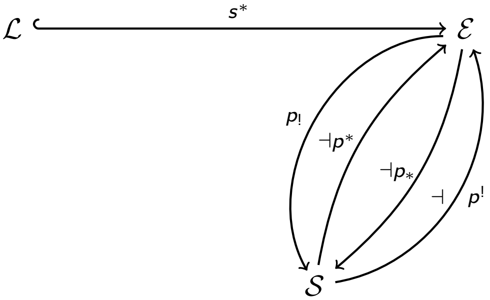

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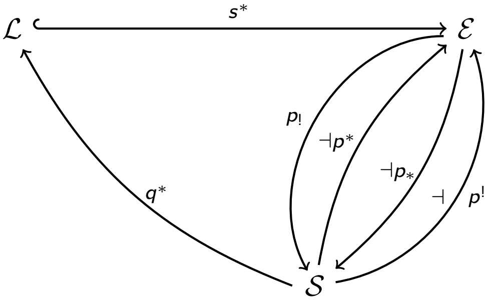

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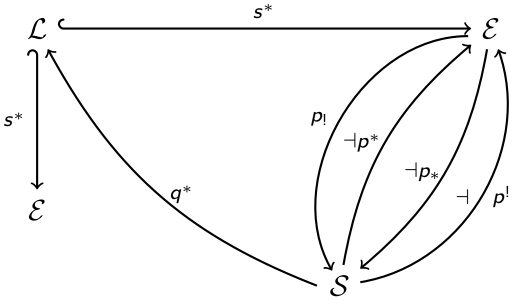

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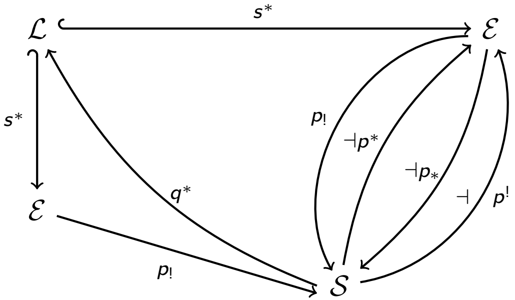

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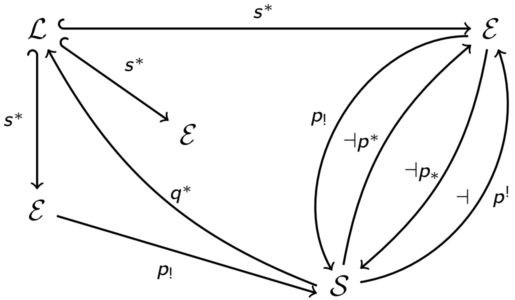

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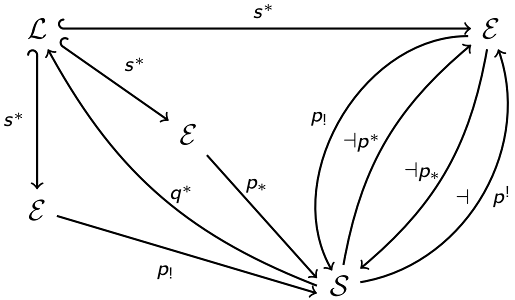

Theorem from Axiomatic Cohesion

Theorem

Any category of cohesion satisfying reasonable completeness conditions has a canonical intensive quality s whose codomain is the subcategory $s^* : \mathcal{L} \to \mathcal{E}$ consisting of those X for which the map $\theta_X : p_* X \to p_! X$ is an isomorphism. Moreover, s^* has a left adjoint $s_!$ and a coproduct-preserving right adjoint s_*.

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Thus $\mathcal{L}$ is a topos.
(Algebras for a left exact comonad.)

Reflexive Graphs Again

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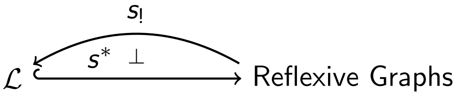
$$\mathcal{L} \xhookrightarrow{\quad s^* \quad} \text{Reflexive Graphs}$$

Reflexive Graphs Again



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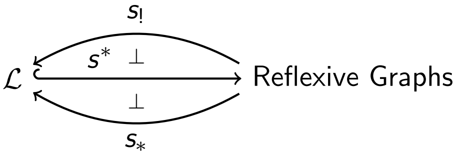

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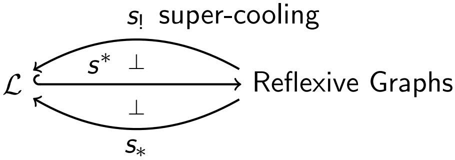

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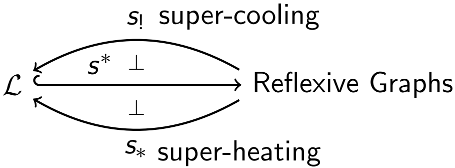

The Actual Construction of the Adjoints

$s_! : \mathcal{E} \to \mathcal{L}$.

$$\begin{array}{ccc} p^* p_* X & \xrightarrow{p^* \theta_X} & p^* p_! X \\ {\scriptstyle \beta_X} \downarrow & & \downarrow \\ X & \longrightarrow & s^* s_! X \end{array}$$

a pushout.

The Actual Construction of the Adjoints

For the right adjoint $s_* : \mathcal{E} \to \mathcal{L}$ we need $\phi : p^* \to p^!$.

$$\begin{array}{ccc} s^* s_* X & \longrightarrow & X \\ \big\downarrow & & \big\downarrow \eta_X \\ p^* p_* X & \xrightarrow[\phi_{p_* X}]{} & p^! p_* X \end{array}$$

a pullback.

Theorem

Let $p : \mathcal{E} \to \mathcal{S}$ be an essential and local geometric morphism between toposes such that the Nullstellensatz holds. Then then the inclusion $s^* : \mathcal{L} \to \mathcal{E}$ of Leibniz objects has a right adjoint. It follows that $\mathcal{L}$ is a topos and p induces an hyperconnected essential geometric morphism $s : \mathcal{E} \to \mathcal{L}$.

Basically consequence of

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Lemma If $p : \mathcal{E} \to \mathcal{S}$ satisfies the Nullstellensatz, then the image of $s^* : \mathcal{L} \to \mathcal{E}$ is closed under subobjects.

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As a consequence

$$s_*(\Omega_{\mathcal{E}}) = \Omega_{\mathcal{L}}.$$

Proof. $L \in \mathcal{L}$, $m : X \rightarrowtail L$ in $\mathcal{E}$.

$$\begin{array}{ccc}
p_* X & \xrightarrow{\theta_X} & p_! X \\
{\scriptstyle p^* m}\big\downarrow & & \big\downarrow{\scriptstyle p_! m} \\
p_* L & \xrightarrow[\theta_L]{\simeq} & p_! L
\end{array}$$

$p : \mathcal{E} \to \mathcal{S}$ essential and local.

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Lemma

If $X \in \mathcal{E}$ is separated for the topology induced by $p_* \dashv p^!$, then $s^* s_* X$ is discrete.

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Lemma

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Lemma

$X \in \mathcal{E}$ is Leibniz if and only if $\beta_X : p^* p_* X \to X$ has a retraction.

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Lemma

Let Ω be the subobject classifier of $\mathcal{E}$. Then $s^* s_* \Omega$ is discrete if and only if $p : \mathcal{E} \to \mathcal{S}$ is an equivalence.

Proposition

Boolean objects in $\mathcal{E}$ are discrete.

Thus, $\mathcal{E}$ Boolean implies that $p : \mathcal{E} \to \mathcal{S}$ is an equivalence.

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$\mathcal{C}$ a small category whose idempotents split.

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$p : \mathbf{Con}^{\mathcal{C}^{\mathrm{op}}} \to \mathbf{Con}$ is precohesive if and only if $\mathcal{C}$ has a terminal object and every object has a point.

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Lemma
For any X in $\mathbf{Con}^{\mathcal{C}^{\mathrm{op}}}$, the counit $s^*(s_* X) \to X$ is

$$(s^*(s_* X))C = \{x \in QC \mid \text{for all } a, b : 1 \to C, x \cdot a = x \cdot b\}$$

for every $C \in \mathcal{C}$.

$\mathcal{C}$ with terminal and every object has a point

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The lemma gives no information as to the nature of $\mathcal{L}$ for $p : \mathbf{Con}^{\mathcal{C}^{\mathrm{op}}} \to \mathbf{Con}$.

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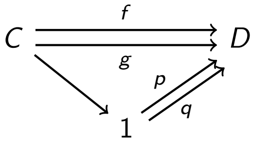
$f \equiv g$ if $f = g$ or both f and g are constant



$s : \mathbf{Con}^{\mathcal{C}^{\mathrm{op}}} \to \mathcal{L}$ is in general not local.

$s_! : \mathcal{L} \to \mathbf{Con}^{\mathcal{C}^{\mathrm{op}}}$ does not in general preserve finite products.

Sites

Theorem
A bounded essential connected geometric morphism $p : \mathcal{E} \to \mathbf{Con}$ satisfies the Nullstellensatz iff $\mathcal{E}$ has a connected and locally connected site of definition $(\mathcal{C}, J)$ such that every object of $\mathcal{C}$ has a point.

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Theorem
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The site $(\mathcal{C}, J)$ is locally connected if each J-covering sieve on C is connected as a full subcategory of $\mathcal{C}/C$. If furthermore $\mathcal{C}$ has a terminal object, then we say that $(\mathcal{C}, J)$ is connected and locally connected.

$(\mathcal{C}, J)$ connected and locally connected

Theorem
Let $\mathcal{C}/\equiv$ be the category that results from identifying all the points, and let $r : \mathcal{C} \to \mathcal{C}/\equiv$ be the quotient functor. If $r_+ J$ is the largest topology on $\mathcal{C}/\equiv$ such that r reflects covers, then

$$\mathcal{L}(\mathsf{Sh}(\mathcal{C}, J)) \simeq \mathsf{Sh}(\mathcal{C}/\equiv, r_+ J).$$

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A sieve S on C in the category $\mathcal{C}/\equiv$ is in $(r_+ J)C$ if and only if the sieve

$$\{g : \mathrm{dom} g \to C \text{ in } \mathcal{C} | r(g) \in S\}$$

is in JC.

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One can use Giraud's theorem to produce a subcanonical site from $\mathsf{Sh}(\mathcal{C}/\equiv, r_+J)$.

If, furthermore, one assumes that every representable is separable, then one application of $(\ )^+$ construction suffices.

Closed intervals and piecewise linear functions

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Closed intervals and piecewise linear functions

The category $\mathcal{C}$.

Objects: $[a, b]$ with $a \leq b$, $a, b \in \mathbb{R}$.

Morphisms: Continuous functions $f : [a, b] \to [c, d]$, such that f is piecewise linear.

The topology is given by a basis K:

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$p : \mathrm{Sh}(\mathcal{C}, K) \to \mathbf{Con}$ is cohesive.

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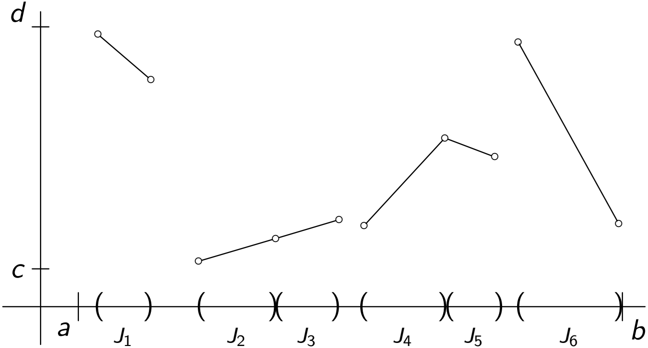

There is an obvious functor $F : \mathcal{C} \to \mathcal{D}$
$\mathrm{Sh}(\mathcal{D}, F_+K) = \mathcal{L}(\mathrm{Sh}(\mathcal{C}, K))$ and it is subcanonical.

Since the sites are subcanonical, $F_! : \mathrm{Sh}(\mathcal{C}, K) \to \mathrm{Sh}(\mathcal{D}, F_+ K)$ preserves representables.

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So

$$\begin{array}{ccc} \mathbf{Con}^{\mathcal{C}^{\mathrm{op}}} & \xrightarrow{F_!} & \mathbf{Con}^{\mathcal{D}^{\mathrm{op}}} \\ \uparrow & & \uparrow \\ \mathrm{Sh}(\mathcal{C}, J) & \xrightarrow[F_!]{} & \mathrm{Sh}(\mathcal{D}, F_+ J) \end{array}$$

commutes.