Graphical calculus in symmetric monoidal (∞–)categories with duals

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July 21, 2017, CT2017, UBC, Vancouver

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Introduction

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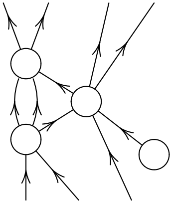
Review on string calculus

Definition 1

A planar graph $\eta : \Gamma \hookrightarrow \mathbb{R}^2$ (with outer-edges) is said to be progressive if for each edge e of Γ, the composition

$$e \xrightarrow{\eta} \mathbb{R}^2 \xrightarrow{\text{proj}_2} \mathbb{R} = y\text{-axis}$$

is strictly increasing along the orientation of the edge e.



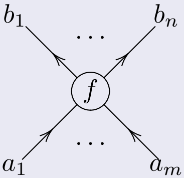
Review on string calculus

$\mathcal{C}$: a monoidal category.

Proposition 2 ([Joyal and Street, 1991])

For a planar progressive graph Γ, *consider a labeling in* $\mathcal{C}$ *subject to the following rules:*

- *each edge of* Γ *is labeled by an object of* $\mathcal{C}$;
- *each vertex of* Γ *is labeled by a morphism of* $\mathcal{C}$ *so that*



$$\leftrightarrow \quad f : a_1 \otimes \cdots \otimes a_m \to b_1 \otimes \cdots \otimes b_n \in \mathcal{C}$$

Then, Γ *together with the labeling determines a morphism in* $\mathcal{C}$. *Moreover, the resulting morphism is invariant under isotopies of planar progressive graphs.*

An extension

Slogan

String calculus in $\mathcal{C}$ = A class of graphs + labeling rules

Question

Is it possible to consider whole planar graphs to obtain new graphical calculus?

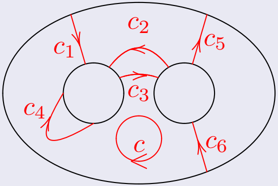
Anser YES!! Let's enjoy more geometry and more duality.

Answer from quantum topology: planar algebras

Definition 3 (modified from [Jones, 1999])

Let C be a set with an involution $(\cdot)^* : C \to C$. Then, a C-colored planar algebra V in a symmetric monoidal category $\mathcal{V}$ consists of

- an object $V(c_1 \dots c_m) \in \mathcal{V}$ for each cycllic sequence in C;
- operations corresponding to labeled pictures such as



$$: V(c_1^* c_2^* c_3 c_4 c_4^*) \otimes V(c_3^* c_2 c_5 c_6^*) \to V(c_1^* c_5 c_6^*),$$

which is compatible with the substitutions of disks.

Goal of the talk

Bad news!

The present definition of planar algebras is more or less algebraic; i.e. by generators and relations.

But... **WE NEED MORE GEOMETRY!!!!!**

Goal

- To define an operad of planar algebras in a purely geometric way.
- Bonus: a priori higher coherence problems.
- Graphical calculus in symmetric monoidal ∞-categories with duals.

Operads of surfaces with strips

1 Introduction

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- Relative objects in smooth category
- Cobordisms of arrangements
- Operads for surfaces with strings
- Algebraic description
- Bonus: Lifts to ∞-contexts

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Relative objects in smooth category

Write $[n] := \{0 < 1 < \cdots < n\}$ the totally ordered set with $(n+1)$-elements.

Definition 4

An arrangement of manifolds of shape $[n]$ is a functor

$$\mathcal{X} : [n] \to \mathbf{Emb}$$

into the category of smooth manifolds (possibly with corners) and smooth embeddings.

$\rightsquigarrow i < j \Rightarrow X(i)$ "is" a submanifold of $X(j)$.

Notation

- The ambient manifold $|\mathcal{X}| := \mathcal{X}(\max[n]) = \mathcal{X}(n)$.
- The dimension $\dim \mathcal{X} := (\dim \mathcal{X}(n), \ldots, \dim \mathcal{X}(0))$.

Cobordisms of arrangements

Definition 5

For a non-increasing sequence $d_n \geq \cdots \geq d_0$ of integers, define a symmetric monoidal category $\mathbf{ArrCob}_{(d_n,\ldots,d_0)}$ as follows:

- **object** arrangements $\mathcal{Y}$ of closed oriented manifolds of shape $[n]$ of dimension $(d_n - 1, \ldots, d_0 - 1)$.
- **morphism** diffeomorphism classes of arrangements $\mathcal{W}$ of compact oriented manifolds with boundaries of shape $[n]$ of dimension $(d_n, \ldots, d_0)$ together with a diffeomorphism

$$\partial\mathcal{W} \cong -\mathcal{Y}_0 \amalg \mathcal{Y}_1 \ .$$

- **composition** gluing (POSSIBLE!!!).
- **$\otimes$-structure** disjoint union $\amalg$.

For our purpose, $\mathbf{ArrCob}_{(2,1)}$!

Operads for surfaces with strings

Definition 6

Define wide subcategories

$$\mathbf{PlTang} \subset \mathbf{SrfTang} \subset \mathbf{ArrCob}_{(2,1)}$$

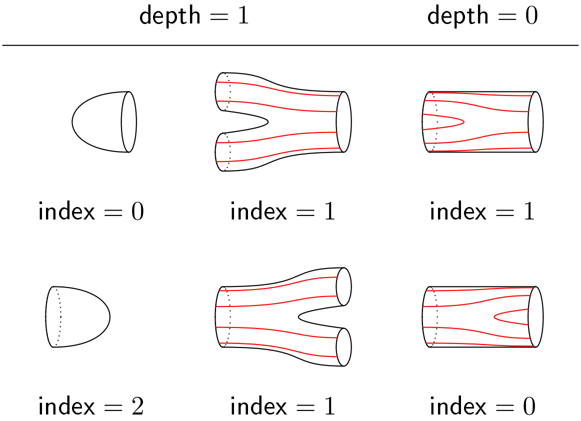
to consist of the following morphisms:

- morphisms in $\mathbf{SrfTang}$ are those $\mathcal{W} : \mathcal{Y}_0 \to \mathcal{Y}_1$ such that $\pi_0(|\mathcal{Y}_1|) \to \pi_0(|\mathcal{W}|)$ is bijective;
- morphisms in $\mathbf{PlTang}$ satisfy in addition that the surface $|\mathcal{W}|$ is of genus 0.

Proposition 7

The subcategories $\mathbf{SrfTang}$ *and* $\mathbf{PlTang}$ *are closed under monoidal products. Moreover, these categories are freely generated by colored operads as symmetric monoidal categories.*

Algebraic description



Remark

The number of strings may varied except for *cup* and *cap*.

Bonus: Lifts to ∞-contexts

Bonus! A construction similar to ∞-category $\mathfrak{Cob}_d$ of cobordisms in [Lurie, 2009] works for arranged cobordisms.

$\rightsquigarrow$ One obtains a symmetric monoidal ∞-category $\mathfrak{ArrCob}_{(2,1)}$ together with a 1-truncation

$$\mathfrak{ArrCob}_{(2,1)} \to \mathbf{ArrCob}_{(2,1)} .$$

Definition 8

$$\begin{array}{ccccc}
\mathfrak{PlTang} & \longrightarrow & \mathfrak{SrfTang} & \longrightarrow & \mathfrak{ArrCob}_{(2,1)} \\
\downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\
\mathbf{PlTang} & \hookrightarrow & \mathbf{SrfTang} & \hookrightarrow & \mathbf{ArrCob}_{(2,1)}
\end{array}$$

Proposition 9

$\mathfrak{PlTang}$ and $\mathfrak{SrfTang}$ are ∞-operads in the sense in [Lurie, 2014].

Labelings and graphical calculus

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Labelings

C: a fixed (small) set with involution.

Definition 10

A C-labeling on an arrangement $\mathcal{X}$ of shape $[1]$ is just a map

$$\mathcal{X}(0) \to C \ .$$

$\rightsquigarrow$ $\mathbf{ArrCob}_{(2,1)/C}$: the category of C-labeled arranged cobordisms.

Definition 11

$$\begin{array}{ccc} \mathbf{SrfTang}_{/C} & \longrightarrow & \mathbf{ArrCob}_{(2,1)/C} \\ \downarrow & & \downarrow \text{forget} \\ \mathbf{SrfTang} & \hookrightarrow & \mathbf{ArrCob}_{(2,1)} \end{array} \qquad \begin{array}{ccc} \mathbf{PlTang}_{/C} & \longrightarrow & \mathbf{ArrCob}_{(2,1)/C} \\ \downarrow & & \downarrow \text{forget} \\ \mathbf{PlTang} & \hookrightarrow & \mathbf{ArrCob}_{(2,1)} \end{array}$$

Remark

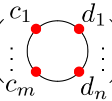
Similarly, one can define $\mathfrak{PlTang}_{/C} \subset \mathfrak{SrfTang}_{/C} \subset \mathfrak{ArrCob}_{(2,1)/C}$.

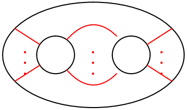
Classification

Theorem 12

An algebra over $\mathbf{PlTang}_{/C}$ is exactly a C-colored planar algebra.

A planar algebra V in $\mathcal{V}$ determines a monoidal $\mathcal{V}$-category $\mathcal{C}_V$ with

- **objects** sequences of elements in C;
- **hom-obj** $\mathcal{C}_V(c_1 \dots c_m, d_1 \dots d_n) = V\Big($  $\Big)$

- **composition** 

Remark

The involution on C extends to a functor $(\,\cdot\,)^* : \mathcal{C}_V \to \mathcal{C}_V^{\mathrm{op}}$ so that

$$c^* \dashv c \,, \quad c^{**} \cong c \,.$$

Graphical calculus

Aspect

Graphical calculi in $\mathcal{C}$
$=$ monoidal functor $\mathcal{C}_V \to \mathcal{C}$ for some $\mathcal{C}_0$-colored planar algebra V.

TODAY: Focus on symmetric monoidal (∞–)categories with duals: the key ingredient is

Theorem 13 (Cobordism Hypothesis in dim 1, folklore, [Baez and Dolan, 1995], [Lurie, 2009])

For a symmetric monoidal ∞-category $\mathcal{C}$, the functor

$$\mathrm{Fun}^{\otimes}(\mathfrak{Cob}_1, \mathcal{C}) \to \mathrm{Core}\,\mathcal{C}\ ; \quad Z \mapsto Z(+)$$

is a categorical equivalence, where the domain is the ∞-category of symmetric monoidal categories from the category of 1-dim cobordisms, and $\mathrm{Core}\,\mathcal{C}$ is the maximal groupoid of $\mathcal{C}$.

Key results

Lemma 14

For each element $c \in C$, there exists a symmetric monoidal functor

$$\mathfrak{ArrCob}_{(2,1)/C} \to \mathfrak{Cob}_1$$

between ∞-categories which does

1. *forgets all strings but ones labeled by $c \in C$; and*
2. *forgets the ambient cobordisms.*

Key results

In particular, we have a functor

$$\psi : \mathfrak{ArrCob}_{(2,1)/C} \to \mathrm{Fun}_0(C, \mathfrak{Cob}_1) \ ,$$

where $\mathrm{Fun}_0(C, \mathfrak{Cob}_1)$ is the ∞-category of functors which values the empty except for finitely many points in C.
In addition, for every map $\lambda : C \to \mathrm{Fun}^{\otimes}(\mathfrak{Cob}_1, \mathcal{C})$, we have

$$\Psi : \mathfrak{ArrCob}_{(2,1)/C} \xrightarrow{(\lambda,\varphi)} \mathrm{Fun}^{\otimes}(\mathfrak{Cob}_1, \mathcal{C}) \times \mathrm{Fun}_0(C, \mathfrak{Cob}_1) \xrightarrow{\mathrm{eval}} \mathrm{Fun}_0(C, \mathcal{C}) \xrightarrow{\otimes} \mathcal{C} \ .$$

Example 15

By Cobordism Hypothesis, we may choose a map

$$\mathcal{C}_0 \hookrightarrow \mathrm{Core}\,\mathcal{C} \to \mathrm{Fun}^{\otimes}(\mathfrak{Cob}_1, \mathcal{C}) \ .$$

Hence, we obtain

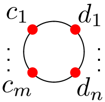
$$\Psi : \mathfrak{ArrCob}_{(2,1)/\mathcal{C}_0} \to \mathcal{C} \ .$$

Main Theorem

Theorem 16

Every symmetric monoidal ∞-category $\mathcal{C}$ with duals gives rise to a $\mathcal{C}_0$-colored planar algebra $Z_{\mathcal{C}} : \mathfrak{SrfTang} \to \infty\mathbf{Grpd}$.

Sketch

- For $\mathcal{Y} =$ 

, put

$$Z_{\mathcal{C}}(\mathcal{Y}) := \mathcal{M}ap_{\mathcal{C}}(\mathbb{1}, \Psi(\mathcal{Y})) \cong \mathcal{M}ap_{\mathcal{C}}(\mathbb{1}, c_m \otimes \cdots \otimes c_1 \otimes d_1 \otimes \cdots \otimes d_n) \ ;$$

- For $\mathcal{W} : \coprod_{i=1}^{n} \mathcal{Y}_i \to \mathcal{Y}$ with each $|\mathcal{Y}_i|$ and $|\mathcal{Y}|$ connected, put

$$Z_{\mathcal{C}}(\mathcal{W}) : Z_{\mathcal{C}}\left(\coprod_{i=1}^{n} \mathcal{Y}_i\right) = \bigotimes_{i=1}^{n} \mathcal{M}ap_{\mathcal{C}}(\mathbb{1}, \Psi(\mathcal{Y}_i)) \to \mathcal{M}ap_{\mathcal{C}}\left(\mathbb{1}, \bigotimes_{i} \Psi(\mathcal{Y}_i)\right) \cong \mathcal{M}ap_{\mathcal{C}}\left(\mathbb{1}, \Psi\left(\coprod_{i} \mathcal{Y}_i\right)\right) \xrightarrow{\Psi(\mathcal{W})} \mathcal{M}ap_{\mathcal{C}}(\mathbb{1}, \mathcal{Y}) = Z_{\mathcal{C}}(\mathcal{Y}) \ .$$

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see Lurie's website http://www.math.harvard.edu/~lurie/.