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A synthetic theory of ∞-categories in homotopy type theory

joint with Michael Shulman

CT2017 UBC, Vancouver, Canada

Motivation

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Plan

1. Homotopy type theory
2. A type theory for synthetic $(\infty, 1)$-categories
3. Segal types and Rezk types
4. The synthetic theory of $(\infty, 1)$-categories

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2. A type theory for synthetic $(\infty, 1)$-categories

3. Segal types and Rezk types

4. The synthetic theory of $(\infty, 1)$-categories

Main takeaway: the dependent Yoneda lemma is a directed analogue of path induction in HoTT.

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Propositions as types:

$A \times B$	A and B	$\sum_{x:A} B(x)$	$\exists x.B(x)$
$A + B$	A or B	$\prod_{x:A} B(x)$	$\forall x.B(x)$
$A \to B$	A implies B	$x =_A y$	x equals y

Dependent sums and products

Formation rules for dependent sums and products

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Semantics

$(a, u) : \sum_{x:A} B(x)$

$$\begin{tikzcd} B(a) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner"] & \sum_{x:A} B(x) \arrow[d, two heads] \\ 1 \arrow[u, dashed, bend left, "u"] \arrow[r, dashed, "a"] & A \end{tikzcd}$$

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Thus, if $x =_A y$ then $B(x) \to B(y)$.

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The ∞-groupoid structure of A with

- terms $x : A$ as objects
- paths $p : x =_A y$ as 1-morphisms
- paths of paths $\alpha : p =_{x=_A y} q$ as 2-morphisms, ...

arises automatically from the path induction principle.

2

A type theory for synthetic $(\infty, 1)$-categories

The intended model

$$
\begin{array}{ccccccc}
\mathcal{S}\mathrm{et}^{\mathbb{\Delta}^{\mathrm{op}} \times \mathbb{\Delta}^{\mathrm{op}}} & \supset & \mathcal{R}\mathrm{eedy} & \supset & \mathcal{S}\mathrm{egal} & \supset & \mathcal{R}\mathrm{ezk} \\
\shortparallel & & \shortparallel & & \shortparallel & & \shortparallel \\
\text{bisimplicial sets} & & \text{types} & & \text{types with} & & \text{types with} \\
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Theorem (Shulman). Homotopy type theory is modeled by the category of Reedy fibrant bisimplicial sets.

Theorem (Rezk). $(\infty, 1)$-categories are modeled by Rezk spaces aka complete Segal spaces.

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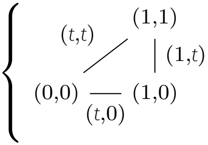

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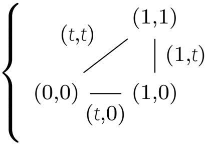
$\Delta^2 :=$ 

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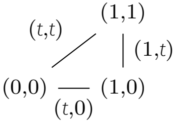

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Because $\phi \wedge \psi$ implies ϕ, there are shape inclusions $\Lambda^2_1 \subset \partial\Delta^2 \subset \Delta^2$.

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$$\frac{\Phi \subset \Psi \text{ shape} \qquad A \text{ type} \qquad a : \Phi \to A}{\left\langle \begin{tikzcd} \Phi \arrow[r, "a"] \arrow[d, tail] & A \\ \Psi \arrow[ur, dashed] & \end{tikzcd} \right\rangle \text{ type}}$$

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The simplicial type theory allows us to *prove* equivalences between extension types along composites or products of shape inclusions.

3

Segal types and Rezk types

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$$\frac{\partial\Delta^1 \subset \Delta^1 \text{ shape} \qquad A \text{ type} \qquad [x, y] : \partial\Delta^1 \to A}{\hom_A(x, y) := \left\langle \begin{tikzcd} \partial\Delta^1 \arrow[r, "{[x,y]}"] \arrow[d, hook] & A \\ \Delta^1 \arrow[ur, dashed] & \end{tikzcd} \right\rangle \text{ type}}$$

A term $f : \hom_A(x, y)$ defines an arrow from x to y.

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Prop. A Reedy fibrant bisimplicial set A is *Segal* if and only if $A^{\Delta^2} \twoheadrightarrow A^{\Lambda^2_1}$ is a Reedy trivial fibration.

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Identity arrows

For any $x : A$, the constant function defines a term

$$\mathrm{id}_x := \lambda t.x : \hom_A(x,x) := \left\langle \begin{tikzcd} \partial\Delta^1 \arrow[r, "{[x,x]}"] \arrow[d, hook] & A \\ \Delta^1 \arrow[ur, dashed] & \end{tikzcd} \right\rangle,$$

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For any $f : \hom_A(x,y)$ in a Segal type A, the term

$$\lambda(s,t).f(t) : \left\langle \begin{tikzcd} \Lambda^2_1 \arrow[r, "{[\mathrm{id}_x, f]}"] \arrow[d, hook] & A \\ \Delta^2 \arrow[ur, dashed] & \end{tikzcd} \right\rangle$$

witnesses the unit axiom $f = f \circ \mathrm{id}_x$.

Associativity of composition

Let A be a Segal type with arrows

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Proof: Consider the composable arrows in the Segal type $\Delta^1 \to A$:

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& y \arrow[dr, "g"] \arrow[dd, "g"'] \arrow[dddr, "h\circ g"] & \\
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 & z \arrow[rd, "h"] & \\
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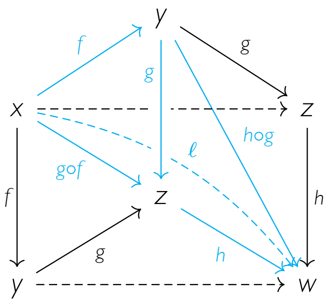

Composing defines a term in the type $\Delta^2 \to (\Delta^1 \to A)$ which yields a term $\ell : \hom_A(x,w)$ so that $\ell = h \circ (g \circ f)$ and $\ell = (h \circ g) \circ f$.

Isomorphisms

An arrow $f\colon \hom_A(x,y)$ in a Segal type is an isomorphism if it has a two-sided inverse $g\colon \hom_A(y,x)$. However, the type

$$\sum_{g\colon \hom_A(y,x)} (g \circ f = \mathrm{id}_x) \times (f \circ g = \mathrm{id}_y)$$

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$$\mathrm{isiso}(f) := \left(\sum_{g \colon \hom_A(y,x)} g \circ f = \mathrm{id}_x \right) \times \left(\sum_{h \colon \hom_A(y,x)} f \circ h = \mathrm{id}_y \right).$$

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For $x, y : A$, the type of isomorphisms from x to y is:

$$x \cong_A y := \sum_{f : \hom_A(x,y)} \mathrm{isiso}(f).$$

Rezk types

By path induction, to define a map

$$\mathsf{id\text{-}to\text{-}iso} \colon (x =_A y) \to (x \cong_A y)$$

for all $x, y : A$ it suffices to define

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Discrete types

Similarly by path induction define

$$\text{id-to-arr}\colon \prod_{x,y:A} (x =_A y) \to \hom_A(x,y) \qquad \text{by} \qquad \text{id-to-arr}(\mathsf{refl}_x) :\equiv \mathrm{id}_x.$$

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Proof:

$$\begin{tikzcd}
x =_A y \arrow[rr, "\text{id-to-arr}"] \arrow[dr, "\text{id-to-iso}"'] & & \hom_A(x,y) \\
& x \cong_A y \arrow[ur, hook] &
\end{tikzcd}$$

4

The synthetic theory of $(\infty, 1)$-categories

Covariant fibrations I

A type family $x : A \vdash B(x)$ over a Segal type A is covariant if for every $f : \hom_A(x, y)$ and $u : B(x)$ there is a unique lift of f with domain u.

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Recall

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Prop. For $u : B(x)$, $f : \hom_A(x,y)$, and $g : \hom_A(y,z)$,

$$g_*(f_*u) = (g \circ f)_*u \qquad \text{and} \qquad (\mathrm{id}_x)_*u = u.$$

Covariant fibrations II

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Yoneda lemma. The maps

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and

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Note. A representable *isomorphism* $\phi : \prod_{x:A} \hom_A(a,x) \cong \hom_A(b,x)$ induces an *identity* $\text{ev-id}(\phi) : b =_A a$ if the Segal type A is Rezk.

The dependent Yoneda lemma

From a type-theoretic perspective, the Yoneda lemma is a "directed" version of the "transport" operation for identity types. This suggests a "dependently typed" generalization of the Yoneda lemma, analogous to the full induction principle for identity types.

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Dependent Yoneda lemma. If A is a Segal type and $B(x, y, f)$ is a covariant family dependent on $x, y : A$ and $f : \hom_A(x, y)$, then evaluation at (x, x, id_x) defines an equivalence

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This is useful for proving equivalences between various types of coherent or incoherent adjunction data.

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Thus, to prove $B(x, y, p)$ it suffices to assume y is x and p is refl_x.

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Dependent Yoneda Lemma: If $B(x, y, f)$ is a covariant family dependent on $x, y : A$ and $f : \hom_A(x, y)$ and A is Segal, then there is a function

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Thus, to prove $B(x, y, p)$ it suffices to assume y is x and f is id_x.

References

For considerably more, see:

Emily Riehl and Michael Shulman, A type theory for synthetic ∞-categories, `arXiv:1705.07442`

To explore homotopy type theory:

Homotopy Type Theory: Univalent Foundations of Mathematics, `https://homotopytypetheory.org/book/`

Michael Shulman, Homotopy type theory: the logic of space, `arXiv:1703.03007`

Thank you!