Group Representations and Matrix Spectral Problems

Ana Paula Santana [1]
João Filipe Queiró [1,2]
Eduardo Marques de Sá [1,2]

Departamento de Matemática
Universidade de Coimbra
3000 Coimbra, Portugal

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Dedicated to the memory of Robert C. Thompson

Abstract

We study a connection, via group representation theory, between the problem of describing the invariant factors of a product of two matrices over a principal ideal domain and the problem of describing the spectrum of a sum of two Hermitian matrices.

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1 Introduction

Consider the following two matrix problems:
- Given the invariant factors of two nonsingular $n \times n$ matrices A and B over a principal ideal domain $\mathcal{R}$, what can be said about the invariant factors of AB?
- Given the eigenvalues of two complex $n \times n$ Hermitian matrices A and B, what can be said about the eigenvalues of $A + B$?

In the past 15 or 20 years, several people have observed a remarkable formal analogy between results known for these two problems, and there is a feeling that they are in some sense the same.

Our goal here is to establish a connection between the two problems. This is done by relating both to the fundamental problem of describing the decomposition of the tensor product of two irreducible polynomial representations of $\mathrm{GL}_n(\mathbf{C})$ into a direct sum of irreducible polynomial representations.

A central role is played by the combinatorial Littlewood-Richardson rule, originally found in connection with the problem of representing the product of two Schur polynomials as a linear combination of Schur polynomials.

The connection between the invariant factor problem and the group representation problem has been made by R.C. Thompson [25], using work by P. Hall, J.A. Green and T. Klein [16]. To pass to the Hermitian eigenvalue problem we use a result of G.J. Heckman [11] concerning representations of compact Lie groups.

In the last sections we collect some additional results and remarks concerning the Littlewood-Richardson rule and matrix spectral problems.

Notation. Throughout the paper n is a fixed positive integer, and the symbol $\Lambda+$ denotes the set of all n-tuples, $\alpha = (\alpha_1, \ldots, \alpha_n)$, of nonincreasing nonnegative integers, $\alpha_1 \geq \ldots \geq \alpha_n \geq 0$.

Given α and β in $\Lambda+$, we denote by $\mathrm{LR}(\alpha, \beta)$ the set of all $\gamma \in \Lambda+$ that can be obtained from α and β according to the Littlewood-Richardson rule (LR rule, for short; see *e.g.* [8, App.A.1]). According to our conventions, the elements of $\mathrm{LR}(\alpha, \beta)$ are n-tuples. The *conjugate* of $\alpha \in \Lambda+$ is denoted by $\tilde{\alpha}$; so $\tilde{\alpha}_t := \max\{i : t \leq \alpha_i\}$.

2 Invariant Factors

Let $\mathcal{R}$ be a principal ideal domain (PID), p a fixed nonzero prime element of $\mathcal{R}$, and $\mathcal{R}_p$ the corresponding local domain. There are some problems involving modules and matrices over $\mathcal{R}$, or $\mathcal{R}_p$, which turn out to be equivalent and have solutions in terms of the LR rule. Let us focus on three of them, that we briefly identify as the problems on: module extensions, products of matrices, and 2-by-2 block matrices.

In the first one we have three finitely generated torsion $\mathcal{R}$-modules, M, N, K, and an exact sequence

$$0 \to N \xrightarrow{\iota} M \xrightarrow{\pi} K \to 0\,. \tag{1}$$

We are asked to find out the relations between the invariant factors of the modules, imposed by the exactness of the sequence. Obviously, for our purposes we may think N is a submodule of M, K is M/N, and ι and π are the inclusion and the projection maps. So the problem is to find all essentially distinct ways a module N can be extended to a supermodule M with prescribed invariant factors.

The 'product problem' consists in describing the invariant factors of a product of two nonsingular matrices A and B over $\mathcal{R}$ in terms of the invariant factors of A and B.

The third one, known among matrix theorists as the "Carlson problem", is the following: given two square matrices, S and T, of orders s and t respectively, with $s + t = n$, with entries on a field $\mathbf{F}$, we are required to describe all possibilities for the similarity invariant polynomials of

$$\begin{bmatrix} S & X \\ 0 & T \end{bmatrix}, \tag{2}$$

where X runs over the set of $s\times t$ matrices with entries in $\mathbf{F}$. This is equivalent to characterizing, over the polynomial ring $\mathbf{F}[z]$, the invariant factors of

$$\begin{bmatrix} S(z) & X(z) \\ 0 & T(z) \end{bmatrix}, \tag{3}$$

where $S(z) = zI - S$, $T(z) = zI - T$ and $X(z)$ runs over the set of $s \times t$ matrices over $\mathbf{F}[z]$. The obvious generalization of this problem is to characterize the invariant factors of a matrix of the type (2) but, this time, with S

and T fixed nonsingular matrices over $\mathcal{R}$ and X running over the set of $s \times t$ $\mathcal{R}$-matrices.

All these problems are "localizable". For the matrix product problem, say, this means that, if we solve the problem when A and B are viewed as matrices over $\mathcal{R}_p$, i.e., if we characterize the power-of-p elementary divisors of a matrix product, for each individual prime p, then we get a solution to the general problem, by just merging together chains of prime power elementary divisors. This is done by R. Thompson in [25] (see also [3], for products of possibly singular matrices). Localization and primary decompositions are well-understood techniques both in abstract commutative algebra and module theory (*e.g.* [1, 13]) as well as in the more concrete matrix equivalence setting (*e.g.* [9, 25]). So in what follows we only consider the local case.

On the basis of previous work by P. Hall and J.A. Green, T. Klein [16] proved the following:

> **Result 1.** *There exists a p-module M and a submodule N over $\mathcal{R}$ with invariant factors $p^{\alpha_1}, \ldots, p^{\alpha_n}$ and $p^{\gamma_1}, \ldots, p^{\gamma_n}$, respectively, such that M/N has invariant factors $p^{\beta_1}, \ldots, p^{\beta_n}$ if and only if $\gamma \in \mathrm{LR}(\alpha, \beta)$.*

As explained at length in [25], using the concept of relations matrix for a module, a matrix result follows from the Klein theorem:

> **Result 2.** *There exist nonsingular $n \times n$ matrices A_p and B_p over the local domain $\mathcal{R}_p$ with invariant factors $p^{\alpha_1}, \ldots, p^{\alpha_n}$ and $p^{\beta_1}, \ldots, p^{\beta_n}$, respectively, such that A_pB_p has invariant factors $p^{\gamma_1}, \ldots, p^{\gamma_n}$ if and only if $\gamma \in \mathrm{LR}(\alpha, \beta)$.*

In section 5 we suggest a simple alternative to the relations matrix approach to show the equivalence of the product problem to the module extension problem and the 2- by-2 block matrix problem. This latter problem is also localizable and, in the local case, the answer is:

> **Result 3.** *There exist matrices S, T and X over $\mathcal{R}_p$, with dimensions as above, S and T nonsingular, such that S, T and* (2) *have invariant factors $p^{\alpha_1}, \ldots, p^{\alpha_s}$, $p^{\beta_1}, \ldots, p^{\beta_t}$ and $p^{\gamma_1}, \ldots, p^{\gamma_n}$, respectively, if and only if $\gamma \in \mathrm{LR}(\alpha, \beta)$.*

(Here we add $n - s$ zeros to the α's and $n - t$ zeros to the β's.)

An interesting fact is that the answer to these problems is independent of the base ring, or the field $\mathbf{F}$. The formulation of the 2-by-2-block matrix problem in terms of module extensions is more or less clear for module theoretically inclined people who perceive (2) as a *concrete* module extension. This formulation has been explicitly made in [5], and therefore Result 3 is at hand by way of the Hall-Green-Klein result. However, the 'Carlson problem', as presented in [5], is a quest for inequalities relating the exponents of the involved elementary divisors. For many years, Robert Thompson had a very intense and fruitful activity on this problem, whereby a huge system of such inequalities has been produced [25] involving the α_i, β_i and γ_i which is believed to completely characterize the assertion $\gamma \in \mathrm{LR}(\alpha, \beta)$.

3 The Main Result

An $n \times n$ Hermitian matrix A has n real eigenvalues that we always represent in nonincreasing order, say $\alpha_1 \geq \ldots \geq \alpha_n$. The n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ is called the *spectrum* of A.

Theorem 3.1. *Let $\alpha, \beta, \gamma \in \Lambda+$. If $\gamma \in \mathrm{LR}(\alpha, \beta)$ there exist $n \times n$ Hermitian matrices A and B with eigenvalues $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ such that $A + B$ has eigenvalues $\gamma_1, \ldots, \gamma_n$.*

Taking into account the comments in section 2, our theorem has some obvious corollaries relating eigenvalues of Hermitian matrices with invariant factors of module extensions, matrix products and partitioned matrices (2) over $\mathcal{R}_p$. We only state the result concerning the matrix product case:

Corollary 3.2. *Let $\alpha, \beta, \gamma \in \Lambda+$. If there exist nonsingular $n \times n$ matrices A_p and B_p over $\mathcal{R}_p$ with invariant factors $p^{\alpha_1}, \ldots, p^{\alpha_n}$ and $p^{\beta_1}, \ldots, p^{\beta_n}$, respectively, such that $A_p B_p$ has invariant factors $p^{\gamma_1}, \ldots, p^{\gamma_n}$, then there exist $n \times n$ Hermitian matrices A and B with eigenvalues $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ such that $A + B$ has eigenvalues $\gamma_1, \ldots, \gamma_n$.* ∎

Before proving the theorem, let us briefly review some basic concepts and notations on the finite-dimensional irreducible representations of $\mathrm{GL}_n = \mathrm{GL}_n(\mathbf{C})$ and their relationship with the LR rule (see, *e.g.*, [8]). A complex representation of dimension m of GL_n is a complex-analytic group homomorphism $\rho : \mathrm{GL}_n \to \mathrm{GL}(V)$, where V is an m-dimensional complex vector space. For simplicity we will sometimes call V itself a representation of GL_n.

For each $\alpha \in \Lambda+$ let V_α denote the irreducible polynomial representation of GL_n with highest weight α relative to the maximal torus of GL_n of diagonal matrices (as usual, these weights are identified with the elements of $\Lambda+$). It is well-known that $\{V_\alpha : \alpha \in \Lambda+\}$ is a full set of irreducible polynomial complex representations of GL_n. For α and β in $\Lambda+$, the tensor product representation $V_\alpha \otimes V_\beta$ has the following fundamental decomposition into irreducible components:

$$V_\alpha \otimes V_\beta \cong \bigoplus_\gamma N_{\alpha\beta\gamma} V_\gamma \,, \tag{4}$$

where γ runs over $\Lambda+$, and $N_{\alpha\beta\gamma}$ is the number of ways the sequence γ can be obtained from α and β according to the Littlewood-Richardson rule. Therefore, the irreducible representation V_γ occurs in the right-hand side of (4) if and only if $\gamma \in \mathrm{LR}(\alpha, \beta)$.

We now turn our attention to the group U_n of $n \times n$ unitary matrices and its continuous representations [17]. U_n is a connected and (maximal) compact subgroup of GL_n. If ρ is a representation of GL_n, then its restriction to U_n is a representation of this group. By Weyl's so-called "unitarian trick" (cf. [27], [8, p.129]), the restriction of the representation V_α to U_n, also denoted by V_α, remains irreducible. Moreover, the V_α's are pairwise non-isomorphic representations of U_n. It follows that the decomposition (4) for tensor products remains valid for U_n.

Let us go back to the eigenvalues of a sum of two Hermitian matrices. Let $\mathbf{u}_n$ be the Lie algebra of U_n, i.e., the elements of $\mathbf{u}_n$ are the $n \times n$ skew-Hermitian matrices. U_n acts on $\mathbf{u}_n$ via the adjoint representation

$$\mathrm{Ad}\, u(X) = uXu^{-1}, \quad u \in \mathrm{U}_n,\ X \in \mathbf{u}_n \,.$$

The dual of the adjoint representation is called the *coadjoint* representation.

We identify $\mathbf{u}_n$ with its dual $\mathbf{u}_n^*$ in the usual way, using a U_n-invariant inner product. With this identification the coadjoint representation of U_n is equivalent to the adjoint representation, and therefore, identifying $\mathbf{u}_n$ with $\frac{1}{i}\mathbf{u}_n$, we can look at the orbits of U_n in the coadjoint representation as conjugation orbits in the space of Hermitian matrices, i.e., classes of unitarily similar Hermitian matrices. These are parametrized by nonincreasing sequences $\alpha = (\alpha_1, \ldots, \alpha_n)$ of real numbers, the spectra of the matrices in each class. Denote by O_α the orbit, or class, corresponding to the sequence α. We are now ready to enter the core of the

Proof of Theorem 3.1. Let $A, B \in \mathbf{u}_n$ with spectra α and β. We are interested in the set

$$\begin{aligned} O_\alpha + O_\beta &= \{M + N :\, M \in O_\alpha, N \in O_\beta\} \\ &= \{uAu^{-1} + vBv^{-1} :\, u, v \in \mathrm{U}_n\} \end{aligned}$$

which, being U_n-invariant, is a union of orbits.

Identifying $\mathbf{u}_n$ with its diagonal embedding $\{(X, X) :\, X \in \mathbf{u}_n\}$ in $\mathbf{u}_n \oplus \mathbf{u}_n$, every linear functional on $\mathbf{u}_n \oplus \mathbf{u}_n$ can be restricted to $\mathbf{u}_n$. Since $\mathbf{u}_n^* \oplus \mathbf{u}_n^* \cong (\mathbf{u}_n \oplus \mathbf{u}_n)^*$, this restriction gives rise to a projection $q : \mathbf{u}_n^* \oplus \mathbf{u}_n^* \to \mathbf{u}_n^*$.

Let φ and ψ be elements of $\mathbf{u}_n^*$ corresponding to Hermitian matrices A and B. Then it is easy to see that $A + B$ is the element of $\mathbf{u}_n$ corresponding to $q(\varphi, \psi)$ under the above mentioned identification. Therefore, given a spectrum γ, the orbit O_γ occurs in the sum $O_\alpha + O_\beta$ if and only if it occurs in $q(O_\alpha \times O_\beta) = q(O_{\alpha\beta})$, where $O_{\alpha\beta}$ denotes the $\mathrm{U}_n \times \mathrm{U}_n$-coadjoint orbit in $\mathbf{u}_n^* \oplus \mathbf{u}_n^*$ corresponding to the orbit associated with the spectrum (α, β) in the adjoint representation of $\mathrm{U}_n \times \mathrm{U}_n$ in its Lie algebra $\mathbf{u}_n \oplus \mathbf{u}_n$.

We now relate the problem of projection of orbits with the problem of the irreducible decomposition of the tensor product of irreducible representations of U_n. Given $\alpha, \beta, \gamma \in \Lambda+$, denote by $V_{\alpha\beta}$ the exterior tensor product $V_\alpha \boxtimes V_\beta$. This is the irreducible representation of $\mathrm{U}_n \times \mathrm{U}_n$ with highest weight (α, β). Identifying U_n with the diagonal subgroup of $\mathrm{U}_n \times \mathrm{U}_n$, we may consider the restriction of $V_{\alpha\beta}$ to U_n, which is nothing but the usual tensor product representation $V_\alpha \otimes V_\beta$.

The situation we have here is considered in G.J. Heckman's paper [11] for an arbitrary compact connected Lie group ($\mathrm{U}_n \times \mathrm{U}_n$ in our case) and a closed connected subgroup (here, the diagonal subgroup of $\mathrm{U}_n \times \mathrm{U}_n$). Heckman is concerned with the following problem: Given an irreducible representation of the group, what can be said about the decomposition into irreducible representations of its restriction to the subgroup? He gives several results on this question, by relating the restriction problem to the projection of coadjoint orbits. His Theorem 7.5 [11, p.352] implies the following:

> *If α and β each have distinct coordinates, and if V_γ occurs in the decomposition of $V_{\alpha\beta}|_{\mathrm{U}_n}$ into irreducible representations, then O_γ occurs in $q(O_{\alpha\beta})$.*

Thus if $\alpha, \beta \in \Lambda+$ each have distinct coordinates, and $\gamma \in \mathrm{LR}(\alpha, \beta)$, by Heckman's theorem above the orbit O_γ occurs in $O_\alpha + O_\beta$.

Introduce the notation $\mathrm{Sp}(\alpha,\beta)$ for the set of nonincreasing sequences γ of n real numbers which are spectra of sums of two $n\times n$ Hermitian matrices with spectra α and β. Then our conclusion reduces to the simple formula

$$\mathrm{LR}(\alpha,\beta)\subseteq \mathrm{Sp}(\alpha,\beta)\cap \mathbf{Z}^n\,, \tag{5}$$

in the case $\alpha,\beta\in\Lambda+$ each have distinct coordinates. To lift this restriction on α and β we first prove a lemma telling that $\mathrm{LR}(\alpha,\beta)$ is a homogeneous set.

Lemma 3.3. *For $\alpha,\beta,\gamma\in\Lambda+$, and r a natural number*

$$\gamma\in \mathrm{LR}(\alpha,\beta)\ \Rightarrow\ r\gamma\in \mathrm{LR}(r\alpha,r\beta)\,.$$

Proof of the lemma. There are many equivalent ways of describing the LR rule. To get an easy proof of our claim we adopt the rule as given in [8, p.456], namely: $\gamma\in\mathrm{LR}(\alpha,\beta)$ iff the Young diagram for α can be *expanded* to the Young diagram for γ by a *strict β- expansion* (see [8, p.456] for definitions). Let x_{it} denote the number of boxes labelled with 't' occurring in row i of the expanded diagram. Then $\gamma\in\mathrm{LR}(\alpha,\beta)$ iff there exist n^2 integers x_{it} satisfying the inequalities

$$x_{it}\geq 0 \tag{6}$$
$$\textstyle\sum_{i=1}^{n} x_{it}=\beta_t \tag{7}$$
$$\textstyle\sum_{t=1}^{n} x_{it}=\gamma_i-\alpha_i \tag{8}$$
$$\textstyle\sum_{i=1}^{k-1} x_{it}\geq \sum_{i=1}^{k} x_{i,t+1}\,, \tag{9}$$
$$\textstyle\alpha_i+\sum_{t=1}^{\tau-1} x_{it}\geq \alpha_{i+1}+\sum_{t=1}^{\tau} x_{i+1,t} \tag{10}$$

where i, t, τ and k run over $\{1,\ldots,n\}$; this goes with the convention $x_{it}=0$ for $i>n$ or $t>n$. (Compare with [26, p.71]. The left hand side of (9) is zero for $k=1$; an easy induction on k shows that (6)&(9) imply $x_{it}=0$ for $i<t$.) Note that (10) expresses Pieri's condition that there are no two boxes in the same column with the same label 'τ'; and (9) means that the expansion is strict. The lemma follows from the fact that the system (6)-(10) behaves well under multiplication of all parameters by r.

Returning to the proof of the theorem, given $\gamma\in\Lambda+$ define $J\gamma=\gamma+(n-1,n-2,\ldots,1,0)$. It is obvious that

$$\gamma\in\mathrm{LR}(\alpha,\beta)\ \Rightarrow\ JJ\gamma\in\mathrm{LR}(J\alpha,J\beta)\,. \tag{11}$$

In fact, the conclusion $J\gamma \in \mathrm{LR}(J\alpha, \beta)$ is trivial in view of (6)-(10); and then (11) follows from $\mathrm{LR}(\alpha,\beta) = \mathrm{LR}(\beta,\alpha)$. If $\|\cdot\|$ denotes the infinity norm, we have

$$\|JJ\gamma - \gamma\| = 2(n-1)\,.$$

Let us now prove (5) for arbitrary $\alpha, \beta \in \Lambda+$. For $\gamma \in \mathrm{LR}(\alpha,\beta)$ and each natural number N, consider the rational n-tuples

$$\alpha_N := \frac{1}{N} JN\alpha, \quad \beta_N := \frac{1}{N} JN\beta \quad \text{and} \quad \gamma_N := \frac{1}{N} JJN\gamma\,.$$

By Lemma 3.3 and (11) above, $JJN\gamma \in \mathrm{LR}(JN\alpha, JN\beta)$. As the n-tuples $JN\alpha$ and $JN\beta$ each have distinct coordinates, $JJN\gamma$ lies in $\mathrm{Sp}(JN\alpha, JN\beta)$. Therefore $\gamma_N \in \mathrm{Sp}(\alpha_N, \beta_N)$. Compute now the distances

$$\begin{aligned} \|\gamma_N - \gamma\| &= \|JJN\gamma - N\gamma\|/N = 2(n-1)/N \\ \|\alpha_N - \alpha\| &= \|\beta_N - \beta\| = (n-1)/N\,. \end{aligned}$$

Hence the n-tuple sequences α_N, β_N and γ_N converge to α, β and γ, respectively. The compactness of U_n gives that $\gamma \in \mathrm{Sp}(\alpha,\beta)$, as desired. ■

G. J. Heckman (private communication) has informed us that the restriction can also be lifted by adapting the argument [11, section 7].

4 Further Results

Given α and β in $\Lambda+$, put $\mathrm{E}(\alpha,\beta) := \mathrm{Sp}(\alpha,\beta) \cap \mathbf{Z}^n$, i.e., $\mathrm{E}(\alpha,\beta)$ is the set of $\gamma \in \Lambda+$ for which there exist $n \times n$ Hermitian matrices A and B with spectra α and β, such that $A + B$ has spectrum γ. Moreover let $\mathrm{H}(\alpha,\beta)$ be the set of those $\gamma \in \Lambda^+$ satisfying the so-called *Horn inequalities*. These are inequalities of the type

$$\gamma_{k_1} + \cdots + \gamma_{k_r} \le \alpha_{i_1} + \cdots + \alpha_{i_r} + \beta_{j_1} + \cdots + \beta_{j_r}\,, \tag{12}$$

where the i's, j's and k's run over a finite set of $3r$-tuples of integers defined by an intricate recursive condition (see [12]). We point out that the so-called *Lidskii inequalities* (from V.I. Lidskii), namely

$$\gamma_{i_1} + \cdots + \gamma_{i_r} \le \alpha_{i_1} + \cdots + \alpha_{i_r} + \beta_1 + \cdots + \beta_r\,, \tag{13}$$

for all $1 \le i_1 < \ldots < i_r \le n$ and $1 \le r \le n$, with equality if $r = n$, form a subsystem of Horn's inequalities. Note that the Lidskii inequalities may be condensed in the formula

$$\gamma - \alpha \preceq \beta\,,$$

where $\gamma - \alpha$ is the coordinatewise difference, and $\preceq$ denotes the *majorization* of Hardy-Littlewood-Polya (see, *e.g.*, [4]). Clearly these definitions as well as the following comments may be partly extended to nonincreasing *real* n-tuples, α, β, γ. But we are mainly interested in the integer case, and anyway from context it will be clear which extensions are allowable.

The relations between the three sets $\mathrm{LR}(\alpha,\beta)$, $\mathrm{E}(\alpha,\beta)$ and $\mathrm{H}(\alpha,\beta)$ have been extensively considered in the past few decades. In 1962, A. Horn [12] conjectured that $\mathrm{E}(\alpha,\beta) \subseteq \mathrm{H}(\alpha,\beta)$, i.e., the eigenvalues of sums of Hermitian matrices satisfy the inequalities we now associate with his name; he in fact proved equality for small values of n ($n \le 4$). The conjecture that

$$\mathrm{LR}(\alpha,\beta) = \mathrm{E}(\alpha,\beta) = \mathrm{H}(\alpha,\beta) \tag{14}$$

has been, for many years, a *leitmotiv* in the work of Robert Thompson. We send the reader to his excelent and very well-documented survey on this and other related problems [26]. There, Thompson tells how the set $\mathrm{LR}(\alpha,\beta)$ appears in this context and reports on the thesis of S. Johnson [14]. In our notation, Johnson and Thompson (see [14], [25, p.433] and also [26, p.73]) presented the result

$$\mathrm{LR}(\alpha,\beta) \subseteq \mathrm{H}(\alpha,\beta)\,. \tag{15}$$

On the other hand, our Theorem 3.1 asserts that

$$\mathrm{LR}(\alpha,\beta) \subseteq \mathrm{E}(\alpha,\beta)\,.$$

In spite of B.V. Lidskii's announcement [19] (see discussion in [26, Lecture 6]), these are the only inclusions we have for granted. Let us report some particular cases for which a sound proof of (14) is available.

Lemma 4.1 . *Let $\delta = (\delta_1, \ldots, \delta_n)$ be an n-tuple of nonnegative integers, and $\beta \in \Lambda+$. Define $m := \beta_1$. The following conditions are pairwise equivalent:*

(a) *$\delta \preceq \beta$;*

(b) *There exists an $n \times m$ 0-1 matrix Z with row sum vector δ and column sum vector $\tilde{\beta}$, satisfying, for $1 \le k \le n$ and $1 \le t < m$,*

$$\textstyle\sum_{i=k}^{n} z_{it} \ge \sum_{i=k}^{n} z_{i,t+1}\,; \tag{16}$$

(c) *There exists an* $n \times n$ *(lower triangular) matrix* X*, with nonnegative integer entries, with row sum vector* δ *and column sum vector* β*, satisfying, for* $1 \le k \le n$ *and* $1 \le t < n$*,*

$$\textstyle\sum_{i=1}^{k-1} x_{it} \ge \sum_{i=1}^{k} x_{i,t+1} \,. \tag{17}$$

Proof. (b)$\Leftrightarrow$(c). Recall a version of the LR rule (see [16]) according to which $\gamma \in \mathrm{LR}(\alpha,\beta)$ iff there exists an $n \times m$ 0-1 matrix Z with row sum vector $\gamma - \alpha$ and column sum vector $\tilde{\beta}$, satisfying (16) and

$$\alpha_i + \textstyle\sum_{t=1}^{\tau} z_{it} \ge \alpha_{i+1} + \sum_{t=1}^{\tau} z_{i+1,t} \,, \tag{18}$$

for the relevant values of i, t, k, τ. Choose any $\alpha \in \Lambda+$ having big enough gaps, $\alpha_i - \alpha_{i+1}$, so that the inequalities (18) become redundant. *E.g.*, take $\alpha_i - \alpha_{i+1} \ge m$ for all $i < n$. Then define $\gamma := \alpha + \delta$. In either case, (b) or (c), we have that $\gamma \in \Lambda+$. So (b) is equivalent to $\gamma \in \mathrm{LR}(\alpha,\beta)$; and so is (c), because (10) is redundant in system (6)-(10).

(b)$\Rightarrow$(a) follows from the Gale-Ryser theorem [4, p.176].

(a)$\Rightarrow$(c). It is obvious that

$$\beta = \mathbf{b}_1 + \cdots + \mathbf{b}_m \,, \tag{19}$$

where $\mathbf{b}_j \in \Lambda+$ is the n-tuple $(1,\ldots,1,0,\ldots,0)$ with $\tilde{\beta}_j$ ones. By the Gale-Ryser theorem, there exists an $n \times m$ 0-1 matrix Z with row sum vector δ and column sum vector $\tilde{\beta}$. Let us fix $j \in \{1,\ldots,m\}$. In column j of Z there are $\tilde{\beta}_j$ 1's which occur, say, in rows $f_j(1),\ldots,f_j(\tilde{\beta}_j)$, written in *increasing* order; define an $n \times n$ 0-1 matrix $X(j)$ having precisely $\tilde{\beta}_j$ ones: one in each position $(f_j(t),t)$. It is easily seen that $X(j)$ is a solution to the system (6)-(9) with β replaced by $\mathbf{b}_j$, and $\gamma - \alpha$ replaced by a permutation of $\mathbf{b}_j$. Trivially, the matrix $X := X(1) + \cdots + X(m)$ has the same row sum vector as Z and, by (19), β is the column sum vector of X. So X is a solution to (6)-(9). ∎

The line of thought of the previous proof yields a quantitative refinement involving the *Kostka numbers* $K_{\mu\lambda}$, as defined, *e.g.*, in [8, p.56], namely: the number of ways to fill in the Young diagram of μ with λ_i symbols 'i', for $i = 1,\ldots,n$, such that the resulting tableau has nondecreasing rows and strictly increasing columns. Here μ is supposed to lie in $\Lambda+$ but the λ_i's may be given in any order: this follows from the commutativity of the tensor product and from the fact that

$$V_{(\lambda_1)} \otimes \cdots \otimes V_{(\lambda_n)} \cong \bigoplus_{\mu} K_{\mu\lambda} V_{\mu} ,$$

where, as usual, (k) denotes the n-tuple $(k, 0, \ldots, 0)$. (See also [8, p.456], [2, Theorem 2.23].)

Theorem 4.2. *Let $\alpha, \beta, \gamma \in \Lambda+$. If*

$$\alpha_i \geq \gamma_{i+1} , \quad \text{for } i = 1, \ldots, n-1 , \tag{20}$$

the irreducible representation V_γ occurs in the irreducible decomposition of $V_\alpha \otimes V_\beta$ with multiplicity $K_{\tilde{\beta}, \gamma - \alpha}$. If $\alpha_i - \alpha_{i+1} \geq \beta_1$ for $1 \leq i < n$, then

$$V_\alpha \otimes V_\beta \cong \bigoplus_{\gamma} K_{\tilde{\beta}, \gamma - \alpha} V_\gamma . \tag{21}$$

Proof. Let $\delta := \gamma - \alpha$. The number of matrices Z satisfying item (b) of Lemma 4.1, and the inequalities (18), is precisely the $N_{\alpha\beta\gamma}$ of (4). The condition (20) makes (18) redundant, so that $N_{\alpha\beta\gamma}$ is the number of ways to fill in the Young diagram of $\tilde{\beta}$ with δ_{n-i+1} symbols 'i', for $i = 1, \ldots, n$, such that the resulting tableau has non decreasing rows and strictly increasing columns. This proves the first part of the theorem. For the second part observe that the 'big gap' condition $\alpha_i - \alpha_{i+1} \geq \beta_1$ implies (20), for all $\gamma \in \mathrm{LR}(\alpha, \beta)$. ■

Remark 4.3. Lemma 4.1 is related to [21, 18]. Note that in [18, Theorem 3.1] the symbol $\lambda - \tau$ denotes, in our componentwise-difference notation, the partition $(\tilde{\lambda} - \tilde{\tau})\tilde{}$.

The equivalence of (b) and (c) of the next theorem is proved by V.I. Lidskii [20] in case $\alpha_i - \alpha_{i+1} \geq \beta_1$ for $1 \leq i < n$.

Theorem 4.4. *If $\alpha, \beta, \gamma \in \Lambda+$ satisfy (20), then the following conditions are pairwise equivalent:*

(a) $\gamma \in \mathrm{LR}(\alpha, \beta)$*;*
(b) $\gamma \in \mathrm{E}(\alpha, \beta)$*;*
(c) α, β, γ *satisfy the Lidskii inequalities;*
(d) α, β, γ *satisfy the Horn inequalities.*

Moreover, if $\alpha_i - \alpha_{i+1} \geq \beta_1$ for $1 \leq i < n$, then (14) holds.

Proof. The equivalence of (a), (b) and (c) is an easy matter, using Theorems 3.1, 4.1, 4.2, and the Lidskii-Wielandt theorem [20, 28]. The Johnson-Thompson result quoted in (15) gives (a)⇒(d); however, under the stringent condition (20), the following simple argument shows that the Lidskii inequalities imply the Horn inequalities. Let (12) be an inequality of the latter type. This means that $i := (i_1, \ldots, i_r)$, $j := (j_1, \ldots, j_r)$ and $k := (k_1, \ldots, k_r)$ are integer r-tuples satisfying Horn's recursion, that is, $(i; j; k)$ belongs to the set T_r^n defined in [12, p.236]. The definition of T_r^n implies $i_v + j_1 \leq k_v + 1$ and, therefore, $i_v \leq k_v$ for $v \in \{1, \ldots, r\}$. Denote the set $\{v : i_v = k_v\}$ by $\{u_1, \ldots, u_s\}$, with the u's in increasing order. As

$$(u_1, \ldots, u_s; 1, \ldots, s; u_1, \ldots, u_s) \in T_s^r \,,$$

we have, by the definition of T_r^n, that $j_1 + \cdots + j_s \leq s(s+1)/2$; therefore, $(j_1, \ldots, j_s) = (1, \ldots, r)$. We have

$$\gamma_{k_{u_1}} + \cdots + \gamma_{k_{u_s}} \leq \alpha_{i_{u_1}} + \cdots + \alpha_{i_{u_s}} + \beta_{j_1} + \cdots + \beta_{j_s} \,,$$

because this is one of the Lidskii inequalities. Then (12) also holds, because $\gamma_{k_v} \leq \alpha_{i_v}$, in case $v \notin \{u_1, \ldots, u_s\}$. ■

It is an easy matter to prove (14) when α and β each have only two distinct entries. We may reduce the problem to the case

$$\alpha = (\underbrace{a, \ldots, a}_{r}, 0, \ldots, 0) \quad \text{and} \quad \beta = (\underbrace{b, \ldots, b}_{s}, 0, \ldots, 0) \tag{22}$$

where a and b are positive and $1 \leq s \leq r < n$.

Lemma 4.5. *For α and β as above, we have $\gamma \in E(\alpha, \beta)$ if and only if*

$$\gamma_i = 0 \,, \qquad (r + s < i \leq n) \tag{23}$$

$$\gamma_i = a \,, \qquad (s < i \leq r) \tag{24}$$

$$\gamma_x + \gamma_{r+s-x+1} = a + b \,, \quad (1 \leq x \leq s) \tag{25}$$

(with the convention $\gamma_i = 0$ for $i > n$).

Proof. Let $\gamma \in \mathrm{E}(\alpha, \beta)$. The inequalities $\leq$ in (23)-(25) follow from the Weyl inequalities [12, Theorem 7] and from [12, Theorem 8]. The identities (23)-(25) are a consequence of this and the trace condition $\gamma_1 + \cdots + \gamma_n = ra + sb$.

For the converse we have to exhibit Hermitian matrices A and B with spectra (22) such that $A+B$ have spectrum γ. But this is easy to do, because (25) reduces our problem to the case $n = 2$. We omit the details. ■

Lemma 4.6. *In the same situation, if a and b are positive integers, we have $\gamma \in LR(\alpha, \beta)$ if and only if conditions* (23)-(25) *hold.*

Proof. The necessity of (23)-(25) follows as in the previous lemma, using a tiny part of Thompson's results [25] (see also [24]). The sufficiency is a nice exercise that we leave to the reader. ■

As a matter of fact, the 'nice exercise' may go a little further. Denote by σ an arbitrary $(n-r)$-tuple of nonnegative integers, $(\sigma_1, \ldots, \sigma_{n-r})$, in nonincreasing order, such that $\sigma_i \leq \beta_i$, for $1 \leq i \leq n-r$, and $\sigma_1 \leq a$. Let

$$\gamma(\sigma) := a^r\sigma + (\beta - \sigma)_{\downarrow}\,,$$

where $a^r\sigma$ denotes the n-tuple $(a, \ldots, a, \sigma_1, \ldots, \sigma_{n-r})$ and $(\beta - \sigma)_{\downarrow}$ is $\beta - \sigma$ in nonincreasing order. What is left to the reader is to show that

$$V_\alpha \otimes V_\beta \;\cong\; \bigoplus_\sigma V_{\gamma(\sigma)}\,.$$

LR rules for Eigenvalues. An interesting fact related to the conjecture (14) is that the eigenvalues of sums of Hermitian matrices satisfy a group of properties closely resembling the LR rules.

Given Hermitian matrices of order n, A and B, with spectra α and β, we may assume for our purpose that $B = \mathrm{Diag}(\beta_1, \ldots, \beta_n)$, and that all eigenvalues of B are nonnegative. For $t \geq 0$ define

$$\begin{aligned}
\beta_i(t) &:= \min\{t, \beta_i\}\\
B(t) &:= \mathrm{Diag}(\beta_1(t), \ldots, \beta_n(t))\\
\lambda_i(t) &:= \lambda_i[A + B(t)]\,,
\end{aligned}$$

where $\lambda_i(X)$ denotes the i-th eigenvalue of X. In a natural way the definition $\tilde{\beta}_t := \max\{i : t \leq \beta_i\}$ may be extended to *real* values of $t \geq 0$, yielding a left continuous step-function of t, having jumps for $t \in \{\beta_1, \ldots, \beta_n\}$.

The function $B(t)$ satisfies the following obvious properties involving the partial order $\leq^*$ induced by the cone of semi- definite Hermitian matrices:

(a) $B(t)$ is nondecreasing with respect to $\leq^*$;
(b) $\lambda_1[B(T) - B(t)] \leq T - t$, for $t \leq T$;
(c) $B(t)$ is **concave*, *i.e.*, for $0 \leq \omega \leq 1$,

$$\omega B(x) + (1-\omega)B(y) \leq^* B[\omega x + (1-\omega)y] \,.$$

As $B(t)$ is continuous, each eigenvalue $\lambda_i(t)$ is also continuous; in fact, it is analytic except at a finite number of points: the possible exceptions are the β_i's and points t' where $A + B(t')$ has a multiple eigenvalue (see [15, p.143]). But even at these exceptional points, $\lambda_i(t)$ has left and right derivatives of all orders. Here, we only need the first left derivative $\frac{d\lambda_i}{dt}(t^-)$.

Theorem 4.7. [LR Rules for Hermitian Matrices] *For all $t \geq 0$ we have:*

Rule 1. $\lambda_1(t) \geq \ldots \geq \lambda_n(t) \geq 0$

Rule 2. $\dfrac{d\lambda_i}{dt}(t^-) \in [0, 1]$

Rule 3. $\tilde{\beta}_t = \displaystyle\sum_{i=1}^{n} \frac{d\lambda_i}{dt}(t^-)$

Rule 4. $\displaystyle\sum_{i=k}^{n} \lambda_i(t)$ *is concave for $k = 1, \ldots, n$.*

Proof. Rule 1 is trivial. Rule 2 follows from (a), (b) and the inequalities $\lambda_i(X+Y) \leq \lambda_i(X) + \lambda_1(Y)$, valid for Hermitian matrices X and Y; in fact, if we let $X := A + B(t)$ and $Y := B(T) - B(t)$, we get for $t \leq T$:

$$\begin{aligned} 0 \leq \lambda_i(T) - \lambda_i(t) &= \lambda_i(X+Y) - \lambda_i(X) \\ &\leq \lambda_1[B(T) - B(t)] \leq T - t \,. \end{aligned}$$

Rule 3 follows from trace considerations and the formula

$$\operatorname{tr}[B(t) - B(t-\Delta)] = \Delta\tilde{\beta}_t \,,$$

for small positive Δ's. Rule 4 is a consequence of Ky Fan inequalities [7]

$$\sum_{i \geq k} \lambda_i(X+Y) \geq \sum_{i \geq k} \lambda_i(X) + \sum_{i \geq k} \lambda_i(Y) \,,$$

and the *concavity of $B(t)$:

$$\begin{aligned}\sum_{i\geq k}\lambda_i[\omega x+(1-\omega)y] &= \sum_{i\geq k}\lambda_i[A+B[\omega x+(1-\omega)y]]\\ &\geq \sum_{i\geq k}\lambda_i[A+\omega B(x)+(1-\omega)B(y)]\\ &\geq \sum_{i\geq k}\left[\omega\lambda_i(x)+(1-\omega)\lambda_i(y)\right].\end{aligned}$$

■

To explain the title we gave to Theorem 4.7, let us say that $f:[0,+\infty[\to \mathbf{R}$ is an *integral piecewise linear* function if f is continuous, $f(t)$ has integer values for integer t, and the graph of f is a polygonal line with vertices of integer abscissæ. Then the LR rule theorem [in the version [16] adopted above (18)] may be paraphrased as

> *The multiplicity $N_{\alpha\beta\gamma}$ in decomposition* (4) *is the number of n-tuples, $\lambda(t)=(\lambda_1(t),\ldots,\lambda_n(t))$, of integral piecewise linear functions λ_i which satisfy the rules* 1 *to* 4 *and the boundary conditions $\lambda(0)=\alpha$ and $\lambda(\beta_1)=\gamma$.*

5 Final Remarks

Remark 5.1 Let us revisit the three problems and results referred to in section 2. Our aim is to show the 'equivalence' of Result 1, Result 2 and Result 3 in that section, in the sense that, once we know one of the results, a short and easy argument is enough to get the other two as 'corollaries'.

For example, we give a simple alternative to the relations matrix approach of [25] to show the equivalence of Results 1 and 2. Let us associate an extension of torsion $\mathcal{R}$-modules to the product AB. We view A and B as mappings,

$$\mathcal{R}^n \xrightarrow{B} \mathcal{R}^n \xrightarrow{A} \mathcal{R}^n\,,$$

which give rise to an exact sequence of $\mathcal{R}$-modules

$$0\to\mathcal{R}^n/\mathrm{Im}(B)\xrightarrow{a}\mathcal{R}^n/\mathrm{Im}(AB)\xrightarrow{\pi}\mathcal{R}^n/\mathrm{Im}(A)\to 0\,. \qquad (26)$$

Here, a is given by $a(x + \mathrm{Im}(B)) = Ax + \mathrm{Im}(AB)$, and π is induced by the natural projection of $\mathcal{R}^n/\mathrm{Im}(AB)$ onto $[\mathcal{R}^n/\mathrm{Im}(AB)]/[\mathrm{Im}(A)/\mathrm{Im}(AB)]$. We then take into account that the invariant factors of A are the invariant factors of the cokernel $\mathcal{R}^n/\mathrm{Im}(A)$, etc. Conversely, any sequence (1) can be presented as (26) because any finitely generated torsion $\mathcal{R}$-module may be presented as $\mathcal{R}^m/\mathrm{Im}(X)$ for some nonsingular $m \times m$ matrix X.

The first version of the 2-by-2 block matrix problem over a field $\mathbf{F}$ [given just above (2)] gives rise to a result that we state below as Result 4, which parallels Result 3, where p is now a monic, irreducible polynomial over $\mathbf{F}$. Note that, in the field case, we cannot escape the obvious degree-dimension consistency condition: if S and T are square $\mathbf{F}$-matrices, having orders s and t, and having, respectively, $p^{\alpha_1}, \ldots, p^{\alpha_s}$ and $p^{\beta_1}, \ldots, p^{\beta_t}$ as elementary divisors, then

$$\alpha_1 + \cdots + \alpha_s \leq s/\deg(p) \quad \text{and} \quad \beta_1 + \cdots + \beta_t \leq t/\deg(p).$$

Assuming this, we have:

Result 4. *There exist $\mathbf{F}$-matrices, S, T and X, with dimensions as above, such that the power-of-p elementary divisors of S, T and* (2) *are $p^{\alpha_1}, \ldots, p^{\alpha_s}$, $p^{\beta_1}, \ldots, p^{\beta_t}$ and $p^{\gamma_1}, \ldots, p^{\gamma_n}$, respectively, if and only if $\gamma \in \mathrm{LR}(\alpha, \beta)$.*

(Here again we add $n - s$ zeros to the α's and $n - t$ zeros to the β's.)

An elementary approach to the localizability of the 2-by-2 block matrix problem goes as follows, in the case of an arbitrary PID. Given nonsingular $\mathcal{R}$-matrices S and T, we wish to describe the set $\mathrm{ED}\,(S, T)$, of all possible nontrivial elementary divisors of the matrices (2). Clearly, $\mathrm{ED}\,(S, T)$ remains the same if we replace S and T by matrices equivalent to S and T, respectively; or by $S \oplus U$, and $T \oplus V$, where U and V are any unimodular matrices. So we may assume that S $[T\,]$ is a diagonal matrix with the elementary divisors of S [of $T\,$] along the main diagonal. In fact we may assume that $S = S_1 \oplus \cdots \oplus S_m$, and $T = T_1 \oplus \cdots \oplus T_m$, where S_i and T_i are (possibly empty) nonsingular matrices, whose determinants are nonnegative powers of a prime p_i, and the involved primes, $p_1, \ldots, p_m$, are non associate in pairs. By a well-known reduction principle, (2) is then equivalent to a direct sum

$$\begin{bmatrix} S_1 & X_1 \\ 0 & T_1 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} S_m & X_m \\ 0 & T_m \end{bmatrix}, \qquad (27)$$

and therefore $\mathrm{ED}\,(S,T)$ is the join of $\mathrm{ED}\,(S_1,T_1),\ldots,\mathrm{ED}\,(S_m,T_m)$. Moreover, $\mathrm{ED}\,(S_i,T_i)$ is the set of nontrivial invariant factors of (2), viewed as a matrix over $\mathcal{R}_{p_i}$, for each i. That is the precise content of *localizing* this kind of problem.

In the 'Carlson problem', when (2) lives in an arbitrary field $\mathbf{F}$, the same argument applies to the characteristic matrices $zI-S$, $zI-T$ and (3). But we may also work inside $\mathbf{F}$, as follows. Assume, as we may, that S and T are direct sums of companion matrices of the respective elementary divisors. The companion blocks may be ordered so that $S=S_1\oplus\cdots\oplus S_m$, and $T=T_1\oplus\cdots\oplus T_m$, where S_i and T_i are (possibly empty) $\mathbf{F}$-matrices, whose characteristic polynomials are nonnegative powers of a monic irreducible polynomial p_i, and the involved primes, $p_1,\ldots,p_m$, are nonassociate in pairs. By [23], (2) is similar to a matrix like (27), etc. This explains what *localization* means for the 'Carlson problem'.

Having for granted that the 2-by-2 block matrix problem is an alternative formulation of the module extension problem, then Result 3 follows by the Hall-Green-Klein Result 1. Another way of doing things is to directly relate (as [22] does in particular cases) the 2-by-2 block matrix problem to the product problem. Let us for example sketch a proof of Result 3 along these lines. The following factorization

$$\begin{bmatrix} S & X \\ 0 & T \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & T \end{bmatrix} \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & I \end{bmatrix},$$

where the middle factor is unimodular, has been noticed in [10]. This, combined with Result 2, proves the *only if* part of Result 3. The converse is not so obvious: we have to show that any product

$$\begin{bmatrix} I & 0 \\ 0 & T \end{bmatrix} \Omega \begin{bmatrix} S & 0 \\ 0 & I \end{bmatrix}, \tag{28}$$

with a unimodular factor Ω is equivalent, over $\mathcal{R}_p$, to a matrix (2) for some X. The redeeming trick is

Lemma 5.1. *Over a local principal domain, any unimodular matrix Ω has an LUL-factorization, $\Omega=L_1UL_2$, where L_1 and L_2 are lower triangular, and U is upper triangular. Besides, the LUL-factorization property for 2-by-2 unimodular matrices characterizes local principal ideal domains.*

Proof. One of the entries in the first row of Ω is a unit of $\mathcal{R}_p$. If ω_{11} is not a unit, then ω_{1j} is a unit for some $j > 1$; adding column j of Ω to the first colunm, we get a unit in the $(1,1)$ position. We may then use the $(1,1)$ entry as a pivot, to eliminate all other entries in the first column, with lower triangular row operations. So the first part of the lemma follows by induction. The rest is a simple exercise. ■

We may combine the lemma with the simple observation that, the L_i's being given, there exist L_1', L_2', S' and T', such that

$$\begin{bmatrix} I & 0 \\ 0 & T \end{bmatrix} L_1 = L_1' \begin{bmatrix} I & 0 \\ 0 & T' \end{bmatrix} \quad \text{and} \quad L_2 \begin{bmatrix} S & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} S' & 0 \\ 0 & I \end{bmatrix} L_2',$$

where the L_i' are lower triangular, and S' and T' are equivalent to S and T, respectively. A moment's thought now shows that (28) is equivalent to (2) for some X. This completes the proof of Result 3, based on Result 2.

Remark 5.2 A recent paper by Dooley, Repka and Wildberger in this journal [6] undertakes a deep investigation of the Hermitian sum eigenvalue problem. An important result of that paper is the proof, using convexity results of Atiyah, Guillemin and Sternberg, and Kirwan, that the set we have denoted by $\mathrm{Sp}(\alpha, \beta)$ is a convex polytope. Later in the paper, the authors consider a generalization of the eigenvalue problem, namely they study the convolution of the invariant measures associated to two adjoint orbits of a compact Lie group. Their description of the support of such a convolution measure, together with Kirillov's character formula, allows them an application to representation theory and multiplicities occurring in decompositions of tensor products. This is close to the material in our paper, but Kirillov's character formula establishes a correspondence between an irreducible representation V_λ and the adjoint orbit $O_{\lambda+\delta}$ (where δ is the half-sum of the positive roots), and does not afford a direct link between V_λ and O_λ. It is, therefore, not clear how that approach might help in establishing Theorem 3.1 and the conjecture discussed in Section 4.

Remark 5.3 After this paper was submitted, we learned of the preprint "Stable bundles, representation theory and Hermitian operators", by A. A. Klyachko, with results closely related to ours, but using very different methods. In general terms, the relation between the two works is as follows: (1)

Klyachko's results imply our Theorem 3.1, but he does not obtain the conjectured identity $\mathrm{LR}(\alpha, \beta) = \mathrm{E}(\alpha, \beta)$, as we don't. (2) The results of our section 4, where we discuss some cases in which the identity holds, are not considered in Klyachko's work. See also "Littlewood-Richardson semigroups", by A. Zelevinsky (an abstract of a talk given in April 1997 at a MSRI workshop on Representation Theory and Symmetric Functions – available from http://www.msri.org).

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