Diameter Minimal Trees

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Abstract

Using the method of seeds and branch duplication, it is shown that for every tree of diameter < 7, there is a Hermitian matrix with as few as the diameter many distinct eigenvalues (a known lower bound). For diameter 7, some trees require 8 distinct eigenvalues, but no more; the seeds for which 7 and 8 are the worst case are classified. For trees of diameter d, it is shown, in general, that the minimum number of distinct eigenvalues is bounded by a function of d. Many trees of high diameter permit as few of distinct eigenvalues as the diameter and a conjecture is made that all linear trees are of this type. Several other specific, related observations are made.

Key Words and Phrases: Branch Duplication, Diameter, Distinct Eigenvalues, Hermitian Matrix, Multiplicities, Tree

AMS/MOS Subject Classification: 15A18, O5CO5, O5C50

1 Introduction

Let T be a tree on n vertices and let $\mathcal{S}(T)$ be the set of all n-by-n Hermitian matrices, the graph of whose off-diagonal entries is T. No restriction, other

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than reality is placed upon the diagonal entries of $A \in \mathcal{S}(T)$. There has been considerable study of the possible lists of multiplicities of the eigenvalues occurring among matrices in $\mathcal{S}(T)$, and there is much structure, based upon the general theory found in [J L-D S 1]. This and [J L-D 1] are good general references for background. The *path cover number* of a tree T, $P(T)$, is the smallest number of vertex disjoint, induced paths of the tree that cover all the vertices, and this number is the *maximum multiplicity* of an eigenvalue occurring in any matrix in $\mathcal{S}(T)$. As mentioned, the possible lists of multiplicities occurring among matrices in $\mathcal{S}(T)$ has been the focus, but there are two sorts of lists: *unordered* (simple partitions of n, the number of vertices); and *ordered* (the same partitions in which the parts are ordered with respect to the numerical order of the underlying eigenvalues). For trees, each ordered list begins and ends with 1. A key fact is that if a list contains a multiplicity bigger than 1 (and sometimes a 1) then the tree must have at least one vertex whose removal gives a principal submatrix in which the multiplicity is one higher. Such vertices are called *Parter vertices* and there may be several of them. To denote a principal submatrix of $A \in \mathcal{S}(T)$ corresponding to a subgraph T' of T, we use the standard notation $A[T']$.

In [J L-D 2], it was shown that the fewest distinct eigenvalues, $c(T)$, that can occur among matrices in $\mathcal{S}(T)$ is, at least, $d(T)$, the diameter of T measured as the number of vertices in a diameter. This generalized the classical fact that an n-by-n irreducible tridiagonal Hermitian matrix (the case of a path) always exhibits n distinct eigenvalues. It raised the natural question of whether $c(T) = d(T)$ for any tree T, i.e., whether as few as $d(T)$ distinct eigenvalues could be realized for some matrix in $\mathcal{S}(T)$, for any T.

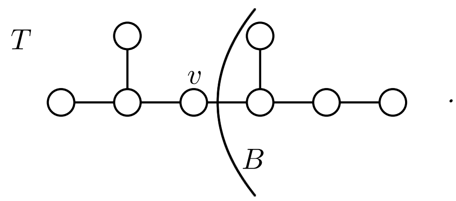
For trees with $d(T) < 6$, this was shown to be the case in [J S], using the technique of “branch duplication” developed by the authors. But, in [B F] an example of a tree with diameter 7 that required 8 distinct eigenvalues was given. This left the question of diameter 6 unresolved.

We call a tree T *di-minimal* (for “diameter minimal”) when $c(T) = d(T)$. The preceding remarks raise the questions (1) how $c(T)$ may be determined and (2) which are the di-minimal trees? We also define the function $C(d) = \max c(T)$, in which the maximum is taken over all T’s of diameter d. Here, we show that $C(d)$ is well-defined and that all trees of diameter 6 are di-minimal. This means that $C(d) = d$ for $d < 7$. We also show that $C(7) = 8$, but that a large fraction of trees of diameter 7 are di-minimal. The first example of a non-di-minimal tree [B F] turns out to be minimal among diameter 7 trees and among all trees, and we note that trees with this one as an induced subgraph can be di-minimal. We also give some infinite families of trees that are di-minimal and further conjectures.

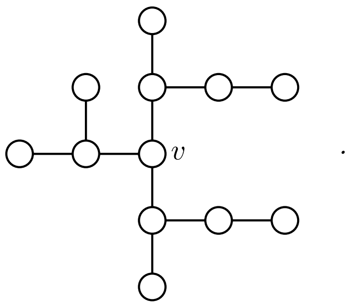
2 Combinatorial Branch Duplication

To accomplish the above, we describe the process of branch duplication and offer a combinatorial version of it. Let T be a tree and B a branch of T at a vertex v. ***C**ombinatorial **b**ranch **d**uplication* (CBD) of B at v results in a new tree $T' = T(B, v)$ in which another copy of B is appended to T at v.

Example 1 *Let T, B and v be as shown*



Then $T' = T(B, v)$ is

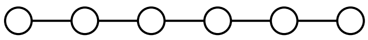


The result of a sequence of CBD's, starting with T (at possibly different or new v's and duplicating possibly different branches) will be called an *unfolding* of T. We will be interested in unfoldings that do not increase the diameter, so that typically v and B will lie in the same "half" of the tree.

By a *seed* of diameter d, we mean a tree of diameter d that is not an unfolding of any smaller tree of diameter d. There are finitely many seeds of diameter d, and any tree of diameter d (that is not a seed) is an unfolding of a unique seed of diameter d.

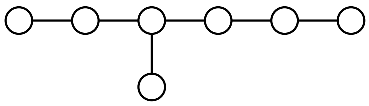
Example 2 *The seeds of diameter* 6 *are*

T_1



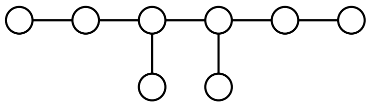
,

T_2



and

T_3



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We call all the diameter d unfoldings of a diameter d seed the *family* of that seed. Of course, the families of the diameter d seeds partition the diameter d trees, but each family is, itself, infinite. We index families by the largest value of $c(T)$ in the family. As we shall see, some families (of a given diameter) are di-minimal, while others may not be.

Interestingly at diameter 7, there begins an explosion in the number of seeds (12 for diameter 7, see Appendix 1). Of course the path of d vertices is always a seed of diameter d and every seed of diameter d has this path as its diameter.

The process of branch duplication endows seeds with some special eigenstructure, so that when certain branches are duplicated (using an algebraic observation), the number of distinct eigenvalues will not increase. The technique will be illustrated in the next section, in which we treat trees of diameter 6. Although this powerful technique was introduced in [J S], curiously it has not been exploited by other authors.

3 Algebraic Branch Duplication and Trees of Diameter 6

We recall here the method presented in [J S] that, from a tree T, a combinatorial branch duplication of a branch T_j at a vertex v results in a new tree $T' = T(T_j, v)$ and, additionally, given $A \in \mathcal{S}(T)$ a matrix $A' \in \mathcal{S}(T')$ is constructed in a way that the eigenvalues of A' are all those of A, together with those corresponding to the duplicated branch, including multiplicities.

Let T be a tree and v be a vertex of degree k with branches $T_1, \ldots, T_k$, and corresponding neighbors $u_1, \ldots, u_k$, and let T' be a combinatorial branch duplication of T_j at v, i.e., $T' = T(T_j, v)$. We denote by u_{k+1} (resp. T'_{k+1}) the new neighbor of v (resp. the new branch at v) in T'.

Let $A = (a_{ij})$ be a matrix in $\mathcal{S}(T)$. We say that a matrix $A' = (a'_{ij})$ in $\mathcal{S}(T')$ is obtained from A by ***a****lgebraic* ***b****ranch* ***d****uplication* (ABD) of summand (branch) $A[T_j]$ at v if A' satisfies the following requirements:

- $A'[T'_i] = A[T_i], i = 1, \ldots, k$, and $A'[T'_{k+1}] = A[T_j]$;
- $a'_{vv} = a_{vv}$;
- $a'_{vu_i} = a_{vu_i},\ i \in \{1, \ldots, k\} \setminus \{j\}$;
- $a'_{vu_j}, a'_{vu_{k+1}} \in \mathbb{C} \setminus \{0\}$ and $\left|a'_{vu_j}\right|^2 + \left|a'_{vu_{k+1}}\right|^2 = |a_{vu_j}|^2$.

An important property [J S, Theorem 1] of matrix A' is that the characteristic polynomial of A', $p_{A'}(t)$, is

$$p_{A'}(t) = p_A(t)p_{A[T_j]}(t)$$

and, therefore, the eigenvalues of A' are all those of A, together with those corresponding to the duplicated summand (branch) $A[T_j]$, including multiplicities.

Here we use combinatorial branch duplication and the associated algebraic branch duplication to show that for any tree with diameter 6 we have $c(T) = d(T)$, i.e., any tree with diameter 6 is di-minimal.

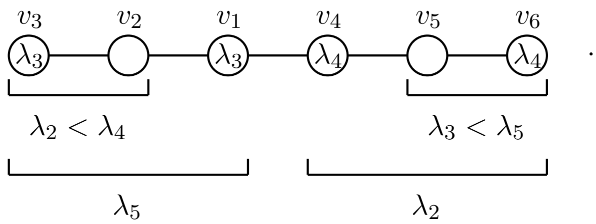
Theorem 3 *Any tree T such that $d(T) < 7$ is di-minimal.*

Proof. In [J S] the result was proved for diameter less than 6, so that we consider here the case in which T is a tree with diameter 6. For this purpose we use the three seeds T_1, T_2 and T_3 in Example 2. Given a tree T with diameter 6 we perform an unfolding of one of the three seeds of diameter 6, that does not increase the diameter, and whose result is T.

For each seed T_i for the combinatorial branch duplication we consider a matrix seed $A_i \in \mathcal{S}(T_i)$ for the algebraic branch duplication with exactly 6 distinct eigenvalues and with an eigenstucture that prevents the number of distinct eigenvalues from changing in any step of the process. Moreover, the distinct eigenvalues in each step of the process remain unchanged. Often, instead of presenting the matrix seed $A_i \in \mathcal{S}(T_i)$ we present T_i indicating the relevant assignments of the eigenvalues in A_i, i.e., what are the eigenvalues of A_i and which are the eigenvalues of the relevant summands (branches) of A_i that may be duplicated and then maintaining the same 6 original distinct eigenvalues.

We start with seed T_1 on vertices $v_1, \dots, v_6$. Consider the following assignment of eigenvalues $\lambda_2 < \lambda_3 < \lambda_4 < \lambda_5$:

T_1



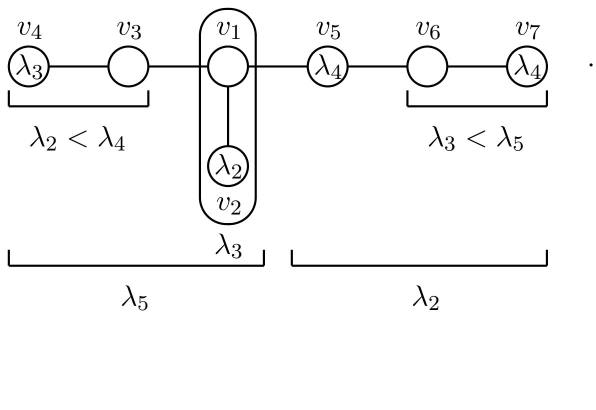
There is a matrix $A_1 \in \mathcal{S}(T_1)$ satisfying these conditions. This assignment imposes $\lambda_2, \lambda_3, \lambda_4$ and λ_5 as eigenvalues of A_1 and A_1 has eigenvalues $\lambda_1 < \cdots < \lambda_6$ for some real numbers λ_1 and λ_6. For example, the matrix

$$A_1 = \begin{bmatrix} 3 & \sqrt{3} & 0 & \sqrt{6} & & \\ \sqrt{3} & 3 & 1 & & & \\ 0 & 1 & 3 & & & \\ \sqrt{6} & & & 4 & \sqrt{3} & 0 \\ & & & \sqrt{3} & 4 & 1 \\ & & & 0 & 1 & 4 \end{bmatrix} \in \mathcal{S}(T_1)$$

has eigenvalues $\lambda_1 = 0$, $\lambda_2 = 2$, $\lambda_3 = 3$, $\lambda_4 = 4$, $\lambda_5 = 5$, $\lambda_6 = 7$ and satisfies the above assignment for T_1.

For seed T_2 on vertices $v_1, \dots, v_7$ we consider the following assignment of eigenvalues $\lambda_2 < \lambda_3 < \lambda_4 < \lambda_5$:

T_2

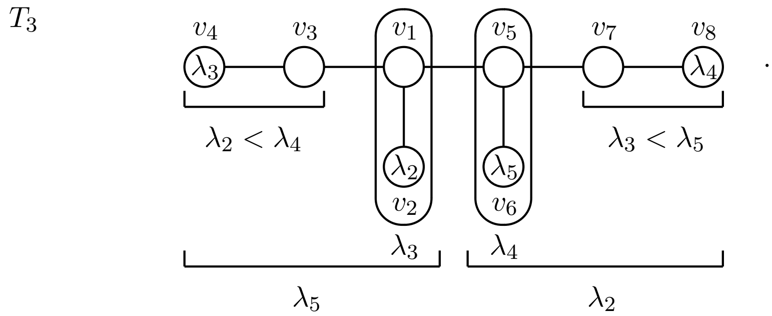


There is a matrix $A_2 \in \mathcal{S}(T_2)$ satisfying these conditions. This assignment produces eigenvalue λ_2 (with multiplicity 2) and eigenvalues $\lambda_3, \lambda_4, \lambda_5$. Some λ_1 and λ_6 will occur as eigenvalues of A_2 such that $\lambda_1 < \cdots < \lambda_6$. For example, the matrix

$$A_2 = \begin{bmatrix} 2 & 1 & 2 & 0 & 4 & & \\ 1 & 2 & 0 & 0 & & & \\ 2 & 0 & 3 & 1 & & & \\ 0 & 0 & 1 & 3 & & & \\ 4 & & & & 4 & \sqrt{3} & 0 \\ & & & & \sqrt{3} & 4 & 1 \\ & & & & 0 & 1 & 4 \end{bmatrix} \in \mathcal{S}(T_2)$$

has eigenvalues $\lambda_1 = -2$, $\lambda_2 = 2$, $\lambda_3 = 3$, $\lambda_4 = 4$, $\lambda_5 = 5$, $\lambda_6 = 8$ (λ_2 has multiplicity 2) and satisfies the above assignment for T_2.

Finally, for seed T_3 on vertices $v_1, \ldots, v_8$ we consider the following assignment of eigenvalues $\lambda_2 < \lambda_3 < \lambda_4 < \lambda_5$:



Again, there is a matrix $A_3 \in \mathcal{S}(T_3)$ satisfying these conditions. This assignment produces eigenvalues λ_2 and λ_5 (both with multiplicity 2) and eigenvalues λ_3, λ_4. Some λ_1 and λ_6 will occur as eigenvalues of A_3 such that $\lambda_1 < \cdots < \lambda_6$. For example, the matrix

$$A_3 = \begin{bmatrix} 2 & 1 & 2 & 0 & \sqrt{8} & & & \\ 1 & 2 & 0 & 0 & & & & \\ 2 & 0 & 3 & 1 & & & & \\ 0 & 0 & 1 & 3 & & & & \\ \sqrt{8} & & & & 5 & 1 & 2 & 0 \\ & & & & 1 & 5 & 0 & 0 \\ & & & & 2 & 0 & 4 & 1 \\ & & & & 0 & 0 & 1 & 4 \end{bmatrix} \in \mathcal{S}(T_3)$$

has eigenvalues $\lambda_1 = -1$, $\lambda_2 = 2$, $\lambda_3 = 3$, $\lambda_4 = 4$, $\lambda_5 = 5$, $\lambda_6 = 8$ (λ_2 and λ_5 have both multiplicity 2) and satisfies the above assignment for T_3.

□

4 Trees of Diameter 7 and $C(7) = 8$

For trees of diameter 7, there are 12 seeds and, thus, 12 families. These seeds are displayed in Appendix 1 along with feasible eigenvalue assignments that support branch duplication. Interestingly 9 of the families consist entirely of di-minimal trees ("di-minimal families") as the given eigenvalue assignments show, via branch duplication, that any unfolding can have as few as 7 distinct eigenvalues. This is not the case for the other 3 seeds (1, 2, and 5), and in Appendix 2 are displayed an unfolding in each of these families for which $c(T)$ is demonstrably 8. The first of these is the example noted in[B F]. However, for these 3 families, the eigenvalue assignments in Appendix 1 show that every unfolding has as few as 8 distinct eigenvalues. This means that we have

Theorem 4 *If T is a tree of diameter 7, then $7 \le c(T) \le 8$, with equality occurring in the right hand inequality for each example in Appendix 2. Thus, $C(7) = 8$.*

We give verification that $c(T) = 8$ for the T in Appendix 2 from family 2 (tree 2). The others are similar. The path cover number [J L-D 1] is 5, so that the maximum multiplicity is 5. For 5 to occur, each of the 3 vertices v_2, v_3, v_4 must be Parter [J L-D S 1, J L-D S 3], with the multiplicity 5 eigenvalue occurring in each of the 8 components resulting from removal of the 3 Parter vertices. In this event the next highest multiplicity that can occur is 3, with the same 3 Parter vertices v_2, v_3, v_4 and, then, at most 3 multiplicity 2 eigenvalues with the vertex v_1 being Parter. This results in a multiplicity list $5, 3, 2, 2, 2, 1, 1, 1$ and 8 distinct eigenvalues. If, instead, the highest multiplicity is 4, the second could be 4 and the list $4, 4, 2, 2, 2, 2, 1, 1, 1$, with 8 distinct, would result. If 3 were the highest multiplicity, then at most 3 multiplicity 3 eigenvalues could occur, again insuring at least 8 distinct, while a maximum multiplicity of 2 would give at least 10 distinct.

Let the *disparity* for a given diameter d be $C(d) - d$. So, the disparity for $d < 7$ is 0 and for $d = 7$ is 1. The 16-vertex example from family 1 (tree 1 in Appendix 2) is interesting in several respects. First, it is minimal in a variety of ways. Not only does it have the smallest diameter for which disparity is positive: $C(d) = d$ for $d < 7$, but it has the fewest vertices among diameter 7 examples that realize a positive disparity. By checking higher diameter examples, we have also seen that this example has the fewest vertices of any case in which a positive disparity is attained. However it is **not** the case that if this example sits as a subgraph in a larger tree, there is necessarily a positive disparity. Consider the 17-vertex tree in which a pendent vertex is appended at the vertex v_1 of the 16-vertex example (tree 1 in Appendix 2). Then, T is in family 3, and $c(T) = 7 = d(T)$. This is also because an additional multiplicity 4 eigenvalue **and** an additional multiplicity 3 eigenvalue can occur. (Applying branch duplication to seed 3 in Appendix 1 in order to obtain the 17-vertex tree in discussion, we get the list of ordered multiplicities $(1, 2, 4, 3, 4, 2, 1)$.) This shows not only that the minimum number of distinct eigenvalues can go down with the addition of a vertex, but also that at least two multiplicities can go up, relative to a list when a vertex is added, contrary to a natural conjecture.

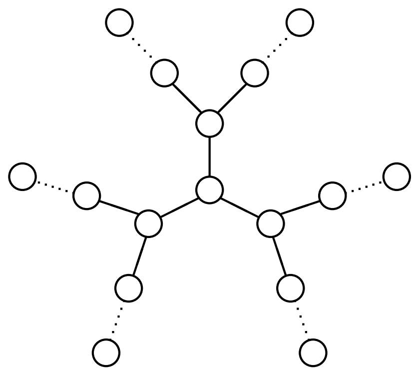
Finally, we note simply that the diameter 7 16-vertex tree with the first realization of positive disparity exists because there is no feasible assignment for the 7-path that allows unlimited branch duplication while only increasing existing multiplicities. Even though only two of three possible branch duplications are used, a forced increase in the number of distinct eigenvalues occurs when the 16 vertices are reached. Further unfoldings of the same type give trees for which $d(T) = 7$ and $c(T) = 8$.

5 Increasing Disparity

We note that disparity, as a function of the diameter d, is unbounded and that it grows at least at the eventual rate of $\frac{d}{2}$. We do not know the actual rate of growth, but it is difficult to construct examples that show a higher rate of growth.

Let T_k denote the tree on $6k+4$ vertices resulting from appending 2 paths of k vertices at each of the 3 pendent vertices of a star on 4 vertices.

$T_k:$



The diameter of T_k is $2k + 3$. The 10 vertex tree of Example 6 is T_1 and tree number 1 of Appendix 2 is T_2. The maximum multiplicity in $\mathcal{S}(T_k)$ is 4 because $P(T_k) = 4$, and the 3 peripheral HDV's of T_k must be Parter for any eigenvalue of multiplicity 4 or 3. There can be at most one of multiplicity 4 and then at most $k - 1$ of multiplicity 3. The center, degree 3, vertex must then be Parter for any multiplicity 2 eigenvalues if there are the k eigenvalues of multiplicity 3 and 4. Thus, there would be at most $k + 1$ multiplicity 2 eigenvalues. Counting and algebra now yield that T_k realizes disparity $k - 1$. This minimum value occurs for exactly one multiplicity list: $4, 3^{k-1}, 2^{k+1}, 1^{k+1}$. Thus, the disparity grows without bound, but at the rate of about $\frac{d}{2}$ in this case.

For completeness, we note that when $d > 6$ is even, the disparity is at least $\frac{d}{2} - 3$, so that beginning at 7, the lower bounds for disparity that we know are $1, 1, 2, 2, 3, 3, \ldots$. The even case results from analysis of the trees T'_k in which at the pendent vertices of the star on 4 vertices are hung 2 paths of k vertices at one and 2 paths of $k-1$ vertices at each of the other two. This results in the even diameter $2k+2$, and the analysis is similar to (but slightly different from) the analysis of T_k. Now, the shortest list is $4, 3^{k-2}, 2^{k+1}, 1^k$ which realizes the disparity of $k - 2$ for $d = 2k + 2$.

6 $C(d)$, Di-Minimal Families and a Conjecture

Here, we show that $C(d)$ is well defined. This is a nice theoretical application of both branch duplication and the notion of a seed.

Despite the fact that the disparity can grow without bound, it can only grow with d. Nonetheless, there are large di-minimal families of trees within which the diameter is unbounded. These and our earlier analysis suggest a conjecture that we make about a very large class of trees.

Theorem 5 *For each positive integer d, $\max_{T:d(T)=d} c(T)$ exists, so that*

$$C(d) = \max_{T:d(T)=d} c(T)$$

is a well-defined function of d.

Proof. Since there are finitely many seeds for d, it suffices to consider only T's in a single family. A given seed has only finitely many branches whose duplication does not increase the diameter. Once a particular branch duplication has been performed, any subsequent duplications of that branch will only increase the multiplicities of the eigenvalues of that branch in the new tree (matrix) relative to the pre-duplication one, and all other eigenvalues stay the same. Thus, the number of distinct eigenvalues will not change (note that this is independent of the cleverness of the eigen-assignment to the seed).

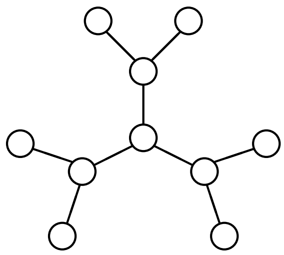
We conclude that our maximum, restricted to a family, will be bounded, at worst, by the number of vertices in an unfolding of the seed in which each possible CBD has been performed once. This is finite and the claim of the theorem follows. □

We note that the estimates in this proof are rather generous. If an eigen-assignment for the seed exists with every eigenvalue of a duplicatable branch occurring as an eigenvalue of the matrix, then the max is no more than the number of vertices in the seed and may well be less if multiple eigenvalues occur. If, in addition, enough multiple eigenvalues occur that the seed is di-minimal, then the family will be di-minimal, as occurs for $d < 7$ and for 9 of 12 diameter 7 seeds. But, even attaining the number of vertices in the seed does not always happen. For example, it never happens for a path of length > 6. Recall that the 16-vertex minimal tree example of disparity 1 is an unfolding of the 7-path. Any further information on the growth of $C(d)$ would be welcome.

There are many infinite collections of types of trees with unbounded diameter that are di-minimal, for example generalized stars and double generalized stars [J L-D S 2].

A vertex in a tree is called *high degree*, HDV for short, if its degree is at least 3. Following [J L W], we call a tree *linear* if all of its HDV's lie on a single path of the tree.

Example 6 *The smallest (fewest vertices) nonlinear tree is*



and each example in Appendix 2 is a nonlinear tree.

In [J L W] it was shown that many eigenvalue statements, not true for general trees, are true for linear trees. Di-minimality was also informally discussed by these authors. Of course generalized stars and double generalized stars (as well as any tree with at most 3 HDV's) are linear, and the only known non-di-minimal trees are nonlinear. We offer the following

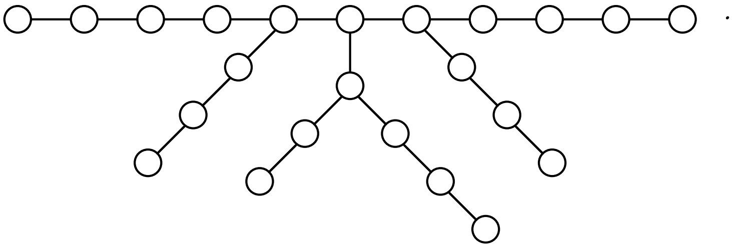
Conjecture 7 *All linear trees are di-minimal.*

7 Additional Remarks

In [B F], it was also noted that for a certain ordered multiplicity list occurring for the 16-vertex tree in Appendix 2, not all spectra corresponding to this list can occur. Thus, the ***i***nverse ***e***igenvalue ***p***roblem (IEP) is not equivalent to the problem of ordered multiplicity lists. The reason is that because of the tight multiplicity list, there are algebraic relations among the eigenvalues due to trace conditions. However, unlike the di-minimality issue, this is not the smallest such example. The first nonlinear tree, the 10-vertex tree displayed in Example 6, exhibits the same behavior, and this is the smallest example for which the IEP and ordered multiplicity problems differ. Indeed, [J L W] gives strong evidence that the two problems are the same for all linear trees.

Even for the three diameter 7 seeds whose families contain trees for which $c(T) = 8$, the seed itself is di-minimal in each case, and many trees in the family are di-minimal as well. However, it can happen that a seed not be di-minimal.

Example 8 *Consider the tree T*



Then, T is a diameter 11 seed. Using reasoning based on Parter vertices, as earlier for the example from diameter 7 family 2 in Appendix 2, the shortest multiplicity list T could have is $4, 3, 3, 2, 2, 2, 2, 1, 1, 1, 1, 1$ and $c(T) = 12$.

It would be of interest to know more about the behavior of the function $C(d)$. It grows faster than d, but still, we suspect, somewhat slowly. Though we do not have strong evidence, it seems worthwhile to focus attention on this with the following conjecture.

Conjecture 9 $C(d) \le d + \max\{0, d - 6\}$.

There are 36 seeds for diameter 8, and many of the families are di-minimal, but it would be quite a task to determine $C(8)$. Our guess is that it is 9. Our best examples, for $d = 6, 7, 8, 9, \ldots$, give disparities $0, 1, 1, 2, 2, 3, 3, \ldots$. This would suggest the stronger

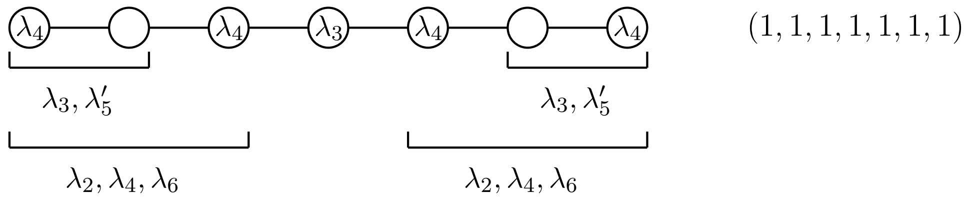
Conjecture 10 $C(d) \le d + \max\left\{0, \left\lceil \frac{d}{2} - 3 \right\rceil\right\}$.

Appendix 1: Diameter 7 Seeds and Classification of their Families using Assignments

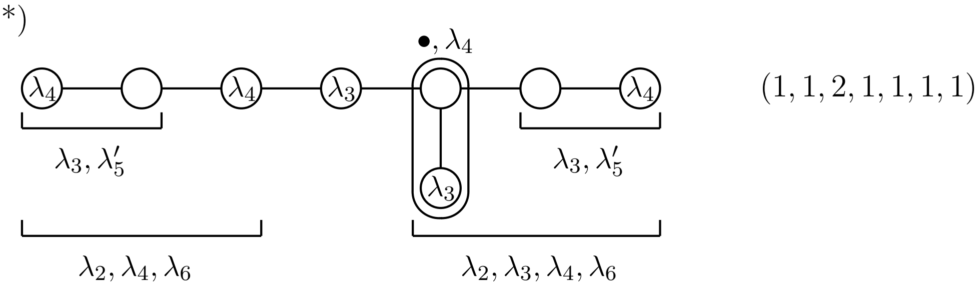
In case of each of the 12 seeds, the graphic depicts how 7 distinct, strictly ordered eigenvalues are assigned to the duplicatable branches or subtrees. When an eigenvalue is assigned to a vertex, it is shown in the vertex. In cases **1**, **2** and **5**, whose families are not di-minimal, one additional, distinct eigenvalue, denoted λ_5' is used. All assignments are realizable. In addition, the ordered multiplicity list is displayed to the right of each seed. Thus, each seed is di-minimal in this case, though three of the families are not.

$$\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7 \in \mathbb{R} \text{ and } \lambda_1 < \lambda_2 < \lambda_3 < \lambda_4 < \lambda_5 < \lambda_6 < \lambda_7.$$

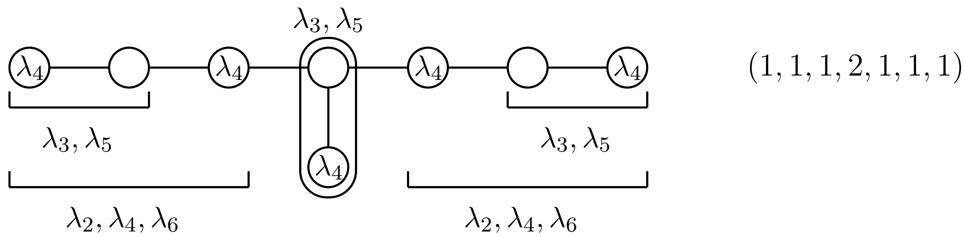
1.(*)


λ_4 λ_4 λ_3 λ_4 λ_4
λ_3, λ_5'
λ_3, λ_5'
$\lambda_2, \lambda_4, \lambda_6$
$\lambda_2, \lambda_4, \lambda_6$
$(1, 1, 1, 1, 1, 1, 1)$

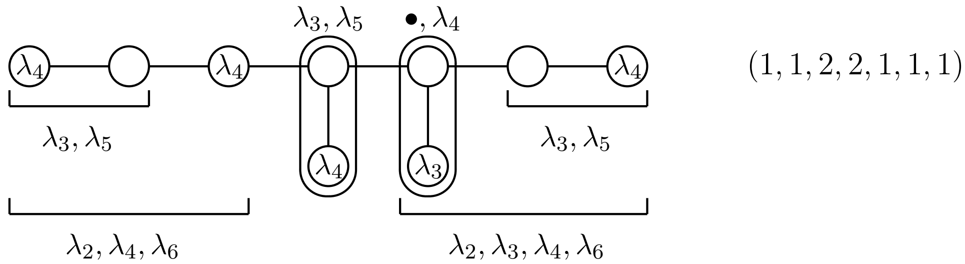
2.(*)


$\bullet, \lambda_4$
λ_4 λ_4 λ_3 λ_3 λ_4
λ_3, λ_5'
λ_3, λ_5'
$\lambda_2, \lambda_4, \lambda_6$
$\lambda_2, \lambda_3, \lambda_4, \lambda_6$
$(1, 1, 2, 1, 1, 1, 1)$

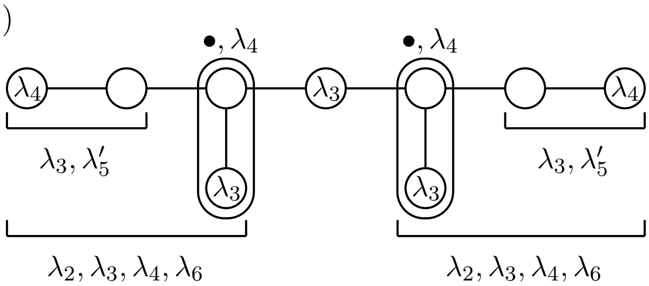
3.


λ_3, λ_5
λ_4 λ_4 λ_4 λ_4 λ_4
λ_3, λ_5
λ_3, λ_5
$\lambda_2, \lambda_4, \lambda_6$
$\lambda_2, \lambda_4, \lambda_6$
$(1, 1, 1, 2, 1, 1, 1)$

4.

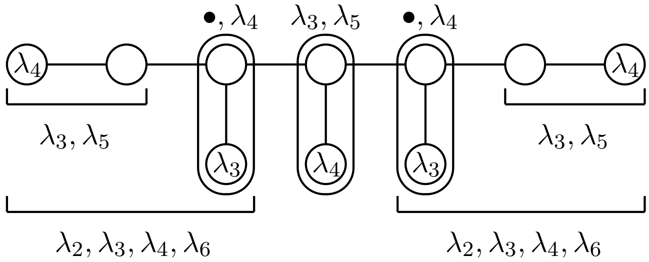

λ_3, λ_5
$\bullet, \lambda_4$
λ_4 λ_4 λ_4 λ_3 λ_4
λ_3, λ_5
λ_3, λ_5
$\lambda_2, \lambda_4, \lambda_6$
$\lambda_2, \lambda_3, \lambda_4, \lambda_6$
$(1, 1, 2, 2, 1, 1, 1)$

5.(*)



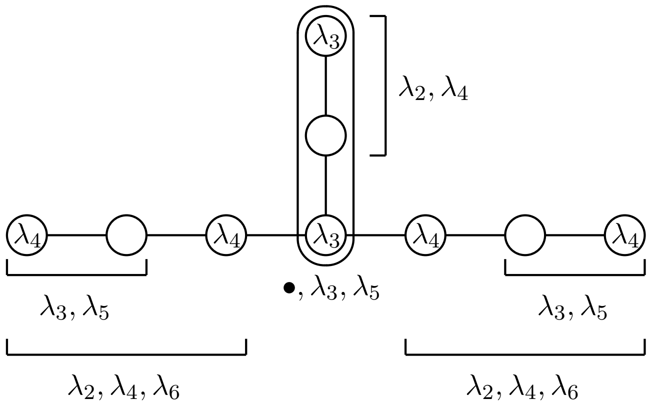
$(1, 1, 3, 1, 1, 1, 1)$

6.



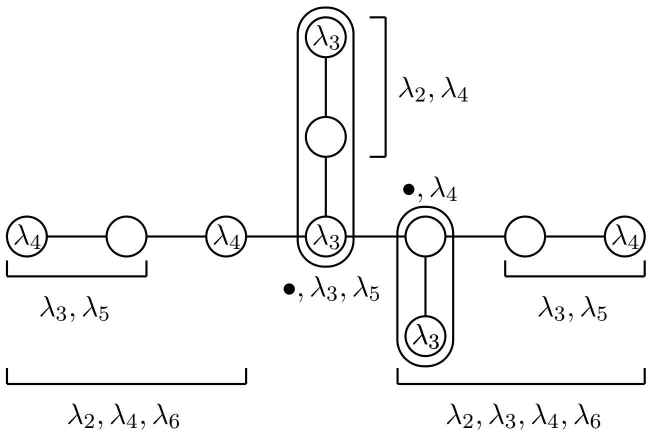
$(1, 1, 2, 2, 2, 1, 1)$

7.



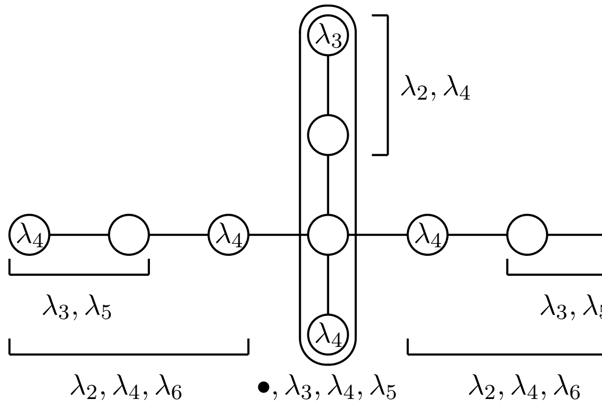
$(1, 2, 1, 2, 1, 1, 1)$

8.



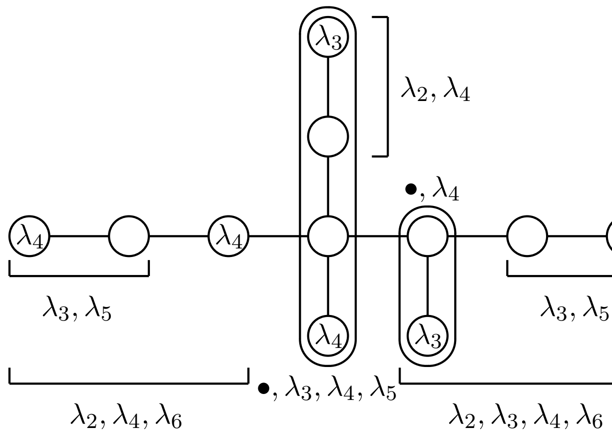
$(1, 2, 2, 2, 1, 1, 1)$

9.



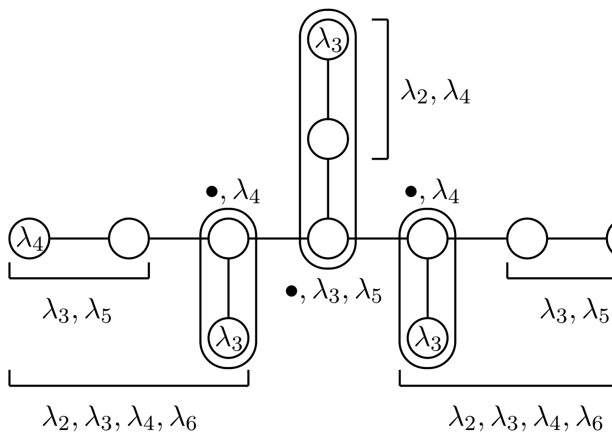
$(1, 2, 1, 3, 1, 1, 1)$

10.



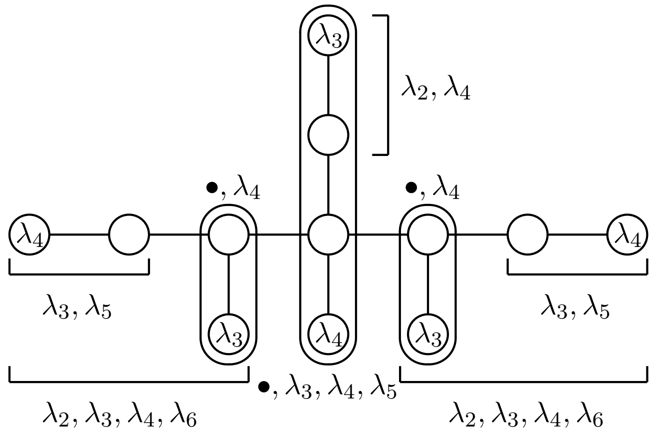
$(1, 2, 2, 3, 1, 1, 1)$

11.



$(1, 2, 3, 2, 1, 1, 1)$

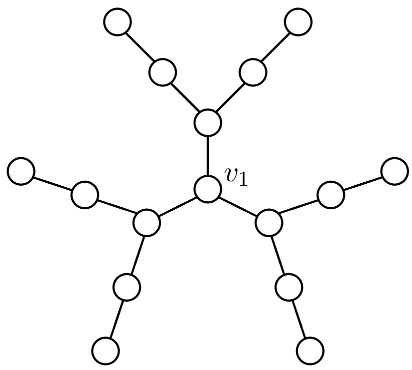
12.



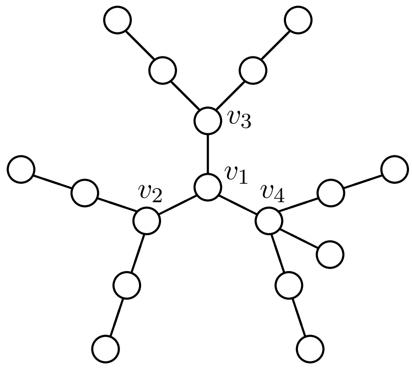
$(1, 2, 3, 3, 1, 1, 1)$

Appendix 2

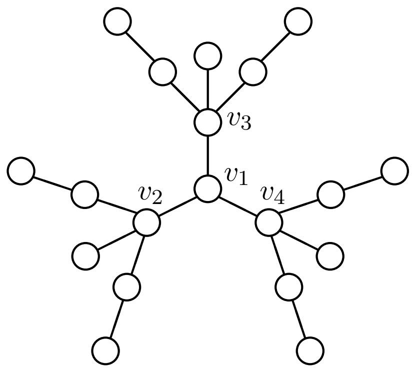
For each of the three diameter 7 families that are not di-minimal, a minimal family-member that is provably not di-minimal is displayed. Note that no example is a linear tree.



1.(*) A tree T of family 1 in Appendix 1 with $c(T) = 8$.



2. A tree T of family 2 in Appendix 1 with $c(T) = 8$.



3. A tree T of family 5 in Appendix 1 with $c(T) = 8$.

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