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JORDANIAN DEFORMATION OF THE OPEN XXX SPIN CHAIN

© **P. P. Kulish,*** **N. Manojlović,**† **and Z. Nagy**‡

We find the general solution of the reflection equation associated with the Jordanian deformation of the $SL(2)$-invariant Yang R-matrix. A special scaling limit of the XXZ model with general boundary conditions leads to the same K-matrix. Following the Sklyanin formalism, we derive the Hamiltonian with the boundary terms in explicit form. We also discuss the structure of the spectrum of the deformed XXX model and its dependence on the boundary conditions.

Keywords: spin chain, boundary condition, quantum group, reflection equation

1. Introduction

The quantum inverse scattering method (QISM) [1]–[4] as an approach for constructing and solving quantum integrable systems led to creating the theory of quantum groups [5], [6]. A particularly interesting feature of quantum groups is a transformation that is called twist [7] and allows constructing new quantum groups from already known ones. Although the twist transformations generate an equivalence relation between quantum groups, they produce different R-matrices. These new R-matrices can in turn lead to new integrable systems [8].

The twist of a quantum group or, more generally, a Hopf algebra $\mathcal{A}$ is a similarity transformation of the coproduct $\Delta\colon \mathcal{A}\to\mathcal{A}\otimes\mathcal{A}$ by an invertible twist element

$$\Delta(a)\to\Delta_t(a)=\mathcal{F}\Delta(a)\mathcal{F}^{-1},\quad a\in\mathcal{A}. \tag{1.1}$$

To guarantee the coassociativity property of the coproduct, the element $\mathcal{F}$ must satisfy a certain compatibility condition, the so-called twist equation

$$\mathcal{F}_{12}(\Delta\otimes\mathrm{id})\mathcal{F}=\mathcal{F}_{23}(\mathrm{id}\otimes\Delta)\mathcal{F}, \tag{1.2}$$

where

$$(\Delta\otimes\mathrm{id})\sum_j f_j^{(1)}\otimes f_j^{(2)}=\sum_j\Delta(f_j^{(1)})\otimes f_j^{(2)}\in\mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A}.$$

Moreover, the transformation law of the coproduct also determines how the corresponding universal R-matrix changes:

$$\mathcal{R}\to\mathcal{R}^{(t)}=\mathcal{F}_{21}\mathcal{R}\mathcal{F}^{-1},\qquad \mathcal{F}_{21}=\sum_j f_j^{(2)}\otimes f_j^{(1)}. \tag{1.3}$$

*St. Petersburg Department of the Steklov Institute of Mathematics, St. Petersburg, Russia, e-mail: kulish@euclid.pdmi.ras.ru.

†Departamento de Matemática, Universidade do Algarve, Campus de Gambelas, Faro, Portugal, e-mail: nmanoj@ualg.pt.

‡Grupo de Física Mathemàtica, Universidade de Lisboa, Lisboa, Portugal, e-mail: zoltan.nagy@m4x.org.

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This new R-matrix allows building and studying new integrable models [9].

A particular solution of the twist equation is provided by the Jordanian twist element for the enveloping algebra of the $sl(2)$ Lie algebra, which was introduced in [10], [11] and was extended to the $sl(N)$ case in [8], [12]. Here, we consider the example of the twisted algebra $sl(2)$ and the corresponding deformation of the Yangian $\mathcal{Y}\big(sl(2)\big)$ [9], [13]. Because the twist preserves the regularity of the R-matrix (i.e., $R(0)=\mathcal{P}$, where $\mathcal{P}$ is the permutation map of the neighboring spaces), we can write the deformed version of the integrable Heisenberg XXX spin chain with periodic boundary conditions [9]:

$$H=\sum_{j=1}^{N}\biggl(\frac{1}{2}(\sigma_j^x\sigma_{j+1}^x+\sigma_j^y\sigma_{j+1}^y+\sigma_j^z\sigma_{j+1}^z)+\theta(\sigma_j^+-\sigma_{j+1}^+)+\theta^2\sigma_j^+\sigma_{j+1}^+\biggr).$$

We note that this operator is non-Hermitian, which results in additional difficulties in applying the algebraic Bethe ansatz to this model. Although it can be seen that the extra terms added to the XXX Hamiltonian do not change the spectrum of the model [9], [14], the explicit form of the Bethe states is not obvious.

Here, we study the deformation by the Jordanian twist of the XXX model with nonperiodic boundary conditions, which are described by reflection matrices $K^{\pm}(\lambda)$ [15]. The result is a classification of reflection matrices compatible with the twisted Jordanian R-matrix. We obtain the general solution of the reflection equation by direct calculation; it is also confirmed by the singular scaling limit from the known reflection matrix of the anisotropic XXZ model. Using the general solution for $K(\lambda)$ and following the Sklyanin approach [15], we construct the Hamiltonian with the general nonperiodic boundary conditions. We conclude with some remarks on the influence of the boundary conditions on the system spectrum.

2. Solutions of the reflection equation

The main QISM relation [1], [3]

$$R_{12}(\lambda-\mu)T_1(\lambda)T_2(\mu)=T_2(\mu)T_1(\lambda)R_{12}(\lambda-\mu) \tag{2.1}$$

can be used to define a special infinite-dimensional quantum algebra, the Yangian. The Yang R-matrix

$$R_{12}(\lambda)=\lambda I+\eta\mathcal{P}\in\operatorname{End}(\mathbb{C}^n\otimes\mathbb{C}^n) \tag{2.2}$$

determines the Yangian $\mathcal{Y}\big(sl(n)\big)$ with the elements of the $n\times n$ matrix $T(\lambda)$ as generating functions of the $\mathcal{Y}\big(sl(n)\big)$ generators (see [16]). We use I for the identity operator and $\mathcal{P}$ for the permutation in $\mathbb{C}^n\times\mathbb{C}^n$, $\mathcal{P}(v\otimes w)=w\otimes v$, $v,w\in\mathbb{C}^n$, and the standard QISM notation $T_1=T\otimes I$ and $T_2=I\otimes T$. The Heisenberg XXX spin chain is related to $\mathcal{Y}\big(sl(2)\big)$, and the universal enveloping algebra of $sl(2)$ is a Hopf subalgebra of the Yangian, $U\big(sl(2)\big)\subset\mathcal{Y}\big(sl(2)\big)$. The two generators h and X^+ of $sl(2)$,

$$[h,X^{\pm}]=\pm2X^{\pm},\qquad [X^+,X^-]=h, \tag{2.3}$$

yield the Jordanian twist element

$$\mathcal{F}=e^{h\otimes\log(1+\theta X^+)}\in U\big(sl(2)\big)\otimes U\big(sl(2)\big), \tag{2.4}$$

which satisfies Drinfeld twist equation (1.2). The matrix form of $\mathcal{F}$ in the spin-1/2 representation ρ is $F_{12}\in\operatorname{End}(\mathbb{C}^2\otimes\mathbb{C}^2)$,

$$F_{12}=(\rho\otimes\rho)\mathcal{F}=e^{\sigma^z\otimes\theta\sigma^+}=\mathbf{1}+\theta\sigma^z\otimes\sigma^+=\begin{pmatrix}1&\theta&0&0\\0&1&0&0\\0&0&1&-\theta\\0&0&0&1\end{pmatrix}, \tag{2.5}$$

where σ^z and $\sigma^{\pm} = (\sigma^x \pm i\sigma^y)/2$ are the Pauli sigma matrices.

Hence, the R-matrix of the twisted Yangian $\mathcal{Y}_\theta\big(sl(2)\big)$ has the form [9]

$$R^{(j)}(\lambda) = F_{21}R_{12}(\lambda)F_{12}^{-1} = \lambda R^{(j)} + \eta\mathcal{P} = \begin{pmatrix} \lambda+\eta & -\lambda\theta & \lambda\theta & \lambda\theta^2 \\ 0 & \lambda & \eta & -\lambda\theta \\ 0 & \eta & \lambda & \lambda\theta \\ 0 & 0 & 0 & \lambda+\eta \end{pmatrix}, \tag{2.6}$$

where $F_{21} = \mathcal{P}F_{12}\mathcal{P}$. This R-matrix is also a solution of the Yang–Baxter equation

$$R_{12}(\lambda-\mu)R_{13}(\lambda)R_{23}(\mu) = R_{23}(\mu)R_{13}(\lambda)R_{12}(\lambda-\mu). \tag{2.7}$$

The unitarity of the R-matrix is unaffected by the twist,

$$R_{12}(\lambda)R_{21}(-\lambda) = g(\lambda) \tag{2.8}$$

with $g(\lambda) = (-\lambda^2 + \eta^2)$, but the PT symmetry is broken,

$$R_{21}(\lambda) \neq R_{12}(\lambda)^{t_1 t_2}, \tag{2.9}$$

where $R_{21}(\lambda) = \mathcal{P}R_{12}(\lambda)\mathcal{P}$ and the indices t_1 and t_2 denote the respective transpositions in the first and second spaces of the tensor product $\mathbb{C}^2 \otimes \mathbb{C}^2$. The R-matrix also does not have the crossing symmetry, but it does satisfy the weaker condition

$$\Big\{\Big\{\big\{R_{12}(\lambda)^{t_2}\big\}^{-1}\Big\}^{t_2}\Big\}^{-1} = \frac{g(\lambda+\eta)}{g(\lambda+2\eta)}M_2R_{12}(\lambda+2\eta)M_2^{-1} \tag{2.10}$$

with the matrix

$$M = \begin{pmatrix} 1 & -2\theta \\ 0 & 1 \end{pmatrix}. \tag{2.11}$$

We note that the more general matrix

$$\widetilde{M} = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \tag{2.12}$$

commutes with the R-matrix,

$$[\widetilde{M}\otimes\widetilde{M}, R(\lambda)] = 0. \tag{2.13}$$

As a general remark, we note that for our purposes, it suffices that the matrix $\Big\{\Big\{\big\{R_{12}(\lambda)^{t_2}\big\}^{-1}\Big\}^{t_2}\Big\}^{-1}$ exists. The fact that this matrix can be written in form (2.10) only implies that we can establish a bijection between solutions $K^-(\lambda)$ and $K^+(\lambda)$ of the left and right reflection equations, as we later show.

A way to introduce nonperiodic boundary conditions compatible with the integrability of the base model was developed in [15]. The boundary conditions at the left and right sites of the system are expressed in the left and right reflection matrices K^- and K^+. The compatibility condition between the bulk and the boundary of the system takes the form of the so-called reflection equation [17], [18]. For the left reflection matrix, it is written in the form

$$R_{12}(\lambda-\mu)K_1^-(\lambda)R_{21}(\lambda+\mu)K_2^-(\mu) = K_2^-(\mu)R_{12}(\lambda+\mu)K_1^-(\lambda)R_{21}(\lambda-\mu). \tag{2.14}$$

The compatibility at the right site of the model is expressed by the dual reflection equation [15], [19]–[21]

$$A_{12}(\lambda-\mu)K_1^{+t}(\lambda)B_{12}(\lambda+\mu)K_2^{+t}(\mu)=K_2^{+t}(\mu)C_{12}(\lambda+\mu)K_1^{+t}(\lambda)D_{12}(\lambda-\mu), \tag{2.15}$$

where the matrices A, B, C, and D are obtained from the R-matrix of reflection equation (2.14) as

$$A_{12}(\lambda)=\bigl(R_{12}(\lambda)^{t_{12}}\bigr)^{-1}=D_{21}(\lambda), \tag{2.16}$$

$$B_{12}(\lambda)=\bigl(\bigl(R_{21}^{t_1}(\lambda)\bigr)^{-1}\bigr)^{t_2}=C_{21}(\lambda) \tag{2.17}$$

or, explicitly,

$$A(\lambda)=\frac{1}{\lambda^2-\eta^2}\begin{pmatrix}\lambda-\eta & 0 & 0 & 0\\ \lambda\theta & \lambda & -\eta & 0\\ -\lambda\theta & -\eta & \lambda & 0\\ \lambda\theta^2 & \lambda\theta & -\lambda\theta & \lambda-\eta\end{pmatrix}, \tag{2.18}$$

$$B(\lambda)=\frac{1}{\lambda(\lambda+2\eta)}\begin{pmatrix}\lambda+\eta & 0 & 0 & 0\\ -\lambda\theta & \lambda+2\eta & -\eta & 0\\ \lambda\theta & -\eta & \lambda+2\eta & 0\\ -(3\lambda+2\eta)\theta^2 & -\lambda\theta & \lambda\theta & \lambda+\eta\end{pmatrix}. \tag{2.19}$$

Using property (2.10), we can write dual reflection equation (2.15) in the equivalent form

$$\begin{aligned}R_{12}(-\lambda+\mu)K_1^{+}(\lambda)M_2R_{21}(-\lambda-\mu-2\eta)M_2^{-1}K_2^{+}(\mu)&=\\ =K_2^{+}(\mu)M_1R_{12}(-\lambda-\mu-2\eta)M_1^{-1}K_1^{+}(\lambda)R_{21}(-\lambda+\mu).&\end{aligned} \tag{2.20}$$

It can then be verified that the map

$$K^+(\lambda)=K^-(-\lambda-\eta)M \tag{2.21}$$

is a bijection between solutions of the reflection equation and the dual equation.

The solutions are classified using the standard, but somewhat tedious, approach. We first note that if $K(\lambda)$ is a solution of this equation, then obviously $f(\lambda)K(\lambda)$ is also. We use this freedom to fix $k_{11}(\lambda)=1$ (and it can be easily verified that the assumption $k_{11}(\lambda)=0$ leads to a noninvertible, rank-1 solution). We seek the general solution in the form

$$K^-(\lambda)=\begin{pmatrix}1 & k_{12}(\lambda)\\ k_{21}(\lambda) & k_{22}(\lambda)\end{pmatrix}. \tag{2.22}$$

Substituting this K-matrix in (2.14) and writing the equations for the matrix elements, we see that by adding elements 21 and 31 of the resulting matrix equation, we obtain

$$k_{21}(\lambda)\bigl(k_{22}(\mu)-1\bigr)=k_{21}(\mu)\bigl(k_{22}(\lambda)-1\bigr). \tag{2.23}$$

This is a functional equation of the form

$$f(\lambda)g(\mu)=f(\mu)g(\lambda). \tag{2.24}$$

We recall its general solution. The equation is obviously satisfied if either function is identically zero. If one of them is not identically zero, then they are proportional to each other. According to this, the solution of (2.23) splits into two cases:

1. $k_{22}(\lambda) = \phi k_{21}(\lambda) + 1$, $\phi \in \mathbb{C}$, or
2. $k_{21}(\lambda) = 0$.

We start with case 1 and substitute $k_{22}(\lambda) = \phi k_{21}(\lambda) + 1$. Entry 21 of (2.14) then yields

$$k_{21}(\lambda)\big(2\eta\mu + \eta\phi\mu k_{21}(\mu) + 2\theta\mu^2 k_{21}(\mu)\big) = k_{21}(\mu)\big(2\eta\lambda + \eta\phi\lambda k_{21}(\lambda) + 2\theta\lambda^2 k_{21}(\lambda)\big), \tag{2.25}$$

which is an algebraic equation for $k_{21}(\lambda)$, whence

$$k_{21}(\lambda) = \frac{2\eta\lambda}{\xi - \eta\phi\lambda - 2\theta\lambda^2}, \tag{2.26}$$

where $\xi \in \mathbb{C}$ is an arbitrary constant.

We have now the expression for two elements, the second being

$$k_{22}(\lambda) = 1 + \frac{2\phi\eta\lambda}{\xi - \eta\phi\lambda - 2\theta\lambda^2}. \tag{2.27}$$

Substituting these expressions in (2.14), we obtain an equation for $k_{12}(\lambda)$,

$$\lambda k_{12}(\mu)\big(\xi - (\eta\phi + 2\theta\mu)\mu\big) = \mu k_{12}(\lambda)\big(\xi - (\eta\phi + 2\theta\lambda)\lambda\big), \tag{2.28}$$

which has the solution

$$k_{12}(\lambda) = \frac{\psi\lambda}{\xi - \eta\phi\lambda - 2\theta\lambda^2} \tag{2.29}$$

with an arbitrary constant $\psi \in \mathbb{C}$.

We now turn to case 2, where $k_{21}(\lambda) = 0$. Substituting this assumption in (2.14) leads to

$$\lambda\big(1 + k_{22}(\lambda)\big)\big(k_{22}(\mu) - 1\big) = \mu\big(1 + k_{22}(\mu)\big)\big(k_{22}(\lambda) - 1\big). \tag{2.30}$$

We can assume that $k_{22} \neq 1$ without loss of generality. Then the general solution depends on one arbitrary parameter ξ,

$$k_{22}(\lambda) = \frac{\xi + \lambda}{\xi - \lambda}. \tag{2.31}$$

Substituting this expression in the reflection equation leads to

$$\lambda k_{12}(\mu)(\mu - \xi) - \mu k_{12}(\lambda)(\lambda - \xi) = 0, \tag{2.32}$$

which has the solution depending on one arbitrary constant ψ:

$$k_{12}(\lambda) = \frac{\psi\lambda}{\xi - \lambda}. \tag{2.33}$$

We can thus identify two families of reflection matrices compatible with a Jordanian R-matrix in the base model. The first family depends on three arbitrary parameters,

$$K^-(\lambda, \psi, \phi, \xi) = \begin{pmatrix} \xi - \phi\eta\lambda - 2\theta\lambda^2 & \psi\lambda \\ 2\eta\lambda & \xi + \phi\eta\lambda - 2\theta\lambda^2 \end{pmatrix}, \tag{2.34}$$

and the second family depends on two,

$$K^{-}(\lambda,\psi,\xi) = \begin{pmatrix} \xi-\lambda & \psi\lambda \\ 0 & \xi+\lambda \end{pmatrix}. \tag{2.35}$$

The obtained form of the solutions after rescaling and redefining the parameters can be transformed into one family with a more familiar form, reminiscent of the general XXX solution,

$$\begin{aligned} K^{-}(\lambda,\xi_-,\phi_-,\psi_-) &= K^{-}_{XXX}(\lambda,\xi_-,\phi_-,\psi_-) - \phi_-\theta\lambda^2\mathbf{1} = \\ &= \begin{pmatrix} \xi_- - \lambda - \phi_-\theta\lambda^2 & \psi_-\lambda \\ \eta\phi_-\lambda & \xi_- + \lambda - \phi_-\theta\lambda^2 \end{pmatrix}. \end{aligned} \tag{2.36}$$

As previously mentioned, because of relation (2.10), the general solution of the dual equation is given by bijection (2.21),

$$K^{+}(\lambda,\xi_+,\phi_+,\psi_+) = K^{-}(-\lambda-\eta,\xi_+,\phi_+,\psi_+)M. \tag{2.37}$$

Many relations of the XXX spin chain can be obtained from the XXZ model by a simple scaling, i.e., a degeneration of trigonometric functions to rational ones. It is known that the Jordanian deformation of the XXX chain can also be obtained by a scaling limit with an additional (singular) similarity transformation of the XXZ model [9]. For this, we start from the R-matrix related to the quantum algebra $U_q\big(sl(2)\big)$,

$$\check{R}(u,q) = u\check{R}(q) - \frac{1}{u}\check{R}^{-1}(q), \qquad \check{R}(q) = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & \omega(q) & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \tag{2.38}$$

where we use the multiplicative parameter $u = e^{\lambda}$ in Yang–Baxter equation (2.7) and in reflection equation (2.14). After the transformation

$$\check{R}(u,q) \to \operatorname{Ad} J(x)\otimes J(x)\check{R}(u,q) \tag{2.39}$$

with the 2×2 matrix

$$J(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \tag{2.40}$$

we consider the scaling limit

$$u = e^{\epsilon\lambda}, \qquad q = e^{\epsilon\eta}, \qquad x = \frac{\theta}{\eta\epsilon}, \quad \epsilon \to 0. \tag{2.41}$$

In the limit, the transformed R-matrix given by (2.39) becomes proportional to the R-matrix of the twisted Yangian $\mathcal{Y}_\theta\big(sl(2)\big)$ [13],

$$\check{R}(\lambda,\eta,\theta) = \lambda\check{R}^{(j)}(\theta) + \eta\mathbf{1}, \tag{2.42}$$

and hence yields a deformed XXX model [9], [14].

Being interested in the solution of reflection equation (2.14), we naturally apply the scaling also to the K-matrix. Although the K-matrix for the XXZ model is well known [22]–[24], the solution corresponding to the R-matrix $\check{R}(u,q)$ differs from that matrix (see [25], [26]),

$$K(u) = \begin{pmatrix} f + u^2 a & (u^2 - u^{-2})b \\ (u^2 - u^{-2})c & f + u^{-2}a \end{pmatrix}, \tag{2.43}$$

where the parameters a, b, c, and f are arbitrary. To obtain a finite solution after the similarity transformation with matrix (2.40),

$$K(u) \to \operatorname{Ad} J(x) K(u), \tag{2.44}$$

we must transform the parameters as

$$f = -a + \epsilon\zeta, \qquad c = \eta\epsilon c_0, \qquad b = \frac{\theta}{\eta\epsilon}(a + \theta c_0) + b_0. \tag{2.45}$$

The limit solution is then

$$K(\lambda) = \begin{pmatrix} \zeta + 2\lambda(a + 2\theta c_0) & 4\lambda b_0 \\ 0 & \zeta - 2\lambda(a + 2\theta c_0) \end{pmatrix}. \tag{2.46}$$

Comparing it with solution (2.34) found previously, we can see that there are still three terms missing. But the scaling approach allows concluding that the spectra of the deformed models with (restricted) K-matrices (2.46) coincides with the spectra of the corresponding nondeformed XXX models.

To obtain the complete K-matrix (2.36), we must use the scaling

$$f = -a + \epsilon\zeta, \qquad a = a_0 - \frac{2\theta c}{\eta\epsilon}, \qquad b = b_0 + \frac{\theta}{\eta\epsilon}\left(a_0 - \frac{\theta}{\eta\epsilon}c\right) \tag{2.47}$$

and also consider the first three terms in the expansion of $u = e^{\varepsilon\lambda}$. We then obtain

$$K(\lambda) = \begin{pmatrix} \zeta + 2a_0\lambda - (4\theta c/\eta)\lambda^2 & 4b_0\lambda \\ 4c\lambda & \zeta - 2a_0\lambda - (4\theta c/\eta)\lambda^2 \end{pmatrix} \tag{2.48}$$

as the limit K-matrix. Obviously, K-matrices (2.36) and (2.48) coincide if $\zeta = \xi_-$, $2a_0 = -1$, $4c = \eta\phi_-$, and $4b_0 = \psi_-$.

3. Construction of open spin chains

To construct deformed integrable open spin chains, we follow the method proposed by Sklyanin [15]. Using two arbitrary solutions $K^-(\lambda)$ and $K^+(\lambda)$ of reflection equations (2.14) and (2.20), we obtain the open chain transfer matrix

$$t(\lambda) = tr_0 K_0^+(\lambda) T_0(\lambda) K_0^-(\lambda) \widehat{T}_0(\lambda), \tag{3.1}$$

where the monodromy matrices are given by

$$T_0(\lambda) = R_{0N}(u) \cdots R_{01}(\lambda), \qquad \widehat{T}_0(\lambda) = R_{10}(\lambda) \cdots R_{N0}(\lambda). \tag{3.2}$$

The index 0 refers to the auxiliary space $\mathbb{C}^2$, while the indices $j = 1, 2, \ldots, N$ refer to the spin-1/2 spaces at the chain sites.

The transfer matrices at different values of the spectral parameter commute,

$$[t(\lambda), t(\mu)] = 0, \tag{3.3}$$

and the open spin chain Hamiltonian is obtained from $t'(0) = dt(\lambda)/d\lambda\big|_{\lambda=0}$. Normalizing the matrix $K^-(0) = 1$, we write the derivative of the transfer matrix in the form (all arguments are zero)

$$t'(0) \propto (tr_0 K_0^{+\prime}) + (tr_0 K_0^+) K_1^{-\prime} + \frac{2}{\eta} tr_0 (K_0^+ \check{R}'_{N0}) + \frac{2}{\eta} tr_0 K_0^+ \sum_{j=1}^{N-1} \check{R}'_{j,j+1}. \tag{3.4}$$

From this expression, we extract the Hamiltonian [15]

$$H=\sum_{j=1}^{N-1}\check{R}'_{j,j+1}+\frac{tr_0 K_0^{+t}\check{R}'_{N0}}{tr_0 K_0^{+}}+\frac{\eta}{2}K_1^{-\prime}. \tag{3.5}$$

Substituting the general boundary matrices in this expression, we obtain the open chain Hamiltonian

$$\begin{aligned}H=&\sum_{j=1}^{N-1}\frac{1}{2}(\sigma_j^x\sigma_{j+1}^x+\sigma_j^y\sigma_{j+1}^y+\sigma_j^z\sigma_{j+1}^z)+\theta(\sigma_j^+-\sigma_{j+1}^+)+\theta^2\sigma_j^+\sigma_{j+1}^+ +\\ &+\frac{\eta}{2}(-\sigma_1^z+\psi_-\sigma_1^++\phi_-\sigma_1^-)+\\ &+\frac{\eta}{2\xi_+}\big((1-\theta\psi_+)\sigma_N^z-\big(\theta(2-\theta\phi_+)+\psi_+\big)\sigma_N^+-\phi_+\sigma_N^-\big).\end{aligned} \tag{3.6}$$

This Hamiltonian has a general set of open boundary parameters compatible with the integrability of the XXX_θ model in the bulk.

Because of similarity transformations (2.39) and (2.44) of the main QISM objects, the quantization conditions (the Bethe equations) and the spectrum of the periodic Hamiltonian are unchanged. Hence, after the passage to the limit of the rational XXX model, the spectrum of the deformed model is unchanged. But this is not the case with the corresponding eigenvectors: some of them are transferred into adjoint vectors.

The obtained reflection matrix (2.36) can be diagonalized. Its eigenvalues are

$$\epsilon_{1,2}(\lambda)=\xi-\phi_-\theta\lambda^2\pm\lambda\sqrt{1+\gamma^2},\qquad \gamma^2=\eta\psi_-\phi_-, \tag{3.7}$$

and the matrix of the corresponding eigenvectors U is independent of λ,

$$K(\lambda)U=U\,\mathrm{diag}\big(\epsilon_1(\lambda),\epsilon_2(\lambda)\big), \tag{3.8}$$

$$U=\begin{pmatrix}1 & -1\\ (x+1)/\psi_- & (x-1)/\psi_-\end{pmatrix} \tag{3.9}$$

with $x=\sqrt{1+\gamma^2}$. But the approach used in [27], for example, to obtain the Bethe equations defining the parameters of the spin Hamiltonian eigenvectors does not work in the case of the deformed model, because the constant matrix K^+ is equal not to unity but to the triangular matrix M given by (2.11) and the R-matrix is not $SL(2)$ invariant.

The spectrum of the free-end deformed model with constant reflection matrices $K^-(\lambda)=1$ and $K^+(\lambda)=M$ coincides with the spectrum of the XXX spin chain because the corresponding Hamiltonians are related by the similarity transformation.

4. Conclusion

We have considered the XXX spin chain with nonperiodic boundary conditions deformed by a Jordanian twist. The obtained solutions $K^\pm(\lambda)$ of the reflection equation and its dual with the Jordanian R-matrix turned out to depend explicitly on the deformation parameter θ. We obtained the solution $K^+(\lambda)$ of the dual reflection equation using the generalized crossing symmetry of the Jordanian R-matrix and the relation $K^+(\lambda)=K^-(-\lambda-\eta)M(\theta)$. We thus obtained the transfer matrix of the model, the generating function of integrals of motion. We wrote the Hamiltonian of the open spin chain with the general boundary terms at the first and the last chain sites explicitly. The boundary algebra and possible symmetries of the model can be studied using these reflection matrices.

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