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Jacobi structures in supergeometric formalism

Paulo Antunes, Camille Laurent-Gengoux *

CMUC, Department of Mathematics, University of Coimbra, Portugal

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ABSTRACT

We use the supergeometric formalism, more precisely, the so-called "big bracket" (for which brackets and anchors are encoded by functions on some graded symplectic manifold) to address the theory of Jacobi algebroids and bialgebroids, following mainly the previous works of Iglesias–Marrero [9] and Grabowski–Marmo [10]. This formalism is efficient to define the Jacobi–Gerstenhaber algebra structure associated to a Jacobi algebroid, to define its Poissonization, and to express the compatibility condition defining Jacobi bialgebroids. It also yields a simple description of the Jacobi bialgebroid associated to a Jacobi structure, and conversely, of the Jacobi structure associated to a Jacobi bialgebroid.

1. Introduction

We consider a Poisson structure $\mathcal{P}$ on a manifold X and an Euler vector field (i.e., a vector field $\mathcal{E}$ on X such that $\mathcal{L}_{\mathcal{E}}\mathcal{P} = \mathcal{P}$). A Lie–Poisson structure, taking for $\mathcal{E}$ the usual Euler vector field, is an example of that situation. More generally, any weight homogeneous Poisson structure [1], of weight $k \neq 2$, together with the weighted Euler vector field divided by $2 - k$ gives an example. In this situation, for every hyper-surface $M \subset X$ transversal to $\mathcal{E}$, i.e., such that

$$T_mM \oplus \mathbb{R}\mathcal{E}_m = T_mX, \quad \forall m \in M,$$

there exists, on the submanifold M, a unique bivector field π and a unique vector field E such that:

$$\mathcal{P}_m = \pi_m + \mathcal{E}_m \wedge E_m, \quad \forall m \in M.$$

It turns out that the pair (π, E) satisfies the following two relations,

$$[\pi, \pi] = -2E \wedge \pi \quad \text{and} \quad [E, \pi] = 0.$$

A bivector field and a vector field on a given manifold M satisfying these conditions form what is called a Jacobi structure on M [2]. It can be shown that any Jacobi structure can be obtained out of a Poisson structure (called "Poissonization") by the above procedure (see [2], or [3] for a more general presentation). Now, it is well-known [4,5] that the cotangent bundle T^*X of a Poisson manifold $(X, \mathcal{P})$ is endowed with a natural Lie algebroid bracket with anchor $\mathcal{P}^\# : T^*X \to TX$, whose restriction to exact one-forms is given by $[\mathrm{d}f, \mathrm{d}g]_{\mathcal{P}} = \mathrm{d}\langle \mathcal{P}, \mathrm{d}f \wedge \mathrm{d}g\rangle$. Also, the pair (TX, T^*X) is what is called a Lie bialgebroid over X (one can consult, for instance, [5] for a detailed introduction to these matters). A natural question, first addressed in [6], is to determine whether a similar construction can be carried out for Jacobi structures on M. The answer goes as follows: the restriction of $T^*X_{|M} \to M$ to M can again be endowed with a Lie algebroid structure (which is not a Lie subalgebroid of $T^*X \to X$), that forms a Jacobi bialgebroid structure, when paired with some natural Lie algebroid structure that appears on the restriction to M of TX. All these constructions can be extended from Poisson structures on Poisson manifolds to Poisson structures on Lie algebroids without any difficulty.

* Corresponding author.
E-mail addresses: pantunes@mat.uc.pt (P. Antunes), laurent@math.univ-poitiers.fr, claurent@mat.uc.pt (C. Laurent-Gengoux).

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In the continuation of [7,8], and in the same spirit of unification and simplification, the purpose of the present short note is to redo, by means of the big bracket, the theory of Jacobi manifolds and Jacobi (bi)-algebroids, exposed mainly in two articles, [9] by Iglesias and Marrero, [10] by Grabowski and Marmo, and continued in [11] by Caseiro et al.. The big bracket is the canonical Poisson structure on the graded cotangent bundle of a given vector bundle (considered itself as a graded manifold, see below) which has been defined and studied by Roytenberg [8], following previous works [12–14] which correspond to the particular case of vector spaces. It is a unifying tool of a remarkable efficiency when it comes to Lie algebroids or one of the many closely related objects (i.e., for instance, bialgebroids, quasi-Lie bialgebroids, triangular Lie bialgebroids, Courant algebroids, to mention a few), yielding both a nice interpretation of their definitions and mechanical proofs of the theorems that they satisfy. More precisely, our claim is that the big bracket reduces to mechanical (but somewhat cumbersome) computations most known results about Jacobi manifolds, once the objects that these articles deal with have been translated in terms of supermanifolds, a translation that we present herein. By mechanical, we mean that it reduces proofs to a play of substitutions involving the Jacobi and Leibnitz identity of the big bracket: we do not claim, however, that the big bracket was the adequate context to guess them.

2. The big bracket

2.1. Definitions

Let $A \to M$ be a vector bundle. There is a natural structure of graded 2-manifold on $T^*\Pi A$ (i.e., the cotangent space of the supermanifold ΠA) [8,15]. Moreover, the (sheaf of) algebra of functions $\mathcal{F}_A := \mathcal{F}(T^*\Pi A)$ is equipped with a graded Poisson bracket which is called the big bracket and denoted by $\{\cdot,\cdot\}$.

We do not intend to give the complete construction of this Poisson algebra, and refer to [8] for a more involved introduction, but we recall a few facts on these structures. First, the (sheaf of) algebra $\mathcal{F}_A$ admits a $(\mathbb{N}\times\mathbb{N})$-valued bi-degree:

$$\mathcal{F}_A = \oplus_{k,l\in\mathbb{N}\times\mathbb{N}}\,\mathcal{F}_A^{k,l}$$

and is graded commutative w.r.t. the total degree, i.e., $F_1F_2 = (-1)^{(k_1+l_1)(k_2+l_2)}F_2F_1$ for every $F_1 \in \mathcal{F}_A^{k_1,l_1}$ and $F_2 \in \mathcal{F}_A^{k_2,l_2}$. The big bracket is a (local) bilinear map $\mathcal{F}_A \times \mathcal{F}_A \to \mathcal{F}_A$, denoted $(F_1, F_2) \mapsto \{F_1, F_2\}$, mapping $\mathcal{F}_A^{k_1,l_1} \times \mathcal{F}_A^{k_2,l_2}$ to $\mathcal{F}_A^{k_1+k_2-1,l_1+l_2-1}$ for all $k_1, k_2, l_1, l_2 \in \mathbb{N}$, (with the understanding that $\mathcal{F}_A^{k,l} = 0$ if $k < 0$ or $l < 0$) and which satisfies the following relations for every $F_i \in \mathcal{F}_A^{k_i,l_i}$, $i = 1, 2, 3$,

$$\{F_1, F_2\} = -(-1)^{(k_1+l_1)(k_2+l_2)}\{F_2, F_1\} \quad \text{(graded skew-symmetry)}$$

$$\{F_1, F_2F_3\} = \{F_1, F_2\}F_3 + (-1)^{(k_1+l_1)(k_2+l_2)}F_2\{F_1, F_3\} \quad \text{(Leibnitz rule)} \tag{2.1}$$

$$\{F_1, \{F_2, F_3\}\} = \{\{F_1, F_2\}, F_3\} + (-1)^{(k_1+l_1)(k_2+l_2)}\{F_2, \{F_1, F_3\}\} \quad \text{(Jacobi).} \tag{2.2}$$

Explicitly, upon fixing local coordinates $x_i, p^i, \xi_j, \theta^j$ (with $i = 1, \dots, n$ and $j = 1, \dots, d$), of respective bi-degrees (0, 0), (1, 1), (0, 1) and (1, 0), the algebra of functions $\mathcal{F}_A$ is the graded commutative algebra in those variables, admitting a polynomial dependence in the even variable $p_1, \dots, p_n$ (by skew-symmetry, the dependence in the variables $\xi_1, \theta^1, \dots, \xi_d, \theta^d$ is also polynomial). The big bracket is given in coordinates by:

$$\{p^i, x_i\} = \{\theta^j, \xi_j\} = 1 \quad i = 1, \dots, n,\ j = 1, \dots, d$$

while all the remaining brackets of coordinate functions vanish.

In this article, we shall use mainly the following points:

1. There is a natural identification between the algebras $\mathcal{F}_A^{0,0}$ and $\mathcal{F}(M)$.
2. There is a natural identification of graded algebra between $\sum_{k\in\mathbb{N}} \mathcal{F}_A^{k,0}$ and $\sum_{k\in\mathbb{N}} \Gamma(\wedge^k A)$. The restriction of $\{.,.\}$ to this subalgebra is trivial.
3. There is a natural identification of graded algebra between $\sum_{k\in\mathbb{N}} \mathcal{F}_A^{0,k}$ and $\sum_{k\in\mathbb{N}} \Gamma(\wedge^k A^*)$. The restriction of $\{.,.\}$ to this subalgebra is trivial.
4. There is therefore a natural inclusion of $\Gamma(\bigwedge(A \oplus A^*)) \simeq \Gamma(\bigwedge A \otimes \bigwedge A^*)$ in $\mathcal{F}_A$. From now, this inclusion shall be implicitly done, and no notational distinctions shall be made between an element in $\Gamma(\bigwedge(A \oplus A^*))$ and its image in $\mathcal{F}_A$. Also, the wedge product of two elements $P, Q \in \Gamma(\bigwedge(A \oplus A^*))$ shall be denoted by PQ instead of $P \wedge Q$.
5. The big bracket between a section of A and a section of A^* is given by the natural pairing; in equation $\{\xi, \theta\} = \{\theta, \xi\} = \xi(\theta)$, for every $\theta \in \Gamma(A)$, $\xi \in \Gamma(A^*)$.
6. There is a canonical isomorphism of graded Poisson algebras $\Phi : \mathcal{F}_A \simeq \mathcal{F}_{A^*}$ mapping $\mathcal{F}_A^{k,l}$ to $\mathcal{F}_{A^*}^{l,k}$ for all $k, l \in \mathbb{N}$,
7. Let $F \in \mathcal{F}_A^{k,l}$, with $k \geq 1$. If $\{\{\{F, a_1\}, \dots\}, a_k\} = 0$ for all $a_1, \dots, a_k \in \Gamma(A)$, then $F = 0$.

There exists a unique function in $\mathcal{F}_A$, that we shall denote id_A, which corresponds to the identity map of the vector bundle A, seen as an element of $\Gamma(A^* \otimes A) \subset \mathcal{F}_A$. It is explicitly defined by the global section of $\Gamma(A^* \otimes A)$ given by

$$\mathrm{id}_A := \sum_{j=1}^{d} \xi_j\theta^j, \tag{2.3}$$

where $\xi_1, \ldots, \xi_d$ and $\theta^1, \ldots, \theta^d$ are local basis of A^* and A dual to each other (see [16]). Taking the bracket with id_A is a manner to count the bi-degree, more precisely, for every $P \in \Gamma(\wedge^k A), \Psi \in \Gamma(\wedge^l A^*)$,

$$\{P\Psi, \mathrm{id}_A\} := (k-l)P\Psi. \tag{2.4}$$

Of course, one can also consider the function $\mathrm{id}_{A^*} \in \mathcal{F}_{A^*}$. Under the canonical isomorphism $\Phi : \mathcal{F}_A \simeq \mathcal{F}_{A^*}$, both functions are related by:

$$\Phi(\mathrm{id}_A) = -\mathrm{id}_{A^*}. \tag{2.5}$$

2.2. Preliminary results

For future purposes, we shall establish several facts about the behavior of the big bracket when one adds a copy of $\mathbb{R}$ either to the base or to the fibers.

A. Enlarging the base. Let $A \to M$ be a vector bundle. Denote by $p^*A \to (M \times \mathbb{R})$ the pull-back of $A \to M$ under the projection onto the first factor $p : M \times \mathbb{R} \to M$. We denote by $t \in \mathcal{F}(M \times \mathbb{R}) \subset \mathcal{F}_{p^*A}$ the projection onto the second factor. There is a canonical inclusion $\imath : \mathcal{F}_A \hookrightarrow \mathcal{F}_{p^*A}$. There exists a unique function $\partial^t \in \mathcal{F}_{p^*A}$ such that

$$\{\partial^t, \imath(F)\} = 0 \quad \text{for all } F \in \mathcal{F}_A \text{ and } \{\partial^t, t\} = 1. \tag{2.6}$$

The bi-degree of this function is $(1, 1)$. With a slight abuse of notations, we shall consider $\mathcal{F}_A$ as a subset of $\mathcal{F}_{p^*A}$, erasing therefore the canonical inclusion $\imath$.

Notice that ∂^t is an even function, and that, for every $F_0, \ldots, F_k \in \mathcal{F}_A$, the relation $\sum_{i=0}^k F_i(\partial^t)^i = 0$ holds if and only if $F_0 = \cdots = F_k = 0$.

B. Enlarging the fibers. Let $B \to M$ be a vector bundle. We call $A = B \oplus \mathbb{R} \to M$ the direct sum of B with the trivial bundle $\mathbb{R} \times M \to M$. There is a natural inclusion $\jmath : \mathcal{F}_B \subset \mathcal{F}_A$, which preserves the big bracket. We define $\phi \in \Gamma(A^*) \subset \mathcal{F}_A$ to be the projection onto the second factor, i.e., the section of $A^* = (B \oplus \mathbb{R})^*$ defined by $\phi(b, f) = f$ for all $b \in \Gamma(B), f \in \mathcal{F}(M)$. The following is immediate:

$$\{\phi, \jmath(F)\} = 0 \quad \text{for all } F \in \mathcal{F}_B \text{ and } \{\phi, \epsilon\} = 1, \tag{2.7}$$

where ϵ is the section of $A := B \oplus \mathbb{R} \to M$ given by $m \to (0_m, 1) \in A_m = B_m \oplus \mathbb{R}$ (in the previous 0_m is the zero element in the vector space B_m). With a slight abuse of notations, we shall consider $\mathcal{F}_B$ as a subset of $\mathcal{F}_A$, omitting the canonical inclusion $\jmath$.

3. Lie and Jacobi algebroids, Lie bialgebroids and Jacobi bialgebroids

3.1. (Pre-)Jacobi algebroids

Lie algebroids and pre-Lie algebroids are in general introduced through brackets and anchors, however, it is well-known [8,17] that the supergeometric point of view is strictly equivalent to those more classical ones.

Definition 3.1 (*[18]*)**.** Let $A \to M$ be a vector bundle. A *pre-Lie algebroid* (*on* A) is a function in $\mathcal{F}_A^{1,2}$ (i.e., a function of bi-degree $(1, 2)$).

The *bracket of a pre-Lie algebroid* μ is the $\mathbb{R}$-bilinear endomorphism of $\Gamma(\bigwedge A) := \oplus_{k\in\mathbb{N}} \Gamma(\wedge^k A)$ defined, for all $P, Q \in \Gamma(\bigwedge A)$ by

$$[P, Q]_\mu = \{\{P, \mu\}, Q\}. \tag{3.8}$$

The *differential* of a pre-Lie algebroid μ is the linear endomorphism of $\Gamma(\bigwedge A^*) := \oplus_{k\in\mathbb{N}} \Gamma(\wedge^k A^*)$, of degree $+1$, defined, for all $\Psi \in \Gamma(\bigwedge A^*)$, by $\mathrm{d}_\mu(\Psi) := \{\mu, \Psi\}$.

Recall that the restriction to $\Gamma(A) \times \mathcal{F}(M) \to \mathcal{F}(M)$ of the bracket $[\cdot, \cdot]_\mu$ is of the form

$$(a, f) \mapsto \rho_\mu(a)[f] \quad \forall a \in \Gamma(A),\ f \in \mathcal{F}(M),$$

for some vector bundle morphism $\rho_\mu : A \to TM$ (over the identity of M) called the *anchor map*. In the previous, the notation $f \mapsto X[f]$ is used to denote the derivation of $\mathcal{F}(M)$ associated to a vector field X on M (a notation that we shall also use for multi-vector fields).

A pre-Lie algebroid μ (on A) is said to be a *Lie algebroid* (*on* A) if $\{\mu, \mu\} = 0$. It is a classical result (see, for instance, [7,8]) that a pre-Lie Algebroid is a Lie algebroid if and only if one of the following equivalent conditions is satisfied: (i) (3.8) satisfies the Jacobi identity (hence is a Gerstenhaber bracket) (ii) $\mathrm{d}_\mu^2 = 0$ or, (iii), the restriction of $[\cdot, \cdot]_\mu$ to $\Gamma(A) \times \Gamma(A) \to \Gamma(A)$ is a Lie algebra bracket.

We now define pre-Jacobi algebroids, following ideas introduced (for Jacobi algebroids) in [19,10,9]. In particular, the definition of the bracket of a pre-Jacobi algebroid below is similar to equation (9) in [19].

Definition 3.2. Let $A \to M$ be a vector bundle. A *pre-Jacobi algebroid* (*on* A) is a function of $\mathcal{F}_A$ of the form $\mu + \phi$, with $\mu \in \mathcal{F}_A^{1,2}$ and $\phi \in \mathcal{F}_A^{0,1}$ (i.e., μ is a pre-Lie algebroid while ϕ is a section of A^*).[1]

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[1] Since μ and ϕ belong to vector spaces with trivial intersection, we can recover both μ and ϕ by knowing only their sum, so that we could also denote the pre-algebroid $\mu + \phi$ as a pair (μ, ϕ). However, we prefer the notation $\mu + \phi$, which is more consistent with Definition 3.4.

The *bracket of a pre-Jacobi algebroid* μ is the bilinear endomorphism of $\Gamma(\bigwedge A) := \oplus_{k\in\mathbb{N}}\Gamma(\wedge^k A)$ defined, for all $P, Q \in \Gamma(\bigwedge A)$ by

$$[P, Q]_{\mu,\phi} := \{\{P, \mu + \mathrm{id}_A\phi\}, Q\} - P\{\phi, Q\} + \{P, \phi\}Q$$

(see Section 2.1 for the definition of id_A). The *differential* of a pre-Jacobi algebroid $\mu + \phi$ is the linear endomorphism $\mathrm{d}_{\mu,\phi}$ of $\Gamma(\bigwedge A^*) := \oplus_{k\in\mathbb{N}}\Gamma(\wedge^k A^*)$ defined, for all $\Psi \in \Gamma(\bigwedge A^*)$, by $\mathrm{d}_{\mu,\phi}(\Psi) := \{\mu, \Psi\} + \phi\Psi$.

Lemma 3.3. *The bracket of a pre-Jacobi algebroid is given explicitly by the following formula, for all* $P \in \Gamma(\wedge^k A)$, $Q \in \Gamma(\wedge^l A)$*:*

$$[P, Q]_{\mu,\phi} = [P, Q]_\mu + (k-1)P(\imath_\phi Q) + (-1)^k(l-1)(\imath_\phi P)Q. \tag{3.9}$$

Proof. Using the definitions of the pre-Lie brackets $[\cdot,\cdot]_{\mu,\phi}$ and $[\cdot,\cdot]_\mu$, we obtain:

$$\begin{aligned}[P, Q]_{\mu,\phi} &= \{\{P, \mu\}, Q\} + \{\{P, \mathrm{id}_A\phi\}, Q\} - P\{\phi, Q\} + \{P, \phi\}Q\\ &= \{\{P, \mu\}, Q\} + \{\{P, \mathrm{id}_A\phi\}, Q\} - P\imath_\phi Q - (-1)^k(\imath_\phi P)Q\\ &= [P, Q]_\mu + \{\{P, \mathrm{id}_A\phi\}, Q\} - P\imath_\phi Q - (-1)^k(\imath_\phi P)Q.\end{aligned}$$

By making several use of the Leibniz identity (2.1), one computes

$$\begin{aligned}\{\{P, \mathrm{id}_A\phi\}, Q\} &= \{\{P, \mathrm{id}_A\}\phi, Q\} + \{\{P, \phi\}\mathrm{id}_A, Q\}\\ &= k\{P\phi, Q\} + \{\{P, \phi\}\mathrm{id}_A, Q\} \quad \text{by (2.4)}\\ &= kP\{\phi, Q\} + \{\{P, \phi\}\mathrm{id}_A, Q\}\\ &= kP\{\phi, Q\} + \{P, \phi\}\{\mathrm{id}_A, Q\}\\ &= kP\{\phi, Q\} - l\{P, \phi\}Q \quad \text{by (2.4)}\\ &= kP\imath_\phi Q + l(-1)^k(\imath_\phi P)Q.\end{aligned}$$

This completes the computation. □

Example. When $\phi \in \Gamma(A^*)$ is exact, i.e., when there exists a function f such that $\phi = \{\mu, f\}$, then the Lie bracket of the pre-Jacobi algebroid $[.,.]_{\mu,\phi}$ is isomorphic to the Lie bracket of the pre-Lie algebroid $[.,.]_\mu$: the isomorphism consists in mapping $P \in \Gamma(\wedge^k A)$ to $e^{(k-1)f}P$.

Remark. In Theorem 3.5 [9] or Equation (25) [10], a bracket is constructed out of the data defining a Jacobi algebroid. Comparing these two brackets, given by quite explicit formulas, with formula (3.9) above proves the coincidence of our bracket with these (more precisely, the match is exact with [10], but is only up to signs with [9] where the Gerstenhaber bracket of a Lie algebroid is given by $P, Q \mapsto \{\{\mu, P\}, Q\}$ and not by (3.8)).

We also easily deduce the following properties from those of the big bracket, for all $a \in \Gamma(\wedge^k A)$, $b \in \Gamma(\wedge^l A)$, $c \in \Gamma(\wedge^m A)$:

$$[b, a]_{\mu,\phi} = -(-1)^{kl}[a, b]_{\mu,\phi}$$
$$[a, bc]_{\mu,\phi} = [a, b]_{\mu,\phi}c + (-1)^{lm}[a, c]_{\mu,\phi}b - \{\phi, a\}bc.$$

Definition 3.4. A pre-Jacobi algebroid $\mu + \phi$ (on A) is said to be a *Jacobi algebroid* (*on* A) when $\{\mu + \phi, \mu + \phi\} = 0$.

Spelled out, the condition $\{\mu + \phi, \mu + \phi\} = 0$ yields the two conditions $\{\mu, \mu\} = 0$ and $\{\mu, \phi\} = \mathrm{d}_\mu\phi = 0$. The first of these conditions means that μ is a Lie algebroid, and the second one means that ϕ is a cocycle of this Lie algebroid, i.e., that $\phi([a, b]_\mu) = \rho_\mu(a)[\phi(b)] - \rho_\mu(b)[\phi(a)]$ for all $a, b \in \Gamma(A)$ ($\rho_\mu : A \to TM$ being the anchor map defined above). In conclusion, a Jacobi algebroid is a Lie algebroid endowed with an algebroid 1-cocycle, which is the usual definition (compare with [9], where such an object is called "Lie algebroid in the presence of a 1-cocycle").

The next result appeared in both [9,10]: we prove it here by means of the super-geometric formalism.

Proposition 3.5. *Let* $A \to M$ *be a vector bundle. For every pre-Jacobi algebroid* $\mu + \phi$ *on A, the following are equivalent:*

(i) $\mu + \phi$ *is a Jacobi algebroid on A;*
(ii) *the operator* $\mathrm{d}_{\mu,\phi}$ *squares to* 0*;*
(iii) $[\cdot,\cdot]_{\mu,\phi}$ *satisfies the graded Jacobi identity:*

$$(-1)^{km}[[a, b]_{\mu,\phi}, c]_{\mu,\phi} + (-1)^{lk}[[b, c]_{\mu,\phi}, a]_{\mu,\phi} + (-1)^{ml}[[c, a]_{\mu,\phi}, b]_{\mu,\phi} = 0,$$

for all homogeneous $a, b, c \in \Gamma(\bigwedge A)$ *of degrees* k, l, m *respectively.*

Proof. The equivalence of (i) and (ii) follows from the relation $\mathrm{d}^2_{\mu,\phi}(\Psi) = \frac{-1}{2}\{\{\mu+\phi, \mu+\phi\}, \Phi\}$, together with item seven in Section 2.1. Let us prove the equivalence of (i) and (iii). A direct computation using usual properties of the big bracket gives the following expression for the Jacobiator of the bracket $[\cdot,\cdot]_{\mu,\phi}$:

$$(-1)^{lm}[[a,b]_{\mu,\phi},c]_{\mu,\phi}+\text{c.p.} = (-1)^{lm}[[a,b]_{\mu},c]_{\mu}+\text{c.p.}$$
$$+(-1)^{lm}(m-1)(\imath_\phi[a,b]_\mu-[\imath_\phi\, a,b]_\mu-[a,\imath_\phi\, b]_y)c+\text{c.p.}$$

(where +c.p. indicates that we add all the terms obtained by permuting the variables a, b, c). Now, the Jacobi identity of the big bracket implies that:

$$\begin{aligned}\imath_\phi[a,b]_\mu-[\imath_\phi\, a,b]_\mu-[a,\imath_\phi\, b]_\mu &= \imath_\phi\{\{a,\mu\},b\}-\{\{\imath_\phi\, a,\mu\}b\}-\{\{a,\mu\}\,\imath_\phi\, b\}\\ &= \{\phi,\{\{a,\mu\},b\}\}-\{\{\{\phi,a\},\mu\},b\}_\mu-\{\{a,\mu\},\{\phi,b\}\}\\ &= \{\{a,\{\phi,\mu\}\},b\}.\end{aligned}$$

As a consequence:

$$(-1)^{lm}[[a,b]_{\mu,\phi},c]_{\mu,\phi}+\text{c.p.} = (-1)^{lm}[[a,b]_\mu,c]_\mu+(-1)^{lm}(m-1)\{\{a,\{\phi,\mu\}\},b\}c+\text{c.p.}$$

If μ is a Lie algebroid, the first term on the right hand side vanishes, and if $\{\phi,\mu\}=0$, the second term vanishes as well, so that the graded Jacobi identity is satisfied. Conversely, if the graded Jacobi identity is satisfied for all homogeneous a, b, c, then, choosing $c=1$, it follows from the previous identity that

$$\{\{a,\{\phi,\mu\}\},b\}=0$$

for all homogeneous a, b. Since $\{\phi,\mu\}\in\mathcal{F}_A^{0,2}$, this implies $\{\phi,\mu\}=0$ (see item seven in Section 2.1). In turn, this implies that the bracket $[\cdot,\cdot]_\mu$ satisfies the graded Jacobi identity, a property that holds true if and only if μ is a Lie algebroid. □

Given a pre-Jacobi algebroid $\mu+\phi$ on $A\to M$, we define a family, indexed by a parameter $c\in\mathbb{R}$, of pre-Lie algebroids on $p^*A\to(M\times\mathbb{R})$, denoted μ_ϕ^c, by

$$\mu_\phi^c := e^{-ct}(\mu+\phi(\partial^t+c\,\mathrm{id}_A))$$

where we use the notations of Section 2.2-A. These anchors and brackets correspond, for $c=0,1$, to (4.16–4.19) in [9]. The pre-Lie algebroid μ_ϕ^1 (on p^*A) is called the *Poissonization* of the pre-Jacobi algebroid $\mu+\phi$ (on A). Notice also that $\mu_\phi^0=\mu+\phi\partial^t$.

Lemma 3.6. *Define* $\mathrm{ad}_{t\,\mathrm{id}_A}$ *to be the linear endomorphism of* $\mathcal{F}_A$ *defined by* $F\mapsto\{t\,\mathrm{id}_A,F\}$. *For all* $c,x\in\mathbb{R}$, *the following relation holds:*

$$\exp(\mathrm{ad}_{-xt\,\mathrm{id}_A})\mu_\phi^c=\mu_\phi^{c+x}.$$

Proof. A direct computation gives

$$\mathrm{ad}_{t\,\mathrm{id}_A}\mu_\phi^0=t\mu_\phi^0-\phi\mathrm{id}_A\quad\text{and}\quad\mathrm{ad}_{t\,\mathrm{id}_A}\phi\mathrm{id}_A=t\phi\mathrm{id}_A.$$

Now, it is a general fact that for every vector space E, every $L\in\mathrm{End}(E,E)$ and every $\mathfrak{a},\mathfrak{b}\in E$, if $L(\mathfrak{a})=t\mathfrak{a}-\mathfrak{b}$ and $L(\mathfrak{b})=t\mathfrak{b}$, we have $\exp(-cL)\mathfrak{a}=e^{-tc}\mathfrak{a}+ce^{-tc}\mathfrak{b}$. Applied to $E:=\mathcal{F}_A$, $L:=\mathrm{ad}_{t\,\mathrm{id}_A}$, $\mathfrak{a}:=\mu_\phi^0$, $\mathfrak{b}:=\phi\mathrm{id}_A$, this gives

$$\exp(\mathrm{ad}_{-ct\,\mathrm{id}_A})\mu_\phi^0=e^{-ct}\mu_\phi^0+ce^{-ct}\mathrm{id}_A\phi=\mu_\phi^c.$$

Applying $\exp(\mathrm{ad}_{-xt\,\mathrm{id}_A})$ to both sides of this relation gives:

$$\exp(\mathrm{ad}_{-xt\,\mathrm{id}_A})\exp(\mathrm{ad}_{-ct\,\mathrm{id}_A})\mu_\phi^0=\exp(\mathrm{ad}_{-xt\,\mathrm{id}_A})\mu_\phi^c.$$

The relation

$$\exp(\mathrm{ad}_{-xt\,\mathrm{id}_A})\ \exp(\mathrm{ad}_{-ct\,\mathrm{id}_A})\ \mu_\phi^0=\exp(\mathrm{ad}_{-(x+c)t\,\mathrm{id}_A})\ \mu_\phi^0=\mu_\phi^{c+x},$$

gives the required result. □

The next lemma is dealt with in [9] for the cases $c=0,1$.

Lemma 3.7. *Let* $A\to M$ *be a vector bundle,* $\mu+\phi$ *a pre-Jacobi algebroid on* $A\to M$. *The following are equivalent* (p^*A, μ_ϕ^c *being as defined above*)*:*

(i) *$\mu+\phi$ is a Jacobi algebroid on A;*
(ii) *There exists $c\in\mathbb{R}$ such that the pre-Lie algebroid μ_ϕ^c is a Lie algebroid on p^*A;*
(iii) *For all $c\in\mathbb{R}$, the pre-Lie algebroid μ_ϕ^c is a Lie algebroid on p^*A.*

Proof. Using the usual properties of the big bracket, one computes

$$\begin{aligned}\{\mu_\phi^0,\mu_\phi^0\} &= \{\mu+\phi\partial^t,\mu+\phi\partial^t\}\\ &= \{\mu,\mu\}+2\{\mu,\phi\}\partial^t.\end{aligned}$$

The vanishing of $\{\mu^0_\phi, \mu^0_\phi\}$ is therefore (in view of Section 2.2-A) tantamount to the vanishing of both $\{\mu, \mu\}$ and $\{\mu, \phi\}$. Hence μ^0_ϕ is a Lie algebroid if and only if $\mu + \phi$ is a Jacobi algebroid. Now, in view of Lemma 3.6, we have for all $c \in \mathbb{R}$

$$\{\mu^c_\phi, \mu^c_\phi\} = \{\exp(\mathrm{ad}_{-ct\ \mathrm{id}_A})\mu^0_\phi, \exp(\mathrm{ad}_{-ct\ \mathrm{id}_A})\mu^0_\phi\} = \exp(\mathrm{ad}_{-ct\ \mathrm{id}_A})\{\mu^0_\phi, \mu^0_\phi\}.$$

In particular, μ^c_ϕ is a Lie algebroid if and only if μ^0_ϕ is a Lie algebroid. This completes the proof. □

Remark 3.8. The assignment from $\sum_{k\in\mathbb{N}} \Gamma(\wedge^k A)$ to $\sum_{k\in\mathbb{N}} \Gamma(\wedge^k p^*A)$ mapping $P \in \Gamma(\wedge^k A)$ to $e^{-(k-1)t}P$ is a Lie algebra morphism, when $\sum_{k\in\mathbb{N}} \wedge^k A$ is equipped with the bracket $[\cdot,\cdot]_{\mu,\phi}$ while $\sum_{k\in\mathbb{N}} \Gamma(\wedge^k p^*A)$ is equipped with the bracket of the Lie algebroid μ^0_ϕ. This claim can be proved directly, and can also be found in [19]. A more general fact is that, for every $c \in \mathbb{R}$, mapping $P \in \Gamma(\wedge^k A)$ to $e^{-(k-1)t-ckt}P$ is a Lie algebra morphism, when $\sum_{k\in\mathbb{N}} \Gamma(\wedge^k p^*A)$ is now equipped with the bracket of the Lie algebroid μ^c_ϕ.

Example 3.9. When M is a point, A becomes a Lie algebra with bracket $[.,.]_A$ equipped with a cocycle $\phi \in A^*$, and sections of $p^*A \to \mathbb{R}$ can be identified to smooth maps from $\mathbb{R}$ to A. The Lie algebroid μ^0_ϕ is then the Lie algebroid with anchor $\rho(X(t)) = \phi(X(t))\frac{\mathrm{d}}{\mathrm{d}t}$ and with bracket

$$[X(t), Y(t)] = [X(t), Y(t)]_A + \phi(X(t))\frac{\mathrm{d}Y(t)}{\mathrm{d}t} - \phi(Y(t))\frac{\mathrm{d}X(t)}{\mathrm{d}t}$$

where $X(t), Y(t)$ are arbitrary maps from $\mathbb{R}$ to A, identified with sections of $p^*A \to \mathbb{R}$.

3.2. Jacobi bialgebroids

Upon identifying $\mathcal{F}_A$ and $\mathcal{F}_{A^*}$ by Φ, it is possible (since Φ exchanges the spaces $\mathcal{F}^{k,l}_A$ and $\mathcal{F}^{l,k}_{A^*}$ for all $k, l \in \mathbb{N}$, see Section 2.1) to consider pre-Lie algebroids on A^* as elements of $\mathcal{F}^{2,1}_A$ and pre-Jacobi algebroids on A^* as functions in $\mathcal{F}_A$ of the form $\nu + X$, with $\nu \in \mathcal{F}^{2,1}_A$ and $X \in \mathcal{F}^{1,0}_A$.

Definition 3.10 (*[7]*)**.** Let $A \to M$ be a vector bundle. A *pre-Lie bialgebroid* is a pair (μ, ν), where

1. μ is a pre-Lie algebroid on A,
2. ν is a pre-Lie algebroid on A^*,

such that d_ν is an odd derivation of the bracket $[\cdot,\cdot]_\mu$, i.e.,

$$\mathrm{d}_\nu[a, b]_\mu = [\mathrm{d}_\nu a, b]_\mu + (-1)^{k-1}[a, \mathrm{d}_\nu b]_\mu \quad \forall a \in \Gamma(\wedge^k A),\ b \in \Gamma\left(\bigwedge A\right), \tag{3.10}$$

where $[\cdot,\cdot]_\mu$, d_ν are as in Definition 3.1. A *Lie bialgebroid* is a pre-Lie bialgebroid (μ, ν) such that μ and ν are Lie algebroids.

Proposition 3.11 (*[7]*)**.** *Let $A \to M$ be a vector bundle, μ and ν be pre-Lie algebroid on A and A^*, respectively.*

1. *The pair (μ, ν) is a pre-Lie bialgebroid if and only if $\{\mu, \nu\} = 0$;*
2. *The pair (μ, ν) is a Lie bialgebroid if and only if $\{\mu + \nu, \mu + \nu\} = 0$.*

Following the same idea, Jacobi bialgebroids are introduced in [10] as follows.

Definition 3.12. Let $A \to M$ be a vector bundle. A *pre-Jacobi bialgebroid* is a pair $(\mu + \phi, \nu + X)$, where

1. $\mu + \phi$ is a pre-Jacobi algebroid on A,
2. $\nu + X$ is a pre-Jacobi algebroid on A^*,

such that $\mathrm{d}_{\nu,X}$ is an odd derivation of the bracket $[\cdot,\cdot]_{\mu,\phi}$, i.e.,

$$\mathrm{d}_{\nu,X}[a, b]_{\mu,\phi} = [\mathrm{d}_{\nu,X} a, b]_{\mu,\phi} + (-1)^{k-1}[a, \mathrm{d}_{\nu,X} b]_{\mu,\phi} \quad \forall a \in \Gamma(\wedge^k A),\ b \in \Gamma\left(\bigwedge A\right), \tag{3.11}$$

where $[\cdot,\cdot]_{\mu,\phi}$, $\mathrm{d}_{\nu,X}$ are as in Definition 3.2. A *Jacobi bialgebroid* is a pre-Jacobi bialgebroid $(\mu + \phi, \nu + X)$ such that $\mu + \phi$ is a Jacobi algebroid and $\mathrm{d}_{\nu,X}$ squares to 0.

We can now give the main result of this section, which is a characterization, in terms of the big bracket, of Jacobi bialgebroid structures.

Theorem 3.13. *Let $A \to M$ be a vector bundle, $\mu + \phi$ a pre-Jacobi algebroid on A and $\nu + X$ a pre-Jacobi algebroid on A^*. Set*

$$\begin{cases} \diamondsuit := \{\phi, X\} = \phi(X) \\ \clubsuit := \{\mu, X\} + \{\nu, \phi\} \\ \spadesuit := \{\mu, \nu\} + \mu X + \nu\phi - \mathrm{id}_A\left(-\phi X + \dfrac{\{\mu, X\} - \{\nu, \phi\}}{2}\right). \end{cases} \tag{3.12}$$

1. *The pair* $(\mu+\phi, \nu+X)$ *is a pre-Jacobi bialgebroid if and only if*

$$\diamondsuit = \clubsuit = \spadesuit = 0.$$

2. *The pair* $(\mu+\phi, \nu+X)$ *is a Jacobi bialgebroid if and only if*

$$\diamondsuit = \clubsuit = \spadesuit = \{\mu+\phi, \mu+\phi\} = \{\nu+X, \nu+X\} = 0.$$

Proof. We prove the first item. The idea is to express with the big bracket the quantity

$$H_{\lfloor .,. \rfloor, D}(a,b) := D\lfloor a,b \rfloor - \lfloor Da, b \rfloor - (-1)^{k-1}\lfloor a, Db \rfloor, \quad a \in \Gamma(\wedge^k A),\ b \in \Gamma\left(\bigwedge A\right)$$

where D stands either for $\mathrm{ad}_\nu := \bigwedge A \mapsto \bigwedge A$ or for the left-multiplication by X, i.e., $m_X(b) := Xb$, and $\lfloor a, b \rfloor$ stands either for the derived bracket $[a,b]_{\mu'} := \{\{a, \mu'\}, b\}$, with $\mu' := \mu + \mathrm{id}_A\phi$, or for the assignment $((a,b))_\phi := \{a,\phi\}b - a\{\phi, b\}$. In fact, by definition, $\mathrm{d}_{\nu,X} := \mathrm{ad}_\nu + m_X$, while $[\cdot,\cdot]_{\mu,\phi} := [\cdot,\cdot]_\mu + ((.,.))_\phi$, hence

$$\begin{aligned} &\mathrm{d}_{A^*,\phi}[a,b]_{A,\phi} - [\mathrm{d}_{A^*,\phi}a, b]_{A,\phi} - (-1)^{k-1}[a, \mathrm{d}_{A^*,\phi}b]_{A,\phi} \\ &\quad = (H_{[\cdot,\cdot]_{\mu'},\mathrm{ad}_\nu} + H_{[\cdot,\cdot]_{\mu'},m_X} + H_{((.,.))_\phi,\mathrm{ad}_\nu} + H_{((.,.))_\phi,m_X})(a,b). \end{aligned} \tag{3.13}$$

A cumbersome but straightforward computation, involving only the Leibnitz and Jacobi identities of the big bracket (and valid for arbitrary $\mu' \in \mathcal{F}_A^{1,2}$, $\nu \in \mathcal{F}_A^{2,1}$, $X \in \mathcal{F}_A^{1,0}$, $\phi \in \mathcal{F}_A^{0,1}$), yields the following results:

$$\begin{cases} H_{[\cdot,\cdot]_{\mu'},\mathrm{ad}_\nu}(a,b) = \mathcal{D}(a, (-1)^k\{\nu,\mu'\}, b) \\ H_{[\cdot,\cdot]_{\mu'},m_X}(a,b) = \mathcal{D}(a, (-1)^k\mu' X, b) + \mathcal{E}(a, (-1)^k\{X,\mu'\}, b) \\ H_{((.,.))_\phi,\mathrm{ad}_\nu}(a,b) = \mathcal{E}(a, (-1)^k\{\nu,\phi\}, b) \\ H_{((.,.))_\phi,m_X}(a,b) = \mathcal{F}(a, 2(-1)^{k+1}\{\phi, X\}, b) + \mathcal{E}(a, -(-1)^k X\phi, b) \end{cases}$$

where, for all $a, b \in \bigwedge A$, for all functions $F \in \mathcal{F}_A$ of bi-degree $(1, 2)$,

$$\mathcal{D}(a,F,b) := \{\{a,F\},b\}, \qquad \mathcal{E}(a,F,b) := \{a,F\}b - a\{F,b\}, \qquad \mathcal{F}(a,F,b) := Fab.$$

Introducing these results in (3.13) yields

$$\mathrm{d}_{A^*,\phi}[a,b]_\phi - [\mathrm{d}_{A^*,\phi}a,b]_\phi - (-1)^{k+1}[a,\mathrm{d}_{A^*,\phi}b]_\phi = (-1)^k\mathcal{D}(a,\hat{\spadesuit},b) + (-1)^k\mathcal{E}(a,\hat{\clubsuit},b) + (-1)^k\mathcal{F}(a,\hat{\diamondsuit},b) \tag{3.14}$$

where

$$\begin{cases} \hat{\spadesuit} := \{\nu,\mu'\} + \mu' X = \spadesuit - \dfrac{1}{2}\mathrm{id}_A\clubsuit \\ \hat{\clubsuit} := \{X,\mu'\} + \{\nu,\phi\} - X\phi = \{\mu, X\} + \{\nu,\phi\} + \mathrm{id}_A\{X,\phi\} = \clubsuit + \mathrm{id}_A\diamondsuit \\ \hat{\diamondsuit} := -2\{\phi, X\} = -2\diamondsuit. \end{cases} \tag{3.15}$$

From these expressions, it follows that, if $\diamondsuit = \clubsuit = \spadesuit = 0$, then $\hat{\spadesuit} = \hat{\clubsuit} = \hat{\diamondsuit} = 0$, hence (3.11) holds true, and the pair $(\mu+\phi, \nu+X)$ is a pre-Jacobi bialgebroid.

Let us prove the converse. First, notice that $\mathcal{D}(a,\hat{\spadesuit},b) = 0$ if $a = 1$ or $b = 1$, and $\mathcal{E}(a,\hat{\spadesuit},b) = 0$ if $a = b = 1$. Hence, if condition (3.11) is satisfied, one sees by plugging $a = b = 1$ in (3.14) that $\hat{\diamondsuit} = 0$. Then plugging $a = 1$ and letting b be an arbitrary section of A, we conclude that $\{\hat{\clubsuit}, b\} = 0$ for all $b \in \Gamma(\bigwedge A)$. Since $\hat{\clubsuit}$ is of bidegree $(1,1)$, according to item seven in Section 2.1, we have $\hat{\clubsuit} = 0$. For similar reasons, we deduce from $\{\{a,\hat{\spadesuit}\},b\} = 0$, valid for a, b arbitrary sections of A, that $\hat{\spadesuit} = 0$. In view of (3.15), we have $\diamondsuit = \clubsuit = \spadesuit = 0$ which completes the proof.

The second item is an immediate consequence of the first and of Proposition 3.5. □

Example 3.14. Let A be a vector space of dimension 4. Denote by Y_+, Y_-, H, I a basis and by Y_+^*, Y_-^*, H^*, I^* the dual basis. Define

$$\begin{aligned} &\mu := -2H^*Y_+^*Y_+ + 2H^*Y_-^*Y_- - Y_+^*Y_-^*H \qquad \phi := I^* \\ &\nu := I^*Y_+Y_- - 2(I-H)Y_+^*Y_+ + 2Y_-^*(I+H)Y_- \qquad X := H. \end{aligned}$$

Then $(\mu+\phi, \nu+X)$ is a Jacobi bialgebroid. Note that that the Lie algebra that corresponds to μ is the Lie algebra of 2×2 matrices (I has to be interpreted as being the identity matrix, H is the diagonal matrix with $(+1,-1)$ on the diagonal, Y_+, Y_- are the upper triangular and lower triangular matrices whose only non-zero term is 1) and that, under this correspondence, I^* is half of the trace map. An interpretation of this structure shall appear in Example 4.7.

In the Eq. (3.12), the roles of A and A^* are symmetric, i.e., applying the canonical isomorphism $\Phi : \mathcal{F}_A \simeq \mathcal{F}_{A^*}$ (and using (2.5)), one obtains equations of the same form, which yields the next corollary, which already appears in both [9,10].

Corollary 3.15. *Let $A \to M$ be a vector bundle, $\mu + \phi$ a pre-Jacobi algebroid on A, and $\nu + X$ a pre-Jacobi algebroid on A^*. The pair $(\mu + \phi, \nu + X)$ is a pre-Jacobi bialgebroid on A if and only if the pair $(\nu + X, \mu + \phi)$ is a pre-Jacobi bialgebroid on A^*.*

In Lemma 3.7 we constructed, out of a Jacobi algebroid $\mu + \phi$ on a vector bundle $B \to M$, a family indexed by $c \in \mathbb{R}$ of Lie algebroid on the pull-back vector bundle $p^*B \to M \times \mathbb{R}$ ($p : M \times \mathbb{R} \to M$ being the projection on the first factor). We denoted these Lie algebroids by μ^c_ϕ. Hence, given a pre-Jacobi algebroid $\mu + \phi$ on a vector bundle $A \to M$ and a pre-Jacobi algebroid $\nu + X$ on the dual bundle $A^* \to M$, we construct, for every $c, d \in \mathbb{R}$:

1. Lie algebroid μ^c_ϕ on $p^*A \to M \times \mathbb{R}$,
2. Lie algebroid ν^d_X on the dual bundle $p^*A^* \to M \times \mathbb{R}$.

Since the vector bundles $p^*A^* \simeq (p^*A)^*$ are dual one to the other, it is natural to ask whether one can pair these Lie algebroids to form Lie bialgebroids. The next corollary gives an answer to this question, and generalizes results obtained in [10,9] for $c = 0, 1$.

Corollary 3.16. *Let $A \to M$ be a vector bundle, $\mu+\phi$ a pre-Jacobi algebroid on A, and $\nu+X$ a pre-Jacobi algebroid on A^*. Choose an arbitrary $c \in \mathbb{R}$.*

1. *The following are equivalent:*
   (i) *the pair $(\mu + \phi, \nu + X)$ is a pre-Jacobi bialgebroid on A;*
   (ii) *the pair $(\mu^c_\phi, \nu^{1-c}_X)$ is a pre-Lie bialgebroid on p^*A;*
   (iii) $\{\mu^c_\phi, \nu^{1-c}_X\} = 0$.
2. *The following are equivalent:*
   (i) *the pair $(\mu + \phi, \nu + X)$ is a Jacobi bialgebroid on A;*
   (ii) *the pair $(\mu^c_\phi, \nu^{1-c}_X)$ is a Lie bialgebroid on p^*A;*
   (iii) $\{\mu^c_\phi + \nu^{1-c}_X, \mu^c_\phi + \nu^{1-c}_X\} = 0$.

Proof. The equivalence of items 1.(ii) and 1.(iii) follows from the first item of Proposition 3.11, and the equivalence of items 2.(ii) and 2.(iii) from the second item of Proposition 3.11. For the remaining equivalences, item 2 is clearly a consequence of item 1, so we only include a proof of item 1. First:

$$\begin{aligned}\exp \mathrm{ad}_{-ct\ \mathrm{id}_A}\{\mu^c_\phi, \nu^{1-c}_X\} &= \{\exp \mathrm{ad}_{-ct\ \mathrm{id}_A}\ \mu^c_\phi, \exp \mathrm{ad}_{-ct\ \mathrm{id}_A}\ \nu^{1-c}_X\}\\ &= \{\exp \mathrm{ad}_{-ct\ \mathrm{id}_A}\ \mu^c_\phi, \exp \mathrm{ad}_{ct\ \mathrm{id}_{A^*}}\ \nu^{1-c}_X\}\\ &= \{\mu^0_\phi, \nu^1_X\} \quad \text{(by Lemma 3.6)}\end{aligned}$$

so that the vanishing of $\{\mu^c_\phi, \nu^{1-c}_X\}$ is equivalent to the vanishing of $\{\mu^0_\phi, \nu^1_X\} = 0$. In view of Theorem 3.13, it suffices to prove that the conditions $\clubsuit = \diamondsuit = \spadesuit = 0$ are satisfied if and only if $\{\mu^0_\phi, \nu^1_X\} = 0$. By a direct computation, we obtain:

$$\begin{aligned}\{\mu^0_\phi, \nu^1_X\} &= \{\mu + \partial^t\phi, e^{-t}(\nu + (\partial^t + \mathrm{id}_{A^*})X)\}\\ &= e^{-t}\{\phi, X\}(\partial^t)^2 + e^{-t}\partial^t(\{\mu, X\} + \{\phi, \nu\} + \mathrm{id}_{A^*}\{\phi, X\}) + e^{-t}(\{\mu, \nu\} + \mu X - \phi\nu + \mathrm{id}_{A^*}(\{\mu, X\} - \phi X))\\ &= e^{-t}\left((\partial^t)^2\diamondsuit + \partial^t(\clubsuit + \mathrm{id}_{A^*}\diamondsuit) + \spadesuit - \frac{1}{2}\mathrm{id}_{A^*}\clubsuit\right).\end{aligned}$$

Now, a function $F \in \mathcal{F}_{p^*A}$ of the form $F = (\partial^t)^2F_1 + \partial^tF_2 + F_3$, with $F_1, F_2, F_3 \in \mathcal{F}_A$ is zero if and only if $F_1 = F_2 = F_3 = 0$ (as stated in Section 2.2-A). We can therefore conclude that $\{\mu^0_\phi, \nu^1_X\} = 0$ if and only if $\diamondsuit = \clubsuit + \mathrm{id}_{A^*}\diamondsuit = \spadesuit - \frac{1}{2}\mathrm{id}_{A^*}\clubsuit = 0$, i.e., if and only if $\spadesuit = \clubsuit = \diamondsuit = 0$, as was to be shown. □

We call the pair (μ^0_ϕ, ν^1_X) the *Poissonified Lie bialgebroid* of the Jacobi bialgebroid $(\mu + \phi, \nu + X)$.

Remark. Corollary 3.16 shows that the conditions listed in item 2 of Theorem 3.13 are equivalent to the single condition $\{\mu^0_\phi + \nu^1_X, \mu^0_\phi + \nu^1_X\} = 0$.

4. Poisson–Jacobi manifolds and their Jacobi bialgebroids

Notations. Throughout this section, we shall use the following conventions.

Let $B \to M$ be a vector bundle endowed with a Lie algebroid $\mu_B \in \mathcal{F}_B$. As in Section 2.2-B, we denote by $A := B \oplus \mathbb{R} \to M$ the direct sum of B with the trivial bundle $\mathbb{R} \times M \to M$, ϵ the section of $A = B \oplus \mathbb{R} \to M$ given by $m \to (0_m, 1) \in A_m = B_m \oplus \mathbb{R}$, and ϕ the section of A^* given by $\phi(b + f\epsilon) := f$ for every $b \in \Gamma(B), f \in \mathcal{F}(M)$. As mentioned in Section 2.2-B, there is a natural inclusion $\mathcal{F}_B \subset \mathcal{F}_A$ that preserves the big bracket. In particular, μ_B, seen as an element in $\mathcal{F}_A$, is a Lie algebroid on $A \to M$. To avoid confusion, we shall denote by μ this Lie algebroid on A.

It is clear that $\phi \in \Gamma(A^*)$ is a cocycle for the Lie algebroid μ, so that $\mu + \phi$ is a Jacobi algebroid on $A \to M$. By Lemma 3.7 therefore, $\mu^0_\phi = \mu + \phi\partial^t$ is a Lie algebroid on $p^*A \to (M \times \mathbb{R})$, where $\partial^t \in \mathcal{F}_{p^*A}$ is as in Section 2.2-B, and $t \in \mathcal{F}(M \times \mathbb{R})$ stands for the parameter on $\mathbb{R}$ (as in Section 2.2-A).

Remark. Explicitly, the Lie algebroid associated to μ on $A = B \oplus \mathbb{R}$ is the Lie algebroid (which appears in [9]) with bracket and anchor:

$$[b_1 + f_1\epsilon, b_2 + f_2\epsilon]_A := [b_1, b_2]_B + \rho_A(b_1)[f_2]\epsilon - \rho_A(b_2)[f_1]\epsilon$$
$$\rho_A(b, f) := \rho_B(b) \quad \forall b_1, b_2, b \in \Gamma(B),\ f_1, f_2, f \in \mathcal{F}(M).$$

4.1. Poissonization

By a *Jacobi structure on the Lie algebroid* μ_B *on* $B \to M$, we mean a pair (π, E) with $\pi \in \Gamma(\wedge^2 B)$ and $E \in \Gamma(B)$ such that:

$$[\pi, \pi]_{\mu_B} = -2E\pi \quad \text{and} \quad [E, \pi]_{\mu_B} = 0.$$

The coefficient -2 is a convention and it could be turned into 1 by replacing E by $-2E$.

Example. As explained in the introduction, given a Poisson structure P on a manifold M and a vector field $\mathcal{E}$ such that $\mathcal{L}_{\mathcal{E}} P = kP$ with $k \in \mathbb{N}^*$, a Jacobi structure can be induced on the tangent Lie algebroid $TM \to M$ for all submanifold $M \subset N$ transverse to $\mathcal{E}$.

Example 4.1. Let B be the Lie algebra $\mathrm{sl}_2(\mathbb{R})$. Recall that $\mathrm{sl}_2(\mathbb{R})$ is generated by elements H, Y_+, Y_- that satisfy $[H, Y_+] = 2Y_+$, $[H, Y_-] = -2Y_-$, $[Y_+, Y_-] = H$. The pair $(Y_- \wedge Y_+, H)$ is a Jacobi structure.

Lemma 4.2. *We use the notations introduced at the beginning of this section. For every* $\pi \in \Gamma(\wedge^2 B)$ *and* $E \in \Gamma(B)$*, the following are equivalent:*

(i) (π, E) *is a Jacobi structure for the Lie algebroid* μ_B *on B,*
(ii) $P_{\pi,E} := e^{-t}(\pi + \epsilon E)$ *is a Poisson structure for the Lie algebroid* μ^0_ϕ *on* $p^*A \to (M \times \mathbb{R})$.

Proof. Recall that, by definition, $P_{\pi,E} \in \Gamma(\wedge^2 p^*A)$ is a Poisson structure for the Lie algebroid μ^0_ϕ if and only if $[P_{\pi,E}, P_{\pi,E}]_{\mu^0_\phi} = 0$. A direct computation gives the following:

$$\begin{aligned}
[P_{\pi,E}, P_{\pi,E}]_{\mu^0_\phi} &= \{\{P_{\pi,E}, \mu^0_\phi\}, P_{\pi,E}\} \\
&= \{\{e^{-t}(\pi + \epsilon E), \mu + \phi\partial^t\}, e^{-t}(\pi + \epsilon E)\} \\
&= \{e^{-t}\{\pi, \mu\} + e^{-t}\epsilon\{E, \mu\} + \{e^{-t}, \partial^t\}\phi(\pi + \epsilon E) - e^{-t}E\partial^t, e^{-t}(\pi + \epsilon E)\} \\
&= \{e^{-t}\{\pi, \mu\} + e^{-t}\epsilon\{E, \mu\} - e^{-t}\phi(\pi + \epsilon E) - e^{-t}E\partial^t, e^{-t}(\pi + \epsilon E)\} \\
&= e^{-2t}\{\{\pi, \mu\} + \epsilon\{E, \mu\} + \phi(\pi + \epsilon E) - E\partial^t, (\pi + \epsilon E)\} + e^{-t}\{-E\partial^t, e^{-t}\}(\pi + \epsilon E) \\
&= e^{-2t}([\pi, \pi]_{\mu_B} + 2E\pi + 2\epsilon[E, \pi]_{\mu_B}).
\end{aligned}$$

Now, a section of $\wedge^2 A$ of the form $\Theta_1 + \epsilon\Theta_2 = 0$, with $\Theta_1 \in \Gamma(\wedge^3 B)$ and $\Theta_2 \in \Gamma(\wedge^2 B)$ vanishes if and only $\Theta_1 = \Theta_2 = 0$, so that $[P_{\pi,E}, P_{\pi,E}]_{\mu^0_\phi} = 0$ if and only if $[\pi, \pi]_{\mu_B} + 2E\pi = [E, \pi]_{\mu_B} = 0$. □

Remark. We could also prove this lemma as follows. It is routine to check that (π, E) is a Jacobi structure if and only if $\pi + \epsilon E$ is a Jacobi element for the Jacobi algebroid $\mu + \phi$ on p^*B, (i.e. $[\pi + \epsilon E, \pi + \epsilon E]_{\mu,\phi} = 0$). Now, according to Remark 3.8, this implies that $P_{\pi,E} := e^{-t}(\pi + \epsilon E)$ satisfies $[P_{\pi,E}, P_{\pi,E}]_{\mu^0_\phi}$.

Remark 4.3. In general [10,9], the Poissonization is constructed on the direct product of the Lie algebroid $B \to M$ with the tangent algebroid $T\mathbb{R} \to \mathbb{R}$. But this Lie algebroid is indeed canonically isomorphic to $p^*(B \oplus \mathbb{R}) \to (M \times R)$, and our construction matches the usual one.

Moreover [20], when $B = TM$ is the tangent Lie algebroid of M, p^*A is isomorphic to $T(M \times \mathbb{R})$, and, under the previous isomorphism, ϵ is the vector field $\frac{\partial}{\partial t}$. Specializing Lemma 4.2, one obtains that a pair (π, E) (with π a bivector field and E a vector field) is a Jacobi structure on TM if and only if $e^{-t}(\pi + \frac{\partial}{\partial t} \wedge E)$ is a Poisson structure on $M \times \mathbb{R}$.

4.2. The Jacobi structure on the base of a Jacobi bialgebroid

Just as a Lie bialgebroid induces a Poisson structure on its base, a Jacobi bialgebroid induces a Jacobi structure on its base, see [9]. Our formalism yields a convenient description of this Jacobi structure on the base.

Proposition 4.4. *Let* $(\mu + \phi, \nu + X)$ *be a Jacobi bialgebroid on a vector bundle* $A \to M$*, then*

1. *The assignment*

$$\mathcal{F}(M) \to \mathcal{F}(M)$$
$$f \mapsto \{\{\mu, X\}, f\}$$

is a vector field $\underline{E}$ *on M. This vector fields is also given by the assignment* $f \mapsto -\{\{\nu, \phi\}, f\}$.

2. *The assignment*

$$\mathcal{F}(M) \times \mathcal{F}(M) \to \mathcal{F}(M)$$
$$(f, g) \mapsto \{\{\mu, f\}, \{\nu, g\}\}$$

is a bivector field $\underline{\pi}$ *on* M.

3. *The pair* $(\underline{\pi}, -\underline{E})$ *is a Jacobi structure on* TM.

Proof. It is clear that both assignments are derivations in each variable for degree reasons. Let us prove that the second one is skew-symmetric. A direct computation yields $\underline{\pi}[f, g] - \underline{\pi}[g, f] = \{\{\{\mu, \nu\}, f\}, g\}$, for every $f, g \in \mathcal{F}(M)$. According to Theorem 3.13, the quantity $\spadesuit = 0$, therefore this relation can be written as:

$$\underline{\pi}[f, g] - \underline{\pi}[g, f] = -\left\{\left\{\mu X + \nu\phi - \mathrm{id}_A\left(-\phi X + \frac{\{\mu, X\} - \{\nu, \phi\}}{2}\right), f\right\}, g\right\},$$

a quantity that vanishes for reasons of bi-degree. This proves the first two items.

We now turn our attention to the third item. According to Corollary 3.16, the pair (μ^0_ϕ, ν^1_X) is a Lie bialgebroid. Hence, according to proposition 3.6 in [21], the assignment $\mathcal{F}(M \times \mathbb{R}) \times \mathcal{F}(M \times \mathbb{R}) \mapsto \mathcal{F}(M \times \mathbb{R})$ given by

$$(f, g) \mapsto \{\{\mu^0_\phi, f\}, \{\nu^1_X, g\}\}$$

is a Poisson structure on $M \times \mathbb{R}$, that we denote by P. In view of Lemma 4.2 (more precisely Remark 4.3), in order to complete the proof it suffices to check that P is precisely the Poissonization of the pair $(\underline{\pi}, -\underline{E})$. In fact, for any functions f, g on $M \times \mathbb{R}$:

$$\begin{aligned}
P[f, g] &= \{\{\mu^0_\phi, f\}, \{\nu^1_X, g\}\} \\
&= \{\{\mu + \partial^t\phi, f\}, \{e^{-t}(\nu + (\partial^t + \mathrm{id}_{A^*})X), g\}\} \\
&= \left\{\{\mu, f\} + \phi\frac{\partial f}{\partial t}, e^{-t}\left(\{\nu, g\} + X\frac{\partial g}{\partial t}\right)\right\} \\
&= e^{-t}\left(\{\{\mu, f\}, \{\nu, g\}\} + \{X, \phi\}\frac{\partial f}{\partial t}\frac{\partial g}{\partial t} + \left\{\{\mu, f\}, X\frac{\partial g}{\partial t}\right\} + \left\{\phi\frac{\partial f}{\partial t}, \{\nu, g\}\right\}\right) \\
&= e^{-t}\left(\{\{\mu, f\}, \{\nu, g\}\} + \left\{\{\mu, f\}, X\frac{\partial g}{\partial t}\right\} + \left\{\phi\frac{\partial f}{\partial t}, \{\nu, g\}\right\}\right) \quad (\text{since } \{\phi, X\} = \diamondsuit = 0\,) \\
&= e^{-t}\left(\{\{\mu, f\}, \{\nu, g\}\} + \{\{\mu, f\}, X\}\frac{\partial g}{\partial t} + \frac{\partial f}{\partial t}\{\phi, \{\nu, g\}\}\right) \\
&= e^{-t}\left(\underline{\pi}[f, g] + \underline{E}(f)\frac{\partial g}{\partial t} - \frac{\partial f}{\partial t}\underline{E}(g)\right) \\
&= e^{-t}\left(\underline{\pi} + \frac{\partial}{\partial t} \wedge (-\underline{E})\right)[f, g].
\end{aligned}$$

This completes the proof. □

4.3. The Jacobi bialgebroid of a Jacobi structure

For every Poisson structure P on a Lie algebroid ν defined on a vector bundle C, the function $\{P, \nu\}$ is a Lie algebroid on C^*. Moreover, the pair $(\nu, \{P, \nu\})$ is a Lie bialgebroid. We refer to [21] for these classical results. We call the Lie algebroid $\{P, \nu\}$ (resp. the Lie bialgebroid $(\nu, \{P, \nu\})$) the *Lie algebroid (resp. the Lie bialgebroid) associated to the Poisson bivector* P. In particular, it follows from Lemma 4.2 that, for every Jacobi structure (π, E) on a Lie algebroid μ_B on $B \to M$, the function in $\mathcal{F}_{p^*A}$ (see Section 2.2 for the notations) defined by

$$\nu_{\mu,\pi,E} := \{P_{\pi,E}, \mu^0_\phi\} = \{P_{\pi,E}, \mu + \phi\partial^t\}$$

is a Lie algebroid on $(p^*A)^* \to (M \times \mathbb{R})$ (which is isomorphic to $p^*A^* \to (M \times \mathbb{R})$). Moreover, the pair $(\mu^0_\phi, \nu_{\mu,\pi,E})$ is a Lie bialgebroid on the vector bundle $p^*A \to (M \times \mathbb{R})$.

Lemma 4.5. *We use the notations introduced at the beginning of Section 4. Let* (π, E) *be a Jacobi structure on the Lie algebroid* μ_B *on* $B \to M$. *Define*

$$\nu := \{\pi, \mu\} + \phi\pi + \epsilon\{\mu, E\} - E\,\mathrm{id}_B \in \mathcal{F}^{1,2}_A.$$

The Poissonization of the pre-Jacobi algebroid $\nu - E$ *on* A *is the Lie algebroid* $\nu_{\mu,\pi,E} := \{P_{\pi,E}, \mu^0_\phi\}$ *associated with the Poisson structure* $P_{\pi,E}$.

Proof. We shall prove that the functions $\nu^1_{-E} = e^{-t}(\nu - E(\partial^t + \mathrm{id}_{A^*}))$ and $\nu_{\mu,\pi,E} = \{P_{\pi,E}, \mu + \phi\partial^t\}$ are equal. This follows from a comparison between

$$\begin{aligned}\nu_{\mu,\pi,E} &= \{e^{-t}(\pi + \epsilon E), \mu + \phi\partial^t\}\\ &= e^{-t}(\{(\pi + \epsilon E), \mu\} + \phi(\pi + \epsilon E) - \partial^t E)\\ &= e^{-t}(\{\pi, \mu\} + \{\epsilon E, \mu\} + \phi(\pi + \epsilon E) - \partial^t E)\\ &= e^{-t}(\{\pi, \mu\} + \epsilon\{E, \mu\} + \phi(\pi + \epsilon E) - \partial^t E)\end{aligned}$$

and

$$\begin{aligned}\nu^1_{-E} &= e^{-t}(\nu + E(\partial^t - \mathrm{id}_A))\\ &= e^{-t}(\{\pi, \mu\} + \phi\pi + \epsilon\{\mu, E\} - E\,\mathrm{id}_B - E(\partial^t - \mathrm{id}_A))\\ &= e^{-t}(\{\pi, \mu\} + \phi\pi + \epsilon\{\mu, E\} - \partial^t E + (\mathrm{id}_A - id_B)E)\\ &= e^{-t}(\{\pi, \mu\} + \phi\pi + \epsilon\{E, \mu\} - \partial^t E + \phi\epsilon E)\end{aligned}$$

where, in the last line, we have used the relation $\mathrm{id}_A = \mathrm{id}_B + \phi\epsilon$. □

Remark. A direct comparison shows that the bracket and anchor of the associated pre-Lie algebroid ν on $A \to M$ are precisely the bracket and anchor of the Lie algebroids that appear in [6,10,9].

The main result of this section is an immediate consequence of Lemma 3.7 and of Lemma 4.5.

Theorem 4.6. *We use the notations introduced at the beginning of Section 4. Let (π, E) be a Jacobi structure on the algebroid μ_B on $B \to M$. Then,*

1. *the pair $(\mu + \phi, \nu - E)$ is a Jacobi bialgebroid on A, where:*

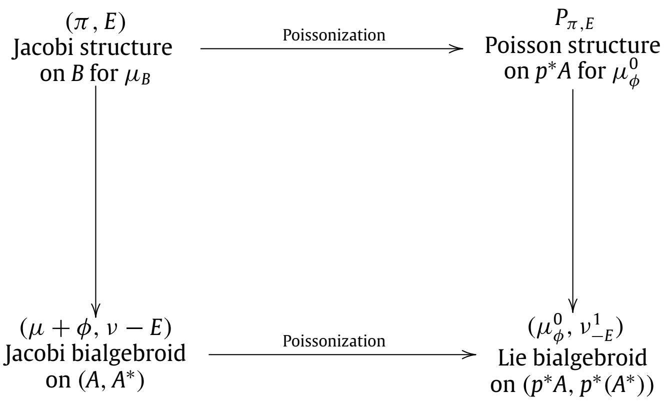
   $$\nu := \{\pi, \mu\} + \phi\pi + \epsilon\{\mu, E\} - E\,\mathrm{id}_B \in \mathcal{F}^{2,1}_A.$$

2. *The Poissonization of this Jacobi bi-algebroid is the Lie bialgebroid $(\mu^0_\phi, \nu_{\mu,\pi,E})$ associated to the Poisson structure $P_{\pi,E}$ (i.e., to the Poissonization of the Jacobi structure (π, E)).*

Remark. A direct computation shows that the bracket of the Jacobi algebroid $\nu - E$ of Theorem 4.6 coincides with the bracket described in [19] corollary 4 applied to the Jacobi element $\pi + \epsilon E \in \Gamma(\wedge^2 p^*A)$ and the Jacobi algebra A, equipped with the Jacobi algebroid $[\cdot,\cdot]_{\mu,\phi}$.

Example 4.7. Applying this construction to the Jacobi structure of Example 4.1, one finds the Jacobi bialgebroid announced in Example 3.14.

We summarize the results of this section in a commutative diagram. Let μ_B be a Lie algebroid on a vector bundle B, equipped with a Jacobi structure (π, E). Recall that A stands for $B \oplus \mathbb{R}$ (and μ is the natural extension of μ_B to that bundle), while p^*A stands for the pull-back of $A \to M$ on $M \times \mathbb{R}$ (and can be equipped with the Lie algebroid $\mu^0_\phi := \mu + \partial^t\phi$). Let $P_{\pi,E}$ be the Poissonization of the Jacobi structure (π, E) (defined on the Lie algebroid μ^0_ϕ on p^*A). Let $\nu := \{\pi, \mu\} + \phi\pi + \epsilon\{\mu, E\} - E\,\mathrm{id}_B$ be the Jacobi algebroid on A defined above. The following diagram commutes



5. Quasi-Jacobi bialgebroids

In [11], quasi-Jacobi bialgebroids are introduced, as follows:

Definition 5.1. Let $A \to M$ be a vector bundle. A quasi-Jacobi bialgebroid is a pre-Jacobi bialgebroid $(\mu+\phi, \nu+X)$, together with a section Z of $\wedge^3 A$, such that $\mu+\phi$ is indeed a Jacobi algebroid, and such that the following compatibility conditions are satisfied

$$\mathrm{d}_{\nu,X}(Z) = 0 \quad \text{and} \quad (\mathrm{d}_{\nu,X})^2(P) + [Z, P]_{\mu,\phi} = 0 \quad \text{for all } P \in \Gamma\left(\bigwedge A\right) \tag{5.16}$$

where $[\cdot,\cdot]_{\mu,\phi}$, $\mathrm{d}_{\nu,X}$ are as in Definition 3.2.

As a slight modification of Theorem 3.13, we obtain:

Theorem 5.2. *Let $A \to M$ be a vector bundle, $\mu+\phi$ a pre-Jacobi algebroid on A, $\nu+X$ a pre-Jacobi algebroid on A^*, and Z a section of $\wedge^3 A$. Define $\diamondsuit$, $\clubsuit$ and $\spadesuit$ as in (3.12). The triple $(\mu+\phi, \nu+X, Z)$ is a quasi-Jacobi bialgebroid if and only if*

$$\diamondsuit = \clubsuit = \spadesuit = \{\mu+\phi, \mu+\phi\} = \{\nu, Z\} + XZ = \{\nu, X\} + \{Z, \phi\} = \maltese = 0,$$

with $\maltese = \frac{1}{2}\{\nu,\nu\} + \{Z,\mu\} + \mathrm{id}_A\{Z,\phi\} + 2Z\phi$.

Proof. According to Theorem 3.13, the pair $(\mu+\phi, \nu+X)$ is a pre-Jacobi bialgebroid if and only if $\diamondsuit = \clubsuit = \spadesuit = 0$. Also, $\{\mu+\phi, \mu+\phi\} = 0$ if and only if $\mu+\phi$ is a Jacobi algebroid. The condition $\mathrm{d}_{\nu,X}(Z) = 0$ is equivalent to $\{\nu, Z\} + XZ = 0$ in view of the expression of the differential given in Definition 3.2.

Spelling out the condition $(\mathrm{d}_{\nu,X})^2(P) = [-Z, P]_{\mu,\phi}$ with the differentials and brackets introduced in Definition 3.2 gives:

$$\{\nu, \{\nu, P\} + XP\} + X(\{\nu, P\} + XP) = -\underline{b}\{Z, \mu + \mathrm{id}_A\phi\}, P + \{Z\phi, P\} - \{Z, \phi\}P,$$

which can be rewritten as

$$\{\maltese, P\} + (\{\nu, X\} + \{Z, \phi\})P = 0.$$

This condition is therefore satisfied if $\{\nu, X\} + \{Z, \phi\} = \maltese = 0$. Conversely, if this condition is satisfied for $P = 1$, then $\{\nu, X\} + \{Z, \phi\} = 0$. If the condition is satisfied for all $P \in \Gamma(\bigwedge A)$ therefore, we have $\{\maltese, P\} = 0$ for all $P \in \Gamma(A)$, which, in view of the seventh item listed in Section 2.1, since $\maltese \in \mathcal{F}_A^{3,1}$, implies that $\maltese = 0$. This completes the proof. □

Recall [8] that a quasi-Lie bialgebroid on a vector bundle B is a triple of functions in, respectively, $\mathcal{F}_A^{1,2}$, $\mathcal{F}_A^{2,1}$, $\mathcal{F}_A^{3,0}$, whose sum commutes with itself. Theorem 5.2 admits the following corollary, to be compared with Corollary 3.16.

Corollary 5.3. *Let $A \to M$ be a vector bundle, $\mu+\phi$ a pre-Jacobi algebroid on A, $\nu+X$ a pre-Jacobi algebroid on A^*, and $Z \in \Gamma(\wedge^3 A)$. The following are equivalent:*

(i) *the triple $(\mu+\phi, \nu+X, Z)$ is a quasi-Jacobi bialgebroid on A;*
(ii) *the triple $(\mu_\phi^0, \nu_X^1, e^{-2t}Z)$ is a quasi-Lie bialgebroid on p^*A.*

(Here, μ_ϕ^0, ν_X^1 are the pre-Lie algebroids on $p^*A \to (M \times \mathbb{R})$ constructed in Lemma 3.7 and $t \in \mathcal{F}(M \times \mathbb{R})$ is, as in Section 2.2-A, the projection on the second factor.)

Proof. The triple $(\mu_\phi^0, \nu_X^1, e^{-2t}Z)$ is a quasi-Lie bialgebroid on p^*A if and only if

$$\{\mu_\phi^0 + \nu_X^1 + e^{-2t}Z, \mu_\phi^0 + \nu_X^1 + e^{-2t}Z\} = 0.$$

For degree reasons, this condition splits into the four conditions:

$$\{\mu_\phi^0, \mu_\phi^0\} = 0 \tag{$*$}$$

$$\{\mu_\phi^0, \nu_X^1\} = 0 \tag{$**$}$$

$$\{\nu_X^1, \nu_X^1\} = -2\{\mu_\phi^0, e^{-2t}Z\} \tag{$***$}$$

$$\{\nu_X^1, e^{-2t}Z\} = 0. \tag{$****$}$$

It follows from Lemma 3.7 that condition $(*)$ holds if and only if $\{\mu+\phi, \mu+\phi\} = 0$. It follows from item (1) in Corollary 3.16 that condition $(**)$ holds if and only if $(\mu+\phi, \nu+X)$ is a pre-Jacobi bialgebroid, i.e., in view of Theorem 3.13, if and only if $\diamondsuit = \clubsuit = \spadesuit = 0$. It follows from a direct computation that condition $(****)$ holds if and only if

$$\{\nu_X^1, e^{-2t}Z\} = e^{-3t}(\{\nu, Z\} + 2ZX - 3ZX) = 0,$$

i.e., if and only if $\{\nu, Z\} + XZ = 0$.

In view of Theorem 5.2, we only have to prove that condition $(***)$ holds if and only if $\{\nu, X\} + \{Z, \phi\} = \maltese = 0$. First, we compute:

$$\begin{aligned}\{\nu^1_X, \nu^1_X\} &= \{e^{-t}(\nu + X(\partial^t + \mathrm{id}_{A^*})), e^{-t}(\nu + X(\partial^t + \mathrm{id}_{A^*}))\}\\ &= e^{-2t}(\{\nu, \nu\} + 2\{\nu, X\}\mathrm{id}_{A^*} + 2\{\nu, X\}\partial^t).\end{aligned}$$

Second, a direct computation gives $\{\mu^0_\phi, e^{-2t}Z\} = e^{-2t}(\{\mu, Z\} - 2\phi Z + \partial^t\{\phi, Z\})$, so that, reordering terms, we obtain that $(***)$ is equivalent to

$$\maltese + \partial^t(\{\nu, X\} + \{Z, \phi\}) = 0,$$

which is itself equivalent to the vanishing of both $\maltese$ and $\{\nu, X\} + \{Z, \phi\}$ (as noted in Section 2.2-A). The result follows. □

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References

[1] C. Laurent-Gengoux, A. Pichereau, P. Vanhaecke, An invitation to Poisson structures (in preparation).
[2] A. Lichnerowicz, Les variétés de Jacobi et leurs algèbres de Lie associées, J. Math. Pures Appl. (9) 57 (4) (1978) 453–488.
[3] J. Grabowski, D. Iglesias, J.C. Marrero, E. Padrón, P. Urbanski, Poisson–Jacobi reduction of homogeneous tensors, J. Phys. A 37 (20) (2004) 5383–5399.
[4] J.-L. Koszul, Crochet de Schouten–Nijenhuis et cohomologie, in: The Mathematical Heritage of Élie Cartan, Astérisque, 1984, pp. 257–271.
[5] K. Mackenzie, General theory of Lie groupoids and Lie algebroids, in: London Mathematical Society Lecture Note Series, vol. 213, Cambridge University Press, Cambridge, 2005.
[6] Y. Kerbrat, Z. Souici-Benhammidi, Variétés de Jacobi et groupoïdes de contact, Comp. Rend. Acad. Sciences Paris 317 (1993) 81–86.
[7] Y. Kosmann-Schwarzbach, Exact Gerstenhaber algebras and Lie bialgebroids, Acta Appl. Math. 41 (1995) 153–165.
[8] D. Roytenberg, Courant algebroids, derived brackets, and even symplectic supermanifolds, Ph.D.-Thesis, UC Berkeley, 1999.
[9] D. Iglesias, J.C. Marrero, Generalized Lie bialgebroids and Jacobi structures, J. Geom. Phys. 40 (2) (2001) 176–199.
[10] J. Grabowski, G. Marmo, Jacobi structures revisited, J. Phys. A 34 (2001) 10975–10990.
[11] R. Caseiro, A. de Nicola, J. Nunes da Costa, On Jacobi quasi-Nijenhuis algebroids and Courant–Jacobi algebroid morphisms, J. Geom. Phys. 60 (6–8) (2010) 951–961.
[12] B. Kostant, S. Sternberg, Symplectic reduction, BRS cohomology, and infinite-dimensional Clifford algebras, Ann. Phys. 176 (1987) 49–113.
[13] Y. Kosmann-Schwarzbach, Jacobian quasi-bialgebras and quasi-Poisson Lie groups, Contemp. Math. 132 (1992) 459–489.
[14] P. Lecomte, C. Roger, Modules et cohomologie des bigèbres de Lie, Comp. Rend. Acad. Sciences Paris 310 (1990) 405–410.
[15] W.M. Tulczyjew, Geometric formulations of physical theories, in: Statics and Dynamics of Mechanical Systems, in: Monographs and Textbooks in Physical Science, Lecture Notes, vol. 11, Bibliopolis, Naples, 1989.
[16] Y. Kosmann-Schwarzbach, V. Rubtsov, Compatible structures on Lie algebroids and Monge–Ampère operators, Acta Appl. Math. 109 (2010) 101–135.
[17] A. Vaintrob, Lie algebroids and homological vector fields, Uspekhi Mat. Nauk. 52 (1997) 161–162.
[18] Y. Kosmann-Schwarzbach, F. Magri, Poisson–Nijenhuis structures, Ann. Inst. H. Poincaré Phys. Théor. 53 (1990) 35–81.
[19] J. Grabowski, G. Marmo, The graded Jacobi algebras and (co)homology, J. Phys. A 36 (2003) 161–181.
[20] P. Dazord, A. Lichnerowicz, C.-M. Marle, Structure locale des variétés de Jacobi, J. Math. Pures Appl. 70 (1991) 101–152.
[21] K. Mackenzie, P. Xu, Lie bialgebroids and Poisson groupoids, Duke Math. J. 73 (1994) 415–452.