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Fibrewise complete posets and continuous lattices

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Abstract. It is well known that categorical injective partially ordered sets and topological T0-spaces are complete posets and continuous lattices, respectively. We analyse their fibrewise counterparts, that is (categorical) injective monotone maps between posets and (categorical) injective continuous maps between T0-spaces, presenting characterizations of these maps and establishing parallelisms between them.

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1. **Introduction**

A partially ordered set X is said to be *injective* if any monotone map $g\colon A \to X$ admits an extension to B whenever $A \subseteq B$ (as a substructure); that is, there

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exists $\overline{g}\colon B \to X$ making the diagram

$$\begin{array}{ccc} A & \xrightarrow{\;g\;} & X \\ {\scriptstyle i}\big\downarrow & \nearrow_{\overline{g}} & \\ B & & \end{array}$$

commutative. It is easy to check that this condition is equivalent to completeness of X.

If one considers the *fibrewise notion* of injective poset one says that a monotone map $f\colon X \to Y$ is *injective* if it is injective as an object of the category of monotone maps over Y. This means that, given any other object over Y, that is a monotone map $b\colon B \to Y$, $A \subseteq B$ and a monotone map $g\colon A \to X$ over Y, so that the following diagram

$$\begin{array}{ccc} A & \xrightarrow{\;g\;} & X \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ B & \xrightarrow[\;b\;]{} & Y \end{array}$$

commutes, there exists a monotone map $\overline{g}\colon B \to X$ extending g, *i.e.* $\overline{g} \cdot i = g$, over Y, meaning $f \cdot \overline{g} = b$. That is, the following diagram is commutative

$$\begin{array}{ccc} A & \xrightarrow{\;g\;} & X \\ {\scriptstyle i}\big\downarrow & \nearrow^{\overline{g}} & \big\downarrow{\scriptstyle f} \\ B & \xrightarrow[\;b\;]{} & Y. \end{array}$$

Injective monotone maps in the category **PoSet**, of posets and monotone maps, have interesting properties, that were studied in [10, 1]. Namely they can be seen as topological functors through the identification of posets and monotone maps as (posetal) categories and functors. Indeed, together with embeddings, injective monotone maps form a weak factorization system, which turns out to be the restriction to **PoSet** of an interesting weak factorization system in the category of small categories and functors (see [1] for details).

Every poset can be endowed with its *Alexandroff topology*: for every $x \in X$,

$$\overline{\{x\}} = {\downarrow} x = \{x' \mid x' \leq x\},$$

and, for every $A \subseteq X$,

$$\overline{A} = \bigcup_{x \in A} \overline{\{x\}}.$$

Monotone maps are continuous with respect to these topologies, so that this identification defines a functor

$$\mathbf{PoSet} \xhookrightarrow{\;\mathrm{Alex}\;} \mathbf{Top}_0.$$

The order of the poset X is the specialization order of the space $\mathrm{Alex}(X)$. We recall that for every T0-space X one can define its *specialization order* $\leq$, given, for $x, y \in X$, by:

$$\begin{array}{rcl} x \leq y & \iff & x \in \overline{\{y\}} \\ & \iff & \dot{y} \to x \\ & \iff & \forall U \in \mathcal{O}(x)\ y \in U \end{array}$$

(where $\dot{y}$ is a net constantly equal to y, and $\mathcal{O}(x)$ is the set of open neighbourhoods of x). Since the topology of every Alexandroff T0-space is completely determined by its specialization order and continuity of maps between Alexandroff spaces is equivalently to monotonicity, injectivity for spaces in the category of Alexandroff T0-spaces and continuous maps means again completeness (as posets).

General injective T0-spaces are more interesting: as Scott showed in [9], a T0-space is injective if, and only if, it is a retract of a power of the Sierpinski space, and if, and only if, its specialization order makes it a continuous lattice. Characterizations of injective continuous maps were obtained much later (see [2, 3, 4]). In [2] the authors make use of a space of continuous sections, that we recall in Section 3, the characterization obtained in [3] is based on the fact that injective continuous maps are algebras for the fibrewise filter monad (see also [8]), while the characterizations presented in [4] deal directly with properties of the topologies, very much like the characterizations of injective monotone maps of [1, 10]. All the three approaches show that, as expected, topology – or 'infinite' convergence – creates a severe obstacle that makes the study of these maps much more demanding.

In this paper we recall characterizations of both injective monotone maps and injective continuous maps and compare them, pointing out the role of convergence.

2. Injective morphisms versus convergence

The characterizations of Theorems 1 and 2 are due to J. Adámek and can be found in [10].

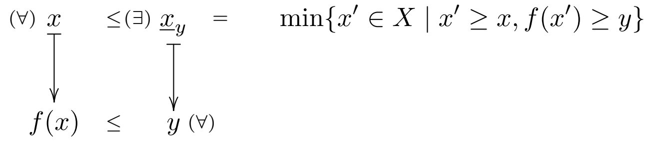
Theorem 1. *A monotone map $f\colon X \to Y$ is injective in* **PoSet** *if, and only if:*

(inj) *for each $y \in Y$, $X_y = f^{-1}(y)$ is injective (*i.e. *a complete poset);*
(fib) *f is a fibration;*
(cofib) *f is a cofibration.*

Here by *fibration* is meant that, for each $x \in X$ and $y \in Y$ with $y \leq f(x)$, there exists $\overline{x}_y = \max\{x' \in X \mid x' \leq x \text{ and } f(x') \leq y\} \in X_y$; this can be depicted by:

$$\begin{array}{ccccc} \max\{x' \in X \mid x' \leq x, f(x') \leq y\} & = (\exists)\ \overline{x}_y & \leq & x\ (\forall) \\ & \downarrow & & \downarrow \\ & (\forall)\ y & \leq & f(x) \end{array}$$

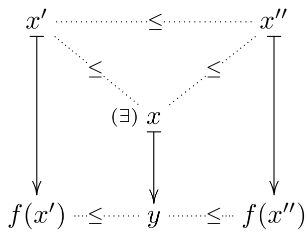
Dually f is a *cofibration* if, for each $x \in X$ and $y \in Y$ with $f(x) \leq y$, there exists $\underline{x}_y = \min\{x' \in X \mid x \leq x' \text{ and } y \leq f(x')\} \in X_y$:



Theorem 2. *A monotone map $f: X \to Y$ is injective in* **PoSet** *if, and only if:*

(inj) *for each $y \in Y$, X_y is injective;*
(exp) *f is convex;*
(ho) *f is homogeneous;*
(coho) *f is co-homogeneous.*

Here a monotone map $f: X \to Y$ is called *convex* if it has the following *interpolation property*: for each $x' \leq x''$ in X and $y \in Y$ with $f(x') \leq y \leq f(x'')$, there exists $x \in X_y$ with $x' \leq x \leq x''$:



We remark that this condition is equivalent to f being *exponentiable* (see, for instance, [10, 7]), reason for the use of (exp) to label it. By *homogeneous* monotone map it is meant that, whenever are given $y \leq y'$ in Y and families $(x_i)_{i \in I}$ and $(x'_i)_{i \in I}$ in X_y and $X_{y'}$ respectively, if, for every $i \in I$, $x_i \leq x'_i$, then $\bigvee_y x_i \leq \bigvee_{y'} x'_i$, where $\bigvee_y x_i$ is the join of (x_i) in the complete poset X_y. The notion of *co-homogeneous* monotone map is defined dually. We point out that in [10, Proposition 3(iii)] co-homogeneity is missing. But it is necessary to add it to (inj), (exp) and (ho) to assure that f is injective. For instance, if we consider the posets $Y = \{0, 1\}$, with $0 \leq 1$, $X = \{0, 1\} \times \{2, 3\}$, with $(0, 2) \leq (0, 3)$, $(1, 2) \leq (1, 3)$ and $(0, 2) \leq (1, 2)$, and the first projection $f: X \to Y$. Then f has complete fibres, it is convex and homogeneous, but it is not a cofibration: $f(0, 3) = 0 \leq 1$ but the set $\{x \in X \mid (0, 3) \leq x \text{ and } 1 \leq f(x)\} = \emptyset$ has no minimum.

From the characterizations of injective continuous maps in T0-spaces obtained in [4] we can derive the next two theorems, which can be compared to the former results for monotone maps.

Theorem 3. *A continuous map $f: X \to Y$ is injective in* $\mathbf{Top}_0$ *if, and only if:*

(inj) *for each $y \in Y$, X_y is injective (*i.e. *a continuous lattice);*

(fib$^+$) *for each net* $(x_\lambda)_{\lambda\in\Lambda}$ *in* X *and* $y \in \lim(f(x_\lambda))$ *in* Y*, there exists* $\overline{x}_y = \max\{x' \in X \mid x_\lambda \to x' \text{ and } f(x') \le y\} \in X_y$*;*

(cofib$^+$) *for each* $x \in X$ *and* $y \in Y$ *with* $f(x) \le y$*, there exists* $\underline{x}_y \in X_y$ *such that* $x \le \underline{x}_y$ *and, for each net* (x_λ) *in* $X_{y'}$ *with* $y \le y'$*, if* $x_\lambda \to x$ *then* $x_\lambda \to \underline{x}_y$*;*

(nho) *for each* $x \in X$ *and each* $U \in \mathcal{O}(x)$*, there exist a continuous section* s *of* f *and* $W \in \mathcal{O}(f(x))$ *such that* $X_W \cap \{x' \in X \mid x' \ge s(f(x'))\}$ *is a neighbourhood of* x *contained in* U*.*

This is the closest counterpart to Theorem 1 we could obtain. Conditions (fib$^+$) and (cofib$^+$) are (stronger) versions of (fib) and (cofib) that use non-principal convergence. We need the extra condition (nho), involving continuity, but this is not surprising when one passes from finite to infinite structures, or from order to topology.

Theorem 4. *A continuous map* $f\colon X \to Y$ *is injective in* $\mathbf{Top}_0$ *if, and only if:*

(inj) *for each* $y \in Y$*,* X_y *is injective;*

(exp) f *is exponentiable;*

(ho$^+$) *for each set* S *of continuous sections, the section* $\overline{s}\colon Y \to X$*, defined by* $\overline{s}(y) = \bigvee_y \{s(y) \mid s \in S\}$*, is continuous;*

(coho$^-$) *for each* $x \in X$ *and* $y \in Y$ *with* $f(x) \le y$*, there exists* $x_y = \min\{x' \in X \mid x' \ge x,\ f(x') = y\}$*.*

3. Injective morphisms versus the space of continuous sections

To exploit further the role of continuous sections of an injective continuous map we start by defining the topological space of continuous sections of f:

$$\mathrm{Sec}(f) = \{s\colon Y \to X \text{ continuous} \mid f \cdot s = 1_Y\},$$

endowed with the subspace topology induced by the inclusion

$$\begin{array}{rcl} \mathrm{Sec}(f) & \longrightarrow & \prod_{y\in Y} X_y \\ s & \longmapsto & (s(y))_{y\in Y}. \end{array}$$

Using results of [2], in [4] it is shown that:

Proposition 5. *If* $f\colon X \to Y$ *is injective in* $\mathbf{Top}_0$*, then:*

(inj*) $\mathrm{Sec}(f)$ *is injective.*

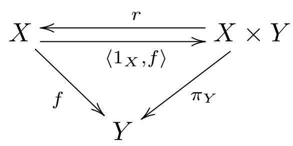
This leads to a new characterization of injectivity, also established in [4].

Theorem 6. *A continuous map* $f\colon X \to Y$ *is injective in* $\mathbf{Top}_0$ *if, and only if:*

(inj) *for each* $y \in Y$*,* X_y *is injective;*

(ho$^+$) *for each set* S *of continuous sections, the section* $\overline{s}\colon Y \to X$*, defined by* $\overline{s}(y) = \bigvee_y \{s(y) \mid s \in S\}$*, is continuous;*

(sec) *the continuous map* $\langle 1_X, f\rangle\colon X \to X\times Y$ *is a section over* Y*, that is there is a continuous map* $r\colon X\times Y \to X$ *with* $r\cdot\langle 1_X, f\rangle = 1_X$ *and making the diagram*



commute.

As already observed in [2, Theorem 2.1], this characterization interpreted in **PoSet** gives:

Theorem 7. *A monotone map* $f\colon X \to Y$ *is injective in* **PoSet** *if, and only if:*

(inj) *for each* $y \in Y$*,* X_y *is injective;*
(sec) *the monotone map* $\langle 1_X, f\rangle\colon X \to X\times Y$ *is a section over* Y*.*

4. Injective morphisms versus topological functors

Finally we mention a characterization of injective monotone maps as topological functors, that was obtained in [1, Proposition 2.5] in the realm of weak factorization systems.

Theorem 8. *The following conditions are equivalent, for a monotone map* $f\colon X\to Y$ *between partially ordered sets:*

(i) f*, considered as a functor between the posetal categories* X *and* Y*, is topological;*
(ii)(adj) f *has a left and a right adjoint;*
(exp) f *is convex;*
(inj) *for each* $y\in Y$*,* X_y *is a complete poset;*
(emb) *for each* $y \in Y$*, the embedding* $X_y \hookrightarrow X$ *preserves non-empty joins and meets.*
(iii) f *is an injective morphism in* **PoSet**.

This result suggests some interesting properties of injective continuous maps. Indeed, if one extends the specialization order to continuous maps between T0-spaces, so that, for continuous maps $f, g\colon X\to Y$,

$$f \le g \iff \forall x\in X\ (f(x)\le g(x)),$$

then one has the notion of adjunction available. That is, for $f\colon X\to Y$ and $g\colon Y\to X$, f is said to be a *left adjoint to* g (and g a *right adjoint to* f) if

$$\forall x\in X\ \forall y\in Y\ (f(x)\le y \iff x\le g(y)).$$

The following result is shown in [4], and in fact follows easily from Theorems 4 and 6.

Proposition 9. *If the continuous map $f\colon X \to Y$ is injective in* $\mathbf{Top}_0$*, then:*
(adj) *f has a left and a right adjoint.*

As a curiosity we add the following result.

Proposition 10. *If the continuous map $f\colon X \to Y$ in* $\mathbf{Top}_0$ *is a surjection and has injective fibres, then:*
(1) *f is closed $\iff$ f has a left adjoint $\iff$ the minimum section of f is continuous;*
(2) *f is open $\iff$ f has a right adjoint $\iff$ the maximum section of f is continuous.*
In particular, every injective continuous map is both open and closed.

In fact the use of the object of continuous sections in [2] led to several characterizations of injective continuous maps in the category **CtLat** of continuous lattices (see [2, Theorem 4.6]). Here we stress that (adj) is enough in this case to guarantee injectivity.

Theorem 11. *A continuous map $f\colon X \to Y$ between continuous lattices is injective in* **CtLat** *if, and only if, it has both a left and a right adjoint.*

Final Remarks. We do not know whether there is a formulation similar to Theorem 8 that allows for a characterization of injective continuous maps as generalized topological functors. The use of 'levels' of ultrafilter convergence, as used in [5, 7, 6] to characterize special classes of continuous maps may be a way of formulating notions of generalized fibrations and cofibrations so that the intricate conditions of Theorems 3 and 4 are better captured and understood.

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