A VARIATION ON THE TABLEAU SWITCHING AND A PAK-VALLEJO'S CONJECTURE

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Abstract. Pak and Vallejo have defined fundamental symmetry map as any Young tableau bijection for the commutativity of the Littlewood-Richardson coefficients $c^{\lambda}_{\mu,\nu} = c^{\lambda}_{\nu,\mu}$. They have considered four fundamental symmetry maps and conjectured that they are all equivalent (2004). The three first ones are based on standard operations in Young tableau theory and, in this case, the conjecture was proved by Danilov and Koshevoy (2005). The fourth fundamental symmetry, given by the author in (1999;2000) and reformulated by Pak and Vallejo, is defined by nonstandard operations in Young tableau theory and will be shown to be equivalent to the first one defined by the involution property of the Benkart-Sottile-Stroomer tableau switching. The proof of this equivalence provides, in the case the first tableau is Yamanouchi, a variation of the tableau switching algorithm which shows *switching* as an operation that takes two tableaux sharing a common border and moves them trough each other by decomposing the first tableau into a sequence of tableaux whose sequence of partition shapes defines a Gelfand-Tsetlin pattern. This property leads to a *jeu de taquin-chain sliding* on Littlewood-Richardson tableaux.

Résumé: Pak et Vallejo ont défini la transformation de la symétrie fondamentale comme une bijection de tableaux de Young pour la comutativité des coefficients de Littlewood-Richardson $c^{\lambda}_{\mu,\nu} = c^{\lambda}_{\nu,\mu}$. Ils ont considéré quatre bijections fondamentaux et ont conjecturé qu'elles sont équivalentes (2004). Les trois premières sont basées sur des opérations standard de la théorie des tableaux de Young et, dans ce cas, la conjecture a été confirmée par Danilov et Koshevoy (2005). La quatrième symétrie fondamentale, donnée par l'auteur (1999;2000) et reformulée par Pak et Vallejo, est définie par des opérations *nonstandard* dans la théorie des tableaux de Young. Cette bijection sera montrée équivalente à la première définie pour la propriété involutoire du *tableau switching* de Benkart-Sottile-Stroomer. La preuve de cette equivalence, dans le cas le premier tableau est de Yamanouchi, donne une variation du algorithme de *tableau switching* qui montre *switching* comme une opération qui prendre deux tableaux avec une même borde et meut un à travers de l'autre en décomposant le premier dans une séquence de tableaux dont la séquence des partitions des formats definit une diagramme de Gelfand-Tsetlin. Cette propriété conduit à un algorithme du type *jeu de taquin-glissements sur chaînes* pour les tableaux de Littlewood-Richardson.

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1. Introduction

Recently, with different approaches, several bijections exhibiting symmetries of Littlewood-Richardson coefficients have been constructed [PV2, KTW, HK, DK]. Also the relationship between different combinatorial objects has been studied [PV1]. In [KTW, HK] hives and octahedron recurrence are the main tools while in [PV2] the bijections are within Young tableaux. The fundamental symmetry map is defined in [PV2] as any bijection between sets of Littlewood-Richardson tableaux of shape λ/μ with weight ν, and of shape λ/ν with weight μ. Namely in [PV2] four fundamental symmetry maps ρ_1, ρ_2, ρ_2^{-1} and ρ_3 are provided and it is conjectured that they are equivalent in the sense that in all of them the outcome is the same. The first three are based on standard algorithms in Young tableau theory, *jeu de taquin*, Schützenberger involution and tableau switching, while ρ_3 uses nonstandard operations in Young tableau theory which exhibits a Gelfand-Tsetlin pattern. In [DK] it is shown that the Henriques-Kamnitzer commuter coincides with the Pak-Vallejo fundamental symmetries ρ_1, ρ_2, ρ_2^{-1}, and $\rho_2 = \rho_2^{-1}$. However the fundamental symmetry ρ_3 in [PV2] is left out. Fundamental symmetry map ρ_3 mentioned in [PV1] and slightly reformulated in [PV2], has appeared earlier in [AZ1, AZ2]. Here we show that ρ_3 is equivalent to ρ_1 defined by the involution switching tableau property [BSS]. The fundamental symmetry map ρ_3 is a *jeu de taquin-chain sliding algorithm* which rectifies a Littlewood-Richardson tableau of shape λ/μ such that the slides along chains decompose the inner shape μ into a sequence of interlacing partitions defining a Gelfand-Tsetlin pattern of type $[\nu, \mu, \lambda]$ with ν the weight of the Littlewood-Richardson tableau.

The paper is divided into four sections. In the next section we give the basic definitions and terminology for what follows. In the third section, Theorem 3.3 stresses the relationship between the tableau switching and the interlacing property. In the fourth section, Theorem 4.1 and Algorithm 4 show that the bijection ρ_3 is equivalent to the tableau switching on Litlewood-Richardson tableaux by exhibiting it as a *jeu de taquin-chain sliding* algorithm. Along the paper several examples are given.

2. Preliminaries

We think of $\mathbb{Z} \times \mathbb{Z}$ as consisting of boxes or blackdots $\bullet$ and we number the rows and columns according the matrix style. Consider x and x' boxes in $\mathbb{Z} \times \mathbb{Z}$. We say that x is to the north of x' if the row containing x is above or equal the row containing x'; and x is to the west of x' if the column containing x is to the left or equal to the column containing x'. The other compass directions are defined analogously. When x and x' are distinct adjacent boxes they are said *neighbours*. For instance, the neighbour to the north of x is the one directly above x. Often we label boxes or dots with integers (or with letters in a totally ordered alphabet) and, in this case, we identify these objects with the corresponding letters.

A *partition* (or normal shape) $\lambda = (\lambda_1, \ldots, \lambda_n)$ is a finite sequence (or infinite sequence of finite support) of nonnegative integers by weakly decreasing order. The null partition is denoted by 0. We ignore the distinction between two partitions that

differ only at a string of zeros at the end. The diagram of λ consists of λ_1 boxes (or blackdots $\bullet$) in the first row, λ_2 boxes in the second row, etc, justified on the left. (The English notation is adopted.) We look at partitions and diagrams indistinctly. If λ and μ are two partitions with $\lambda_i \geq \mu_i$ for all i, we write $\mu \subseteq \lambda$. The skew-diagram of shape λ/μ is the difference set of λ and μ. Whenever $\mu \subseteq \lambda$ we say λ/μ *extends* μ and the outer border of μ is the inner border of λ/μ. The *tableau* T of shape λ/μ, written $shT = \lambda/\mu$, is a filling (or labeling) of the skew-diagram λ/μ using letters of a totally ordered alphabet such that the entries increase weakly along rows and strictly down columns. The *weight* of a tableau is $\nu = (\nu_1, \ldots, \nu_n)$ where ν_i is the multiplicity of the letter i in the filling of the tableau. A *tableau* of (normal) shape λ is a tableau of shape $\lambda/0$. We say the tableau T extends the tableau S of normal shape if the shape of T extends the shape of S.

A word is a finite sequence of letters over a totally ordered alphabet. Knuth congruence $\equiv$ [K] on words is the congruence generated by the so-called elementary transformations, where x, y, z are letters and u, v are words $uxzxv \equiv uzxxv$, $uzzxv \equiv uzxzv$, $x < z$, $uxzyv \equiv uzxyv$, $x < y < z$, $uyzxv \equiv uyxzv$, $x < y < z$. We define the word of a tableau by row reading the entries from left to right and bottom to top. The *Yamanouchi tableau* of shape λ, denoted $Y(\lambda)$, is the tableau whose shape and weight is λ, that is, the tableau obtained by filling the first row of λ with λ_1 1's, the second with λ_2 2's etc. A Yamanouchi word of weight λ is any word Knuth equivalent with $Y(\lambda)$. A Littlewood-Richardson (LR for short) tableau of type $[\mu, \nu, \lambda]$ is a tableau of shape λ/μ and weight ν whose word is Yamanouchi. We denote by $LR[\mu, \nu, \lambda]$ the set of all LR tableaux of type $[\mu, \nu, \lambda]$. The cardinal of this set is the *Littlewood-Richardson coefficient* $c^{\lambda}_{\mu,\nu}$ [LR, F, LLT, Sa, S].

Definition 2.1. [PV2] The fundamental symmetry is a bijection

$$\rho : LR[\mu, \nu, \lambda] \longrightarrow LR[\nu, \mu, \lambda].$$

In [PV2] the version ρ_1 of the fundamental symmetry is based on the involution property of the tableau switching [BSS]. In the last section we shall present the version of the fundamental symmetry ρ_3 [AZ1, AZ2, PV2] in terms of a *jeu de taquin-chain sliding* and this allows us to conclude that ρ_1 and ρ_3 produce the same outcome.

We recall now the elementary operations of the *jeu de taquin slides*. Let us consider a black dot $\bullet$ with two possible south-east letter neighbours a and b, $\begin{array}{cc} \bullet & \mathbf{b} \\ \mathbf{a} & \end{array}$. A *contracting slide* into the blackdot $\bullet$ is performed according to the following rules: if the blackdot has (a) only one neighbour, swap with that neighbour; (b) two different neighbours, swap with the smaller one; (c) equal neighbours, swap with the one to the south. In the case of two possible north-west neighbours $\begin{array}{cc} & \mathbf{d} \\ \mathbf{c} & \bullet \end{array}$, an *expanding slide* into the black dot $\bullet$ is performed analogously. If the blackdot has (a) only one neighbour, swap with that neighbour; (b) two different neighbours, swap with the bigger one; (c) equal neighbours, swap with the one to the north.

Let λ/μ be a skew-diagram and x an empty box which shares at least the south (north) or east (west) edge with λ/μ. The box x is an inner (outer) corner of T if $\{x\} \cup \lambda/\mu$ is still a skew-diagram. Let T be a tableau and x an inner (outer) corner. A

contracting (expanding) *jeu de taquin* slide of T into the blackdot or empty box x is a sequence of contracting (expanding)slides starting in x through T until the blackdot becomes an outer (inner) corner. The words of the intermediate arrays, produced by the contracting and expanding slides, are Knuth equivalent [F]. Two tableaux are said *Knuth equivalent* if one of them can be transformed by contracting and expanding *jeu de taquin* slides into the another one, equivalently if their words are Knuth equivalent. Occasionally we shall denote by T^{n} the rectification of T. Thus another perspective for Litlewood-Richardson coefficients is that $c^{\lambda}_{\mu,\nu}$ counts the number of LR tableaux of type $[\mu,\nu,\lambda]$ that are Knuth equivalent to $Y(\nu)$ [F, LLT, Sa, S]. This point of view will be explored here.

Gelfand-Tsetlin (GT for short) patterns are related with LR tableaux as follows [GZ].

Definition 2.2. A Gelfand-Tsetlin pattern of size n is a map $G : \{(i,j) : 1 \leq j \leq i \leq n\} \to \mathbb{Z}_{\geq 0}$ such that $G(i,j) \geq G(i-1,j) \geq G(i,j+1)$ for all i and j.

Thus a Gelfand-Tsetlin pattern of size n is a sequence of partitions $\nu^{(s)} = (\nu_1^{(s)}, \ldots, \nu_s^{(s)})$, $s = 1, ..., n$, satisfying the interlacing inequalities

$$\nu_j^{(i)} \geq \nu_j^{(i-1)} \geq \nu_{j+1}^{(i)},\ 1 \leq j < i \leq n. \tag{2.1}$$

We represent a Gelfand-Tsetlin pattern of size n by a triangular array of nonnegative integers $(\nu_j^{(i)})_{1\leq j\leq i\leq n}$,

$$\begin{array}{ccccccccccc}
 & & & & \nu_1^{(1)} & & & & & & \\
 & & & \nu_1^{(2)} & & & \nu_2^{(2)} & & & & \\
 & & \nu_1^{(3)} & & & \nu_2^{(3)} & & & \nu_3^{(3)} & & \\
 & & \cdots & & & \cdots & & & \cdots & & \\
 & \nu_1^{(n-1)} & & & \nu_2^{(n-1)} & & & \nu_3^{(n-1)} & \cdots & & \nu_{n-1}^{(n-1)} & \\
\nu_1^{(n)} & & & \nu_2^{(n)} & & & \nu_3^{(n)} & \cdots & & \nu_{n-1}^{(n)} & & & \nu_n^{(n)}.
\end{array}$$

The *base* of the Gelfand-Tsetlin pattern is the sequence of integers that appears in the bottom row $\nu = (\nu_1^{(n)}, \ldots, \nu_n^{(n)})$, and the *weight* of the GT pattern is the sequence $(\gamma_1, \ldots, \gamma_n)$ defined by the first row followed by the differences of row sums from top to bottom, that is, $\gamma_1 + \cdots + \gamma_i = \nu_1^{(i)} + \cdots + \nu_i^{(i)}$, $1 \leq i \leq n$.

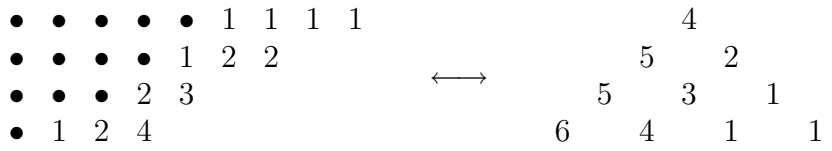
A GT pattern of size n, base ν and weight $\lambda - \mu$ is said of type $[\mu,\nu,\lambda]$ if the sequence of partitions $\nu^{(s)} = (\nu_1^{(s)}, \ldots, \nu_s^{(s)})$, $s = 1, ..., n$, with $\nu^{(n)} = \nu$, satisfy the system of linear inequalities

$$\mu_{i-1} + \sum_{j=1}^{r-1}(\nu_j^{(i-1)} - \nu_j^{(i-2)}) \geq \mu_i + \sum_{j=1}^{r}(\nu_j^{(i)} - \nu_j^{(i-1)}),\ r = 1, \ldots, i-1,\ i = 2, \ldots, n, \tag{2.2}$$

$$\mu_i + \sum_{j=1}^{i}(\nu_j^{(i)} - \nu_j^{(i-1)}) = \lambda_i,\ i = 1, \ldots, n, \tag{2.3}$$

with the convention $\nu_i^{(i-1)} = 0$. Sometimes μ is called a border of the GT pattern. There is a standard bijection between Littlewood-Richardson tableaux of type $[\mu, \nu, \lambda]$ and the GT patterns of type $[\mu, \nu, \lambda]$ [GZ]. This bijection sends an LR tableau T to the GT pattern whose value at (i, j) is the number of $j's$ in the first i rows of T.

Here is an LR tableau of type $[\mu = (5, 4, 3, 1); \nu = (6, 4, 1, 1); \lambda = (9, 7, 5, 4)]$ and the corresponding GT pattern with weight ν, weight $\lambda - \mu = (4, 3, 2, 3)$ and border μ,



We recall now the *switching procedure* and some related terminology [BSS]. A *perforated tableau* T of shape λ/μ is a labeling of some of the boxes satisfying some restrictions: whenever x and x' are letters in T and x is north-west of x', $x \leq x'$; within each column of T the letters are distinct. If S and T are perforated tableaux of some given shape λ and together they completely label λ such that no box is labeled twice, then $S \cup T$ (as union of sets of $\mathbb{Z} \times \mathbb{Z}$) is called a perforated pair of shape λ. In particular, given two tableaux S and T of shapes μ and λ/μ respectively, $S \cup T$ is a perforated pair of shape λ by glueing S and T. For convenience, when considering pairs of perforated tableaux $S \cup T$, the letters in S and T belong to the alphabets $\bar{\mathbf{1}} < \cdots < \bar{\mathbf{n}}$ and $1 < \cdots < n$ respectively. Let S be a perforated tableau and suppose $\bar{\mathbf{s}}$ is a neighbour to the north or to the west of an empty box. The swap of $\bar{\mathbf{s}}$ with that empty box is called an *expanding slide* in S whenever a new perforated tableau is produced. A *contracting slide* in T is defined analogously. Contrary to the slides in tableaux starting in an inner or outer corner, the intermediate words are not in general Knuth equivalent. (Assuming that we read words from bottom to top and from left to right along rows.) Let $S \cup T$ be a perforated pair and assume that $\bar{\mathbf{s}}$ and $\mathbf{t}$ are two adjacent letters $\bar{\mathbf{s}}\ \mathbf{t}$ or $\begin{matrix}\bar{\mathbf{s}}\\ \mathbf{t}\end{matrix}$ from S and T respectively. Swapping $\bar{\mathbf{s}}$ and $\mathbf{t}$ is called a *switch* whenever we have simultaneously a contracting slide in T and an expanding slide in S. The *switching procedure* starts with two tableaux S and T such that T extends S and by switching letters from S with letters from T transforms $S \cup T$ into a pair of tableaux $P \cup Q$ such that Q extends P, S is Knuth equivalent to Q and T is Knuth equivalent to P. We say that $P \cup Q$ is the switching of S and T. The switching transformation is an involution.

3. Tableau switching and shape interlacing property

A subtableau is a tableau obtained by suppressing some rows in a tableau. A GT-pattern of size n and base ν encodes the normal shape sequence of the rectified subtableaux of an LR tableau of weight ν defined by the first i rows, for $1 \leq i \leq n$. Now let $S \cup T$ be a pair of tableaux of shapes respectively μ and λ/μ which switching procedure transforms into a pair $P \cup Q$ where P is a tableau of normal shape ν. The sequence of shapes of the rectified subtableaux defined by the first i rows of T, for $i \geq 1$, is a GT pattern of type $[\mu, \nu, \lambda]$, and the sequence of shapes of the rectified

subtableaux defined by the first i rows of Q, for $i \geq 1$, is a GT pattern of type $[\nu, \mu, \lambda]$. This may easily be seen from Haiman results on dual equivalence [H] (see also [BSS]). We recall that two tableaux of the same shape are dual equivalent if, whenever a particular choice of the order for performing *jeu de taquin*, the shapes of the two tableaux are the same throughout the *jeu de taquin* process (see also [F]).

Theorem 3.1. [H] *Let U and V be tableaux of the same shape. If W is any tableau that extends U (or V), then switching transforms $U \cup W$ into $P \cup Q$ and $V \cup W$ into $P \cup R$ where Q and R are dual equivalent.*

Given a tableau T and $i \geq 1$, we denote by $T^{(i)}$ the subtableau defined by the first i rows of T. In fact if we replace P with $Y(\nu)$ in $P \cup Q$ and apply switching to $Y(\nu) \cup Q$, then we get the pair $S \cup L$ where L is an LR tableau of type $[\mu, \nu, \lambda]$ dual equivalent to T. Since the subtableaux defined by the first i rows of L and T are also dual equivalent, it is now clear that $(\mathrm{L}^{(i)})^{\mathrm{n}}$ and $(\mathrm{T}^{(i)})^{\mathrm{n}}$ have the same shape.

Corollary 3.2. *Let* T *be a tableau with n rows whose rectification has shape μ. Let* $\tilde{\mathrm{T}}$ *be the tableau defined by the first $n-1$ rows of T whose rectification has shape $\tilde{\mu}$. Then $\tilde{\mu}$ interlaces with μ, that is,*

$$\mu_i \geq \tilde{\mu}_i \geq \mu_{i+1}, \;\; 1 \leq i \leq n-1.$$

Remark 3.1. This corollary also follows directly from [F], Row Bumping Lemma, and exercise 3 in Appendix A.2.

By passing we observe that interlacing inequalities occur in other contexts as matrix theory or module theory: invariant factors of matrices over principal ideal domains and eigenvalues of Hermitian matrices [FP, EMSa, TH]. It is not a surprise that we have the same phenomenon in the combinatorics of Young tableaux as the unified explanation for some analogies between those numbers is combinatorial. For instance in [QSSA] an explanation for the analogy of the interlacing property of those numbers is given, and in [AZ1, AZ2] and [AW] a relationship between Littlewood-Richardson combinatorics and invariant factors of a product of matrices is also discussed.

We may thus state the following results

Theorem 3.3. *Let Y and T be tableaux of shape μ and λ/μ respectively. Suppose switching transforms $Y \cup T$ into $T^{\mathrm{n}} \cup L$. Let $(T^{(1)}, \cdots, T^{(n)} = T)$ and $(L^{(1)}, \cdots, L^{(n)} = L)$ be the sequences of tableaux defined respectively by the first i rows of T and L, for all i. Then $(sh\,(T^{(1)})^{\mathrm{n}}, \cdots, sh\,(T^{(n)})^{\mathrm{n}})$ and $(sh\,(L^{(1)})^{\mathrm{n}}, \cdots, sh\,(L^{(n)})^{\mathrm{n}})$ define GT patterns respectively of types $[\mu, sh\,T^{\mathrm{n}}, \lambda]$ and $[sh\,T^{\mathrm{n}}, \mu, \lambda]$.*

Corollary 3.4. *Let Y be the Yamanouchi tableau of shape μ and T a skew-tableau of shape λ/μ. Suppose switching transforms $Y \cup T$ into $T^{\mathrm{n}} \cup L$. The following conditions hold*

1. *T^{n} of shape ν is Knuth equivalent to T, and L is a Littlewood-Richardson tableau of type$[\nu, \mu, \lambda]$.*
2. *Let $M_i = (m_1^{(i)}, \ldots, m_i^{(i)})$ with $m_k^{(i)}$ the number of letters slid down from the k-th row of Y (that is, the number of k's) to the i-th row of L, $1 \leq k \leq i \leq n$. Then*

(a) *If* $\mu^{(i)} = \mu^{(n)} - \sum_{j=i+1}^{n} M_j$, $1 \leq i \leq n$, $(\mu^{(1)}, \ldots, \mu^{(n)})$ *is the GT-pattern of type* $[\nu, \mu, \lambda]$ *defining* L.
(b) *If* $(T^{\mathrm{n}})^{(i)} \cup L^{(i)}$ *is the pair of tableaux defined by the first* i *rows of* $T^{\mathrm{n}} \cup L$, $L^{(i)}$ *is Knuth equivalent to* $Y(\mu^{(i)})$, *the Yamanouchi tableau of shape* $\mu^{(i)}$.

For each $i < n$, $Y(\mu^{(i)})$ is the Yamanouchi tableau obtained by sliding down $m_k^{(i+1)}$ k's from the Yamanouchi tableau $Y(\mu^{(i+1)})$, for $k = 1, \cdots, i+1$. That is, $\mu^{(i)} = \mu^{(i+1)} - M_{i+1}$, for all $i < n$. We claim that the interlacing property between the normal shape of a rectified tableau and the normal shape of any rectified subtableau gives rise to *jeu de taquin-like* operations and they are a shortcut in the tableau switching procedure on Littlewood-Richardson tableaux. Before going to the next section we explain the idea behind this claim.

For what follows we consider the following variation of the *jeu de taquin* on a two-row tableau T and assume for simplicity that there is no $\bullet$ to the left of the second row: (1) Switch horizontally the letters of the first row of T with the $\bullet$'s such that the letters get the leftmost possible positions; (2) Then the letters of the second row with $\bullet$ as a neighbour to the north are switched with those $\bullet$'s; (3) Finally slide horizontally the letters of the second row completely to the left. At this point we get the rectification of T. This can be seen as the *Schensted insertion* of the first row of T into the second one.

Example 3.1.

$$\begin{array}{ccccccccccccccccccccc} \bullet & \bullet & \bullet & 1 & 3 & & \longleftrightarrow & \bullet & 1 & \bullet & 3 & \bullet & \longleftrightarrow & 1 & 1 & 3 & 3 & \bullet & \longleftrightarrow & 1 & 1 & 3 & 3 & \bullet \\ 1 & 2 & 3 & 4 & & & & 1 & 2 & 3 & 4 & & & \bullet & 2 & \bullet & 4 & & & 2 & 4 & \bullet & \bullet & \end{array} .$$

This procedure can be generalized to tableaux with more than two rows as follows:

Algorithm 3.5. *Let* T *be a tableau with* n *rows.*(1) *Switch horizontally the letters of the first* $n-1$ *rows of* T *with the* $\bullet$*'s such that the letters in each row get the leftmost possible positions;* (2) *Then the letters in the* n*th row with a* $\bullet$ *as a neighbour to the north are switched with those* $\bullet$*'s; and* (3) *the letters of the* n*th row are completely slid to the left. Let* T_{n-1} *be the outcome array.*

Apply steps 1 *and* 2 *to the first* $n-1$ *rows of* T_{n-1}. *Then apply steps* 1, 2 *and* 3 *by this order to the last two rows. Next apply steps* 1 *and* 2 *to the first* $n-2$ *rows of the outcome array. Then steps* 1 *and* 2 *to the* $n-1$*th and* $n-2$*th rows, and again steps* 1, 2 *and* 3 *to the last two rows.*

When we reach the first row we apply steps 1 *and* 2 *to the first two rows, then steps* 1, 2 *to each pair of consecutive rows downwards and when we are in the last two rows we apply finally steps* 1 2 *and* 3. *At this point we get the rectification of* T.

In the previous algorithm if we replace the $\bullet$'s in the ith row of the inner shape of T with i's, for all i, then we have a switching procedure for a tableau pair $Y \cup T$ with Y a Yamanouchi tableau where a certain choice of the order of the switches was made.

Remark 3.2. This algorithm can be seen as a Schensted insertion in two consecutive rows starting with the insertion of the two bottom rows and going upwards.

Example 3.2.

$$T = \begin{array}{cccccc} \bullet & \bullet & \bullet & \bullet & 1 & 2 \\ \bullet & \bullet & \bullet & 1 & 3 & \\ 1 & 2 & 3 & 4 & & \end{array} \leftrightarrow \begin{array}{cccccc} \bullet & \bullet & 1 & 2 & \bullet & \bullet \\ \bullet & 1 & \bullet & 3 & \bullet & \\ 1 & 2 & 3 & 4 & & \end{array} \leftrightarrow \begin{array}{cccccc} \bullet & \bullet & 1 & 2 & \bullet & \bullet \\ 1 & 1 & 3 & 3 & \bullet & \\ \bullet & 2 & \bullet & 4 & & \end{array} \leftrightarrow \begin{array}{cccccc} \bullet & \bullet & 1 & 2 & \bullet & \bullet \\ 1 & 1 & 3 & 3 & \bullet & \\ 2 & 4 & \bullet & \bullet & & \end{array}$$

$$\leftrightarrow \begin{array}{cccccc} 1 & 1 & 1 & 2 & \bullet & \bullet \\ \bullet & \bullet & 3 & 3 & \bullet & \\ 2 & 4 & \bullet & \bullet & & \end{array} \leftrightarrow \begin{array}{cccccc} 1 & 1 & 1 & 2 & \bullet & \bullet \\ \bullet & 3 & 3 & \bullet & \bullet & \\ 2 & 4 & \bullet & \bullet & & \end{array} \leftrightarrow \begin{array}{cccccc} 1 & 1 & 1 & 2 & \bullet & \bullet \\ 2 & 3 & 3 & \bullet & \bullet & \\ \bullet & 4 & \bullet & \bullet & & \end{array} \leftrightarrow \begin{array}{cccccc} 1 & 1 & 1 & 2 & \bullet & \bullet \\ 2 & 3 & 3 & \bullet & \bullet & \\ 4 & \bullet & \bullet & \bullet & & \end{array} .$$

Example 3.3.

$$Y \cup T = \begin{array}{cccccc} \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & 1 & 2 \\ \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & 1 & 3 & \\ 1 & 2 & 3 & 4 & & \end{array} \leftrightarrow \begin{array}{cccccc} \bar{\mathbf{1}} & \bar{\mathbf{1}} & 1 & 2 & \bar{\mathbf{1}} & \bar{\mathbf{1}} \\ \bar{\mathbf{2}} & 1 & \bar{\mathbf{2}} & 3 & \bar{\mathbf{2}} & \\ 1 & 2 & 3 & 4 & & \end{array} \leftrightarrow \begin{array}{cccccc} \bar{\mathbf{1}} & \bar{\mathbf{1}} & 1 & 2 & \bar{\mathbf{1}} & \bar{\mathbf{1}} \\ 1 & 1 & 3 & 3 & \bar{\mathbf{2}} & \\ \bar{\mathbf{2}} & 2 & \bar{\mathbf{2}} & 4 & & \end{array} \leftrightarrow \begin{array}{cccccc} \bar{\mathbf{1}} & \bar{\mathbf{1}} & 1 & 2 & \bar{\mathbf{1}} & \bar{\mathbf{1}} \\ 1 & 1 & 3 & 3 & \bar{\mathbf{2}} & \\ 2 & 4 & \bar{\mathbf{2}} & \bar{\mathbf{2}} & & \end{array}$$

$$\leftrightarrow \begin{array}{cccccc} 1 & 1 & 1 & 2 & \bar{\mathbf{1}} & \bar{\mathbf{1}} \\ \bar{\mathbf{1}} & \bar{\mathbf{1}} & 3 & 3 & \bar{\mathbf{2}} & \\ 2 & 4 & \bar{\mathbf{2}} & \bar{\mathbf{2}} & & \end{array} \leftrightarrow \begin{array}{cccccc} 1 & 1 & 1 & 2 & \bar{\mathbf{1}} & \bar{\mathbf{1}} \\ \bar{\mathbf{1}} & 3 & 3 & \bar{\mathbf{1}} & \bar{\mathbf{2}} & \\ 2 & 4 & \bar{\mathbf{2}} & \bar{\mathbf{2}} & & \end{array} \leftrightarrow \begin{array}{cccccc} 1 & 1 & 1 & 2 & \bar{\mathbf{1}} & \bar{\mathbf{1}} \\ 2 & 3 & 3 & \bar{\mathbf{1}} & \bar{\mathbf{2}} & \\ \bar{\mathbf{1}} & 4 & \bar{\mathbf{2}} & \bar{\mathbf{2}} & & \end{array} \leftrightarrow \begin{array}{cccccc} 1 & 1 & 1 & 2 & \bar{\mathbf{1}} & \bar{\mathbf{1}} \\ 2 & 3 & 3 & \bar{\mathbf{1}} & \bar{\mathbf{2}} & \\ 4 & \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & & \end{array} .$$

Example 3.4. Consider example 3.3. The pair $Y \cup T = \begin{array}{cccccc} \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & 1 & 2 \\ \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & 1 & 3 & \\ 1 & 2 & 3 & 4 & & \end{array}$ by tableau-switching was transformed into $T^{\mathrm{n}} \cup L = \begin{array}{cccccc} 1 & 1 & 1 & 2 & \bar{\mathbf{1}} & \bar{\mathbf{1}} \\ 2 & 3 & 3 & \bar{\mathbf{1}} & \bar{\mathbf{2}} & \\ 4 & \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & & \end{array}$. The GT pattern defined by the LR tableau L is $(\mu^{(1)} = (2); \mu^{(2)} = (3,1); \mu^{(3)} = (4,3,0))$. We construct now a sequence of arrays having in the up left corner the Yamanouchi tableau of shape $\mu^{(i)}$ as follows : for each $1 \leq i < n$, transform by switching $(T^{\mathrm{n}})^{(i)} \cup L^{(i)}$ into $Y(\mu^{(i)}) \cup X^{(i)}$, and define the n-row array Q_i whose last $n-i$ rows are equal to the corresponding rows of $T^{\mathrm{n}} \cup L$, and the first i rows are equal to $Y(\mu^{(i)}) \cup X^{(i)}$. *(Notice that these arrays are recording devices that do not obey the usual rules about entries strictly increasing down columns.)*

$$Y \cup T = \begin{array}{cccccc} \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & 1 & 2 \\ \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & 1 & 3 & \\ 1 & 2 & 3 & 4 & & \end{array} \rightarrow Q_1 = \begin{array}{cccccc} \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & 1 & 1 & 2 \\ \bar{\mathbf{2}} & 1 & 2 & 3 & 3 & \\ 4 & \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & & \end{array} \rightarrow Q_2 = \begin{array}{cccccc} \bar{\mathbf{1}} & \bar{\mathbf{1}} & 1 & 1 & 1 & 2 \\ 2 & 3 & 3 & \bar{\mathbf{1}} & \bar{\mathbf{2}} & \\ 4 & \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & & \end{array}$$

$$\rightarrow T^{\mathrm{n}} \cup L = \begin{array}{cccccc} 1 & 1 & 1 & 2 & \bar{\mathbf{1}} & \bar{\mathbf{1}} \\ 2 & 3 & 3 & \bar{\mathbf{1}} & \bar{\mathbf{2}} & \\ 4 & \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & & \end{array} .$$

In the next section we define operations which clarify the meaning of these recording arrays.

4. A *jeu de taquin-chain sliding* algorithm for Litllewood-Richardson tableaux

We shall now develop operations that shall make use of the information given in Corollary 3.4. The following technical statement defines the *chain-sliding* operations as a *jeu de taquin-like* operation and relate them with the Benkart-Sotille-Stroomer switching transformation. This explains the nonstandard operations on the basis of the involution ρ_3 described with different flavours in [AZ1, AZ2] and [PV2].

Theorem 4.1. *(*Chain-sliding operation.) *Consider the following skew-tableau with inner shape* μ

$$T = \begin{array}{cccccccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & z_1 \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \delta_1 & z_2 & \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \gamma_2 & \theta_2 & \delta_2 & z_3 & & \\
\bullet & \alpha_3 & \mathbf{x_3} & \gamma_3 & \mathbf{y_3} & \theta_3 & \mathbf{w_3} & \delta_3 & z_4 & & & \\
\lambda_4 & \alpha_4 & \gamma_4 & \theta_4 & \delta_4 & & & & & & &
\end{array}$$

such that $z_i \geq \delta_i$, *for all* i, *and the following chain conditions are satisfied*
(a) $\delta_4 > \delta_3 > \delta_2 > \delta_1$; $\quad \theta_4 > \theta_3 > \theta_2$; $\quad \gamma_4 > \gamma_3 > \gamma_2$; $\quad \alpha_4 > \alpha_3$;
(b) $\mathbf{w_3} \geq \theta_4 > \mathbf{y_3} \geq \gamma_4 > \mathbf{x_3} \geq \alpha_4$.
Then it holds

(1) T *is Knuth equivalent to*

$$T' = \begin{array}{cccccccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \underline{\delta_1} & z_1 \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \underline{\gamma_2} & \underline{\theta_2} & \underline{\delta_2} & z_2 & \\
\bullet & \bullet & \bullet & \bullet & \bullet & \underline{\alpha_3} & \underline{\gamma_3} & \underline{\theta_3} & \underline{\delta_3} & z_3 & & \\
\underline{\lambda_4} & \underline{\alpha_4} & \mathbf{x_3} & \underline{\gamma_4} & \mathbf{y_3} & \underline{\theta_4} & \mathbf{w_3} & \underline{\delta_4} & z_4 & & &
\end{array} , \tag{4.1}$$

where T' *is obtained from* T *by sliding one row up the chains* $\delta_4 > \delta_3 > \delta_2 > \delta_1$; $\theta_4 > \theta_3 > \theta_2$; $\gamma_4 > \gamma_3 > \gamma_2$; $\alpha_4 > \alpha_3$ *and* λ_4. *The inner shape* μ' *of* T' *interlaces with the inner shape* μ *of* T,

$$\mu_i \geq \mu'_i \geq \mu_{i+1},$$

such that $\mu_i - \mu'_i$ *is equal to the number of chains that have reached row* i *of* T. *(The underlines indicate the slid chains while the non underlined letters were kept fixed.) We call these chain-sliding operations* jeu de taquin-chain slides.

(2) *Suppose* $Y(\mu) \cup T$ *is by switching transformed into* $T^{\mathrm{n}} \cup L$. *Then the last row of* $T^{\mathrm{n}} \cup L$ *has* $(\mu_i - \mu'_i)$ i*'s, for all* i, *and* $(T^{\mathrm{n}})^{(n-1)} \cup L^{(n-1)}$, *defined by the first* $n-1$ *rows of* $T^{\mathrm{n}} \cup L$, *can be transformed by switching into* $Y(\mu') \cup T'$, *with* T' *as in (4.1).*

Proof. Apply Algorithm 3.5 to T with z_1, z_2, z_3 and z_4 fixed. Without lost of generality we shall assume that $\mathbf{w_3} > \delta_2 \geq \theta_3$; $\mathbf{y_3} > \theta_2 > \delta_1 \geq \gamma_3$; $\mathbf{x_3} > \gamma_2 \geq \alpha_3 \geq \lambda_4$. Considering these inequalities and the ones above, by horizontal switches, as in our algorithm, $Y(\mu) \cup T$ can be transformed into

$$\begin{array}{cccccccccccc}
\bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & z_1 \\
\bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \delta_1 & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & z_2 & \\
\bar{\mathbf{3}} & \bar{\mathbf{3}} & \gamma_2 & \bar{\mathbf{3}} & \theta_2 & \bar{\mathbf{3}} & \delta_2 & \bar{\mathbf{3}} & \bar{\mathbf{3}} & z_3 & & . \\
\bar{\mathbf{4}} & \alpha_3 & \mathbf{x_3} & \gamma_3 & \mathbf{y_3} & \theta_3 & \mathbf{w_3} & \delta_3 & z_4 & & & \\
\lambda_4 & \alpha_4 & \gamma_4 & \theta_4 & \delta_4 & & & & & & &
\end{array}$$

Then again by the following sequence of switches

$$\begin{array}{cccccccccccc}
\bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & z_1 \\
\bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \delta_1 & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & z_2 & \\
\lambda_4 & \alpha_3 & \gamma_2 & \gamma_3 & \theta_2 & \theta_3 & \delta_2 & \delta_3 & \bar{\mathbf{3}} & z_3 & & \\
\bar{\mathbf{3}} & \bar{\mathbf{3}} & \mathbf{x_3} & \bar{\mathbf{3}} & \mathbf{y_3} & \bar{\mathbf{3}} & \mathbf{w_3} & \bar{\mathbf{3}} & z_4 & & & \\
\alpha_4 & \gamma_4 & \theta_4 & \delta_4 & \bar{\mathbf{4}} & & & & & & &
\end{array}
\quad \rightarrow \quad
\begin{array}{cccccccccccc}
\bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & z_1 \\
\bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \delta_1 & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & z_2 & \\
\lambda_4 & \alpha_3 & \gamma_2 & \gamma_3 & \theta_2 & \theta_3 & \delta_2 & \delta_3 & \bar{\mathbf{3}} & z_3 & & \\
\bar{\mathbf{3}} & \mathbf{x_3} & \mathbf{y_3} & \mathbf{w_3} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & z_4 & & & \\
\alpha_4 & \gamma_4 & \theta_4 & \delta_4 & \bar{\mathbf{4}} & & & & & & &
\end{array}$$

$$\begin{array}{cccccccccccc}
 & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & z_1 \\
 & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \delta_1 & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & z_2 & \\
\rightarrow & \lambda_4 & \alpha_3 & \gamma_2 & \gamma_3 & \theta_2 & \theta_3 & \delta_2 & \delta_3 & \bar{\mathbf{3}} & z_3 & & \\
 & \alpha_4 & \mathbf{x_3} & \mathbf{y_3} & \mathbf{w_3} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & z_4 & & & \\
 & \bar{\mathbf{3}} & \gamma_4 & \theta_4 & \delta_4 & \bar{\mathbf{4}} & & & & & & &
\end{array}
\quad \rightarrow \quad
\begin{array}{cccccccccccc}
\bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & z_1 \\
\bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \delta_1 & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & z_2 & \\
\lambda_4 & \alpha_3 & \gamma_2 & \gamma_3 & \theta_2 & \theta_3 & \delta_2 & \delta_3 & \bar{\mathbf{3}} & z_3 & & \\
\alpha_4 & \mathbf{x_3} & \mathbf{y_3} & \mathbf{w_3} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & z_4 & & & \\
\gamma_4 & \theta_4 & \delta_4 & \bar{\mathbf{3}} & \bar{\mathbf{4}} & & & & & & &
\end{array}$$

$$\begin{array}{cccccccccccc}
 & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & z_1 \\
 & \lambda_4 & \alpha_3 & \gamma_2 & \gamma_3 & \delta_1 & \theta_3 & \delta_2 & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & z_2 & \\
\rightarrow & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \theta_2 & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \delta_3 & \bar{\mathbf{3}} & z_3 & & \\
 & \alpha_4 & \mathbf{x_3} & \mathbf{y_3} & \mathbf{w_3} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & z_4 & & & \\
 & \gamma_4 & \theta_4 & \delta_4 & \bar{\mathbf{3}} & \bar{\mathbf{4}} & & & & & & &
\end{array}
\quad \rightarrow \quad
\begin{array}{cccccccccccc}
\bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & z_1 \\
\lambda_4 & \alpha_3 & \gamma_2 & \gamma_3 & \delta_1 & \theta_3 & \delta_2 & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & z_2 & \\
\alpha_4 & \mathbf{x_3} & \theta_2 & \mathbf{w_3} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \delta_3 & \bar{\mathbf{3}} & z_3 & & \\
\bar{\mathbf{2}} & \bar{\mathbf{2}} & \mathbf{y_3} & \bar{\mathbf{2}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & z_4 & & & \\
\gamma_4 & \theta_4 & \delta_4 & \bar{\mathbf{3}} & \bar{\mathbf{4}} & & & & & & &
\end{array}$$

Recall that $\mathbf{w_3} > \mathbf{y_3} > \theta_2 > \gamma_3 > \gamma_2 \geq \mathbf{x_3}$; $\theta_4 > \mathbf{y_3} > \mathbf{x_3}$ and $\mathbf{y_3} \geq \gamma_4$.

$$\begin{array}{cccccccccccc}
 & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & z_1 \\
 & \lambda_4 & \alpha_3 & \gamma_2 & \gamma_3 & \delta_1 & \theta_3 & \delta_2 & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & z_2 & \\
\rightarrow & \alpha_4 & \mathbf{x_3} & \theta_2 & \mathbf{w_3} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \delta_3 & \bar{\mathbf{3}} & z_3 & & \\
 & \gamma_4 & \mathbf{y_3} & \delta_4 & \bar{\mathbf{2}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & z_4 & & & \\
 & \bar{\mathbf{2}} & \theta_4 & \bar{\mathbf{2}} & \bar{\mathbf{3}} & \bar{\mathbf{4}} & & & & & & &
\end{array}
\quad \rightarrow \quad
\begin{array}{cccccccccccc}
\lambda_4 & \alpha_3 & \gamma_2 & \gamma_3 & \delta_1 & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & z_1 \\
\bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \theta_3 & \delta_2 & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & z_2 & \\
\alpha_4 & \mathbf{x_3} & \theta_2 & \mathbf{w_3} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \delta_3 & \bar{\mathbf{3}} & z_3 & & \\
\gamma_4 & \mathbf{y_3} & \delta_4 & \bar{\mathbf{2}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & z_4 & & & \\
\theta_4 & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{3}} & \bar{\mathbf{4}} & & & & & & &
\end{array}$$

$$\begin{array}{cccccccccccc}
 & \lambda_4 & \alpha_3 & \gamma_2 & \gamma_3 & \delta_1 & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & z_1 \\
 & \alpha_4 & \mathbf{x_3} & \theta_2 & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \theta_3 & \delta_2 & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & z_2 & \\
\to & \gamma_4 & \mathbf{y_3} & \bar{\mathbf{1}} & \mathbf{w_3} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \delta_3 & \bar{\mathbf{3}} & z_3 & & . \\
 & \theta_4 & \bar{\mathbf{1}} & \delta_4 & \bar{\mathbf{2}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & z_4 & & & \\
 & \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{3}} & \bar{\mathbf{4}} & & & & & & &
\end{array}$$

According to Corollary 3.4, in the last row of the previous perforated pair, the multiplicity of a letter $\bar{\mathbf{k}}$ is precisely the number of letters slid from the kth row of $Y(\mu)$ to the nth-row of $Y(\mu) \cup T$. Despite the z_i's were kept fixed in the application of Algorithm 3.5, once the letters of Y get the last row they stay till the end of the switching process. Thus the last perforated tableau pair with the last row suppressed by switching is transformed into $(T^{\mathrm{n}})^{(n-1)} \cup L^{(n-1)}$ the tableau pair defined by the first $n-1$ rows of $T^{\mathrm{n}} \cup L$

$$\begin{array}{ccccccccccc}
\lambda_4 & \alpha_3 & \gamma_2 & \gamma_3 & \delta_1 & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & z_1 \\
\alpha_4 & \mathbf{x_3} & \theta_2 & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \theta_3 & \delta_2 & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & z_2 & & \\
\gamma_4 & \mathbf{y_3} & \bar{\mathbf{1}} & \mathbf{w_3} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \delta_3 & \bar{\mathbf{3}} & z_3 & & \longrightarrow (T^{\mathrm{n}})^{(n-1)} \cup L^{(n-1)}. \\
\theta_4 & \bar{\mathbf{1}} & \delta_4 & \bar{\mathbf{2}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & z_4 & & & & (4.2)
\end{array}$$

It remains to show that (4.2) can be transformed by switches into

$$Y(\mu') \cup T' = \begin{array}{cccccccccccc}
\bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \underline{\delta_1} & z_1 \\
\bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \underline{\gamma_2} & \underline{\theta_2} & \underline{\delta_2} & \underline{z_2} & \\
\bar{\mathbf{3}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & \underline{\alpha_3} & \underline{\gamma_3} & \underline{\theta_3} & \underline{\delta_3} & z_3 & & \\
\lambda_4 & \underline{\alpha_4} & \mathbf{x_3} & \underline{\gamma_4} & \mathbf{y_3} & \underline{\theta_4} & \mathbf{w_3} & \underline{\delta_4} & z_4 & & &
\end{array} . \qquad (4.3)$$

Performing the following sequence of switches we get the wished result

$$\begin{array}{cccccccccccc}
\lambda_4 & \alpha_3 & \gamma_2 & \gamma_3 & \delta_1 & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & z_1 \\
\alpha_4 & \mathbf{x_3} & \theta_2 & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \theta_3 & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \delta_2 & z_2 & \\
\gamma_4 & \mathbf{y_3} & \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & \delta_3 & z_3 & & \\
\theta_4 & \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & \mathbf{w_3} & \delta_4 & z_4 & & &
\end{array} \to \begin{array}{cccccccccccc}
\lambda_4 & \alpha_3 & \gamma_2 & \gamma_3 & \delta_1 & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & z_1 \\
\alpha_4 & \mathbf{x_3} & \theta_2 & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \delta_2 & z_2 & \\
\gamma_4 & \mathbf{y_3} & \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & \theta_3 & \delta_3 & z_3 & & \\
\bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & \theta_4 & \mathbf{w_3} & \delta_4 & z_4 & & &
\end{array}$$

$$\begin{array}{cccccccccccc}
\lambda_4 & \alpha_3 & \gamma_2 & \gamma_3 & \delta_1 & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & z_1 \\
\alpha_4 & \mathbf{x_3} & \theta_2 & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \delta_2 & z_2 & \\
\bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \mathbf{y_3} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & \theta_3 & \delta_3 & z_3 & & \\
\gamma_4 & \bar{\mathbf{2}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & \theta_4 & \mathbf{w_3} & \delta_4 & z_4 & & &
\end{array} \to \begin{array}{cccccccccccc}
\lambda_4 & \alpha_3 & \gamma_2 & \gamma_3 & \delta_1 & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & z_1 \\
\alpha_4 & \mathbf{x_3} & \theta_2 & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \delta_2 & z_2 & \\
\bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & \theta_3 & \delta_3 & z_3 & & \\
\gamma_4 & \bar{\mathbf{2}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & \mathbf{y_3} & \theta_4 & \mathbf{w_3} & \delta_4 & z_4 & & &
\end{array}$$

$$\begin{array}{cccccccccccc}
\lambda_4 & \alpha_3 & \gamma_2 & \gamma_3 & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \delta_1 & z_1 \\
\alpha_4 & \mathbf{x_3} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \theta_2 & \delta_2 & z_2 & \\
\bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & \theta_3 & \delta_3 & z_3 & & \\
\bar{\mathbf{2}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & \gamma_4 & \mathbf{y_3} & \theta_4 & \mathbf{w_3} & \delta_4 & z_4 & & &
\end{array} \to \begin{array}{cccccccccccc}
\lambda_4 & \alpha_3 & \gamma_2 & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \delta_1 & z_1 \\
\alpha_4 & \mathbf{x_3} & \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \theta_2 & \delta_2 & z_2 & \\
\bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{2}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & \gamma_3 & \theta_3 & \delta_3 & z_3 & & \\
\bar{\mathbf{2}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & \gamma_4 & \mathbf{y_3} & \theta_4 & \mathbf{w_3} & \delta_4 & z_4 & & &
\end{array}$$

$$\begin{array}{l} \\ \rightarrow \\ \\ \\ \end{array} \begin{array}{cccccccccccc} \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \delta_1 & z_1 \\ \lambda_4 & \alpha_3 & \gamma_2 & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \theta_2 & \delta_2 & z_2 & \\ \alpha_4 & \mathbf{x_3} & \bar{\mathbf{2}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & \gamma_3 & \theta_3 & \delta_3 & z_3 & & \\ \bar{\mathbf{2}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & \gamma_4 & \mathbf{y_3} & \theta_4 & \mathbf{w_3} & \delta_4 & z_4 & & & \end{array} \begin{array}{l} \\ \rightarrow \\ \\ \\ \end{array}$$

$$\begin{array}{l} \\ \rightarrow \\ \\ \\ \end{array} \begin{array}{cccccccccccc} \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \delta_1 & z_1 \\ \lambda_4 & \alpha_3 & \gamma_2 & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \theta_2 & \delta_2 & z_2 & \\ \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & \gamma_3 & \theta_3 & \delta_3 & z_3 & & \\ \bar{\mathbf{3}} & \alpha_4 & \mathbf{x_3} & \gamma_4 & \mathbf{y_3} & \theta_4 & \mathbf{w_3} & \delta_4 & z_4 & & & \end{array} \begin{array}{l} \\ \rightarrow \\ \\ \\ \end{array} \begin{array}{cccccccccccc} \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \delta_1 & z_1 \\ \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \bar{\mathbf{2}} & \gamma_2 & \theta_2 & \delta_2 & z_2 & \\ \bar{\mathbf{3}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & \bar{\mathbf{3}} & \alpha_3 & \gamma_3 & \theta_3 & \delta_3 & z_3 & & \\ \lambda_4 & \alpha_4 & \mathbf{x_3} & \gamma_4 & \mathbf{y_3} & \theta_4 & \mathbf{w_3} & \delta_4 & z_4 & & & \end{array} .$$

□

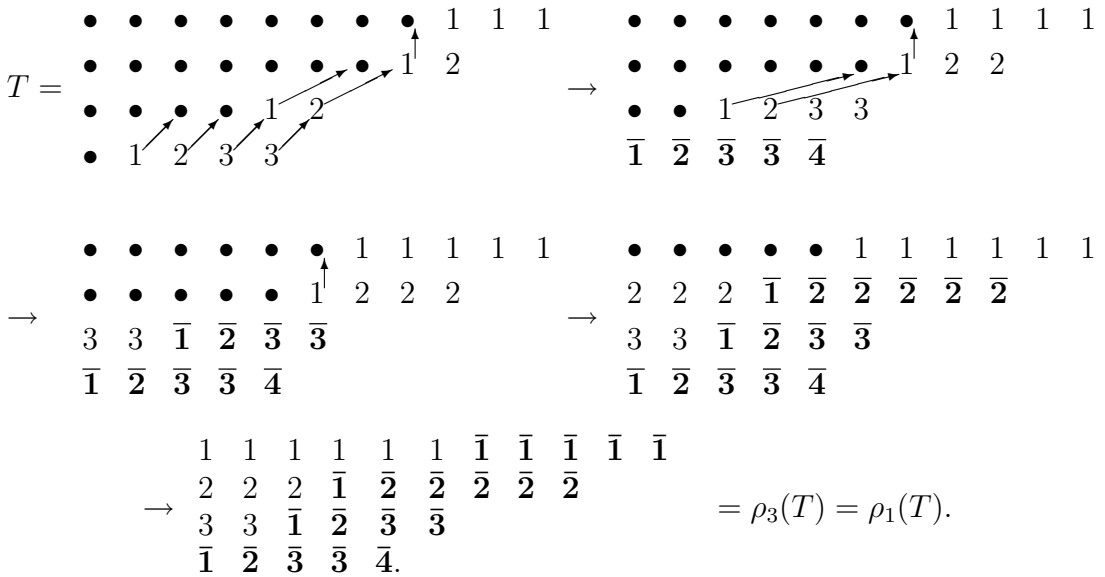
Example 4.1. (*a*) Example 3.4. (*b*) We may use the *jeu de taquin-chain sliding* to conclude that the following tableaux are Knuth equivalent

$$\begin{array}{cccccc} \bullet & \bullet & \bullet & \bullet & 4 & \\ \bullet & \bullet & 2 & 4 & & \\ 3 & 5 & 6 & & & \end{array} \quad \rightarrow \quad \begin{array}{cccccc} \bullet & \bullet & 2 & 4 & 4 & \\ \bullet & 3 & 5 & 6 & & \end{array} .$$

According to our previous study we now reformulate the algorithms presented in [AZ1, AZ2] and [PV2].

Algorithm 4.2. Consider an LR tableau T. Replace rows one by one in T, beginning with the bottom row, as follows. In each row i to be replaced, build a sequence of chains: one letter x in row i and the other letters in rows above, starting with the rightmost letter of i and going to the leftmost one. For each such element x, find the largest $y < x$ in the above row, not used by the previous chains starting from row containing x, then the largest element $z < y$ in the row above that of y not used by the previous chains, etc. This chain will finish either in a $\bullet$, in row k of the inner shape of T whenever the length of the chain is $i-k+1$, or in the first row of T in a 1. This last situation occurs whenever the length of the chain is i. Now replace y with x, z with y, etc, until the top element of the chain removes a $\bullet$ in row k of the inner shape of T unless the chain reaches the first row of T. In this case no replacement is made along the chain but the starting letter i that remains in row i is moved as far as possible to the left. The removed $\bullet$ is recorded as a letter $\overline{\mathbf{k}}$ in the row i of T to be replaced. Note that each entry of the inner shape of T, in the row i to be replaced, forms a chain of length 0 which will be recorded in the same row as $\overline{\mathbf{i}}$.

Example 4.2. Consider the LR tableau T of type $[\mu = (8,7,4,1), \nu = (6,3,2); \lambda = (11,9,6,5)]$ and apply the *jeu de taquin-like*



The outcome is a tableau-pair defined by the Yamanouchi tableau $Y(\nu)$, the rectification of the LR tableau T of type $[\mu, \nu, \lambda]$, and an LR tableau of type $[\nu, \mu, \lambda]$.

In [AZ1, AZ2, PV2] instead of *replacements* of rows by new ones recording what has been removed in the inner shape, rows are *removed* and what has been removed in the inner shape is recorded in a matrix. This is the original idea since as we have discussed what is removed in the inner shape of T defines a GT-pattern which encodes the LR tableau type $[\nu, \mu, \lambda]$. But here one wishes to stress the relationship between the involutions ρ_1 and ρ_3.

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