EFFECTIVE KEY MAPS AND ORTHOGONAL EVACUATION

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ABSTRACT. We prove that the key map on crystals of any classical type can be reduced to a key map on simply-laced types by using virtualization of crystals. As a direct application we obtain new algorithms to compute evacuation, keys, and Demazure atoms in type B_n in terms of Kashiwara–Nakashima tableaux. In particular, we are able to use type A_n and C_n methods. For type C_n, we apply the results obtained by Azenhas–Tarighat Feller–Torres, Azenhas–Santos and Santos.

1. INTRODUCTION

Given an irreducible finite-dimensional representation $V(\lambda)$ of finite complex Lie algebra $\mathfrak{g}$ with highest weight λ, a dominant integral weight, the ***crystal graph*** $\mathcal{B}(\lambda)$ of $V(\lambda)$ is a finite directed graph with vertices given by the crystal basis of $V(\lambda)$ and edges corresponding to deformations of the Chevalley operators of the representation [22]. ***Demazure modules*** are certain Borel modules that arose in connection to Schubert varieties [15] and generalize the highest weight modules $V(\lambda)$ of a Lie algebra $\mathfrak{g}$. It was shown by Littelmann and Kashiwara [24, 31] that any Demazure module $V_w(\lambda)$ has an associated ***Demazure crystal*** $\mathcal{B}_w(\lambda)$ that arises as an induced subgraph of $\mathcal{B}(\lambda)$. Thus, for fixed $w \in W$ in the Weyl group W, it is natural to inquire whether a given vertex $b \in \mathcal{B}(\lambda)$ belongs to the Demazure $\mathcal{B}_w(\lambda)$. The answer to this question is provided by the ***right key map***, which associates to $b \in \mathcal{B}(\lambda)$ an *extremal* element $b_{w\lambda}$ in the W-orbit of the highest weight vector called the ***right key*** of b (with the similar notion of ***left key*** analogously defined). Correspondingly, the ***Demazure atom*** $\mathring{\mathcal{B}}_w(\lambda)$ is the subset of $\mathcal{B}(\lambda)$ consisting of $b \in \mathcal{B}(\lambda)$ whose right key is $b_{w\lambda}$. The study of various properties of Demazure crystals and atoms is an extremely active topic of research today, for instance [2, 1, 5, 6, 3, 12, 13, 17, 19].

Finding algorithms that *effectively* compute the key maps has captured the interest of mathematicians over the last few decades. Indeed, although right keys can be computed from Lakshimbai-Seshadri paths [30] or the alcove path model [29, Definition 5.2, Remark 5.3, Corollary 6.2] in a type independent fashion, such procedures are generally more difficult than via type-specific tableux models. For instance, in type A_n there is a well-known algorithm based on *jeu de taquin* for semi-standard Young tableaux [27], a recursive generalization in [19], and other alternative algorithms provided in [3, 12, 26, 32, 39]. For type C_n Kashiwara–Nakashima tableaux, there are two known procedures due to Santos [36, 37] and a third by Azenhas–Santos [6]. Alternative ortho-symplectic

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constructions are also found in [13]. In particular, all known algorithms for computing keys that utilize *tableaux* models are type dependent.

The Lusztig–Schützenberger involution is the unique set map ξ on $\mathcal{B}(\lambda)$ induced by the automorphism of the Dynkin diagram of $\mathfrak{g}$ given by left multiplication by the longest element of W. The process of combinatorially computing the image $\xi(b)$ for a given $b \in \mathcal{B}(\lambda)$ is known as ***evacuation***. In types A_n and C_n, respectively, algorithms were developed in [10] and [36]. In particular, in each of these models it was shown that the Lusztig–Schützenberger involution exchanges left and right keys. This is also known in the model of Lakshmibai-Seshadri paths and alcove path model.

1.1. Main Results.

In this paper, we introduce a new type-crystal model-independent technique for computing both the key maps and the Schützen-berger–Lusztig involution via *virtualization* of crystals. ***Virtualization*** is a method introduced by Kashiwara [23] that embeds a highest weight crystal inside another of (potentially) different Lie type, provided the associated Dynkin diagrams are related via so-called diagram *folding*. The image of such an embedding equipped with an induced crystal structure is termed a ***virtual crystal***.

For any positive integer m, the natural embedding $\mathcal{B}(\lambda) \hookrightarrow \mathcal{B}(m\lambda)$ is a distinguished virtualization map. This particular map, introduced as ***dilation*** by Kashiwara [23], is the basis for the definition of the key map. Despite providing minimal bounds for m for this construction in Theorem 3.5, this formulation is not computationally effective.

Inspired by the results in [6, 4] for the Baker embedding $C_n \hookrightarrow A_{2n-1}$ [8], in this paper we prove that virtualization preserves the left and right key maps (see Theorem 3.8). It is important to note that while cases of these results were known for specific Lie types, our results constitute a type-independent and crystal model-independent generalization. In particular, (provided the virtualization has a well-defined left inverse) our theorem reduces the computation of keys, atoms, and evacuation for non-simply laced types to simply laced cases via Table 1.

We then prove the effectiveness of such methods by applying it to type B Kashiwara–Nakashima tableaux via different virtualization maps, for instance that of Baker [9], Fujita [16], and Pappe–Pfannerer–Schilling–Simone [34]. This, in turn, provides a new effective algorithmic description of evacuation and key maps in type B (see Theorem 4.4).

This extended abstract is organized in four sections. We refer the reader to [7] for details and proofs, containing the results hereby presented.

2. Crystals

Let $\mathfrak{g}$ be a finite complex semisimple Lie algebra with usual Cartan data given by the weight lattice P, simple roots α_i, fundamental weights ω_i, canonical pairing $\langle \cdot, \cdot \rangle : P^\vee \times P \to \mathbb{Z}$, and Weyl group W (endowed with the strong Bruhat order) [11]. We review Kashiwara's theory of $\mathfrak{g}$-crystals but refer the reader to [21, 20] for details.

Definition 2.1. A (normal) $\mathfrak{g}$-***crystal*** is a nonempty finite set $\mathcal{B}$ with a *weight* map $\mathrm{wt} : \mathcal{B} \to P$, *string operators* $\varepsilon_i, \varphi_i : \mathcal{B} \to \mathbb{Z}$, and *crystal operators* $e_i, f_i : \mathcal{B} \to \mathcal{B} \sqcup \{0\}$ where $0 \notin \mathcal{B}$ is an auxiliary symbol, subject to the following conditions for all $i \in I$ and $b, b' \in \mathcal{B}$:

- $\varphi_i(b) - \varepsilon_i(b) = \langle \alpha_i^\vee, \mathrm{wt}(b)\rangle$,
- $\mathrm{wt}(e_i(b)) = \mathrm{wt}(b) + \alpha_i$ if $e_i(b) \in \mathcal{B}$,
- $\mathrm{wt}(f_i(b)) = \mathrm{wt}(b) - \alpha_i$ if $f_i(b) \in \mathcal{B}$,
- $b' = e_i(b)$ if and only if $b = f_i(b')$,
- $\varepsilon_i(b) = \max\{k \geq 0 | e_i^k(b) \in \mathcal{B}\}$,
- $\varphi_i(b) = \max\{k \geq 0 | f_i^k(b) \in \mathcal{B}\}$.

Let $\mathcal{E}$ and $\mathcal{F}$ be the semigroups generated by $\{e_i\}_{i\in I}$ and $\{f_i\}_{i\in I}$, respectively and for any $w \in W$ with reduced expression $s_{i_1}\dots s_{i_k}$ define $\mathcal{F}_w := \bigcup_{m_i\in\mathbb{Z}_{\geq 0}}\{f_{i_1}^{m_1}\dots f_{i_k}^{m_k}\} \subset \mathcal{F}$ and $\mathcal{E}_w := \bigcup_{m_i\in\mathbb{Z}_{\geq 0}}\{e_{i_1}^{m_1}\dots e_{i_k}^{m_k}\} \subset \mathcal{E}$.

We say a element $b \in \mathcal{B}$ is a ***highest weight vector*** (resp. ***lowest weight vector***) if $\mathcal{E}\{b\} = 0$ (resp. $\mathcal{F}\{b\} = 0$). So then, for λ a dominant integral weight denote by $\mathcal{B}(\lambda)$ the crystal graph of the associated highest weight $\mathfrak{g}$-module $V(\lambda)$ with highest weight vector b_λ, so that $\mathrm{wt}(b_\lambda) = \lambda$, $\mathcal{E}\{b_\lambda\} = 0$, and $\mathcal{F}\{b_\lambda\} = \mathcal{B}(\lambda)$. We call a vertex $b \in \mathcal{B}(\lambda)$ ***extremal*** if it lies in the W-orbit of b_λ.

Definition 2.2. Given any $\lambda \in P^+$, the ***Lusztig–Schützenberger involution*** $\xi = \xi_{\mathcal{B}(\lambda)} : \mathcal{B}(\lambda) \to \mathcal{B}(\lambda)$ is the unique set involution such that for all $i \in I$ and $b \in \mathcal{B}(\lambda)$: $e_i\xi(b) = \xi f_{\theta(i)}(b)$, $f_i\xi(b) = \xi e_{\theta(i)}(b)$, $\mathrm{wt}(\xi(b)) = w_0\mathrm{wt}(b)$, where θ is the automorphism of I defined by applying the longest element $w_0 \in W$ to the simple roots: $w_0\alpha_i = -\alpha_{\theta(i)}$. More generally, on a given normal $\mathfrak{g}$-crystal $\mathcal{B}$, $\xi_\mathcal{B}$ acts locally on each connected component of $\mathcal{B}$.

For example, in types B_n and C_n we have $\theta = Id$, whereas in type A_n $\theta(i) = n - i$ (see [4]).

2.1. **Virtual Crystals.** For any Dynkin diagram D, denote by P_D the corresponding integral weight lattice and by ω_i^D the corresponding fundamental weights. Let X and Y be two Dynkin diagrams and let *aut* be an automorphism of Y such that distinct nodes of Y in the same *aut*-orbit are not connected by an edge. We say there is an *embedding* $\psi : X \hookrightarrow Y$ if there exists a bijection $\Psi : X \to Y/aut$ inducing a map $P_X \to P_Y$ given by the assignment $\omega_i^X \mapsto \sum_{j\in\Psi(i)}\gamma_i(\omega^Y)_j$, with γ_i given as in the Table 1. Consequently, we have a natural embedding of the Weyl groups W^X into W^Y, identifying W^X with the set of elements $\widetilde{W}^X$ in W^Y that are fixed under the Dynkin symmetry:

$$W^X \cong \widetilde{W}^X := \langle \Pi_{j\in\psi(i)}\tilde{s}_j \mid i \in I^X\rangle \subset W^Y = \langle \tilde{s}_j \mid j \in I^Y\rangle,$$

via the group isomorphism $s_i \mapsto \Pi_{j\in\psi(i)}\tilde{s}_j$. We abuse notation and use ψ to also denote the induced maps on weight lattices, Weyl groups, and indices $\psi : I^X \to I^Y$. In particular, ψ preserves strong Bruhat order and reduced expressions for elements.

Definition 2.3. Suppose X and Y are Dynkin diagrams with an embedding $\psi : X \hookrightarrow Y$ as above. Let $(\tilde{\mathcal{B}}; \tilde{e}_j, \tilde{f}_j, \tilde{\varphi}_j, \tilde{\varepsilon}_j)_{j\in I^Y}$ be a normal $\mathfrak{g}_Y$-crystal. A ***virtual*** $\mathfrak{g}_X$***-crystal*** is a subset $\mathcal{V} \subset \tilde{\mathcal{B}}$ such that $\mathcal{V}$ has a normal $\mathfrak{g}_X$-crystal structure where for any $i \in I^X$ the crystal operators are given by:

$$e_i^{\mathbf{v}} := \prod_{j\in\psi(i)}\tilde{e}_j^{\gamma_i}, \qquad f_i^{\mathbf{v}} := \prod_{j\in\psi(i)}\tilde{f}_j^{\gamma_i}, \tag{1}$$

and for any choice of $j \in \psi(i)$, the string operators defined as: $\varepsilon_i := \gamma_i^{-1}\tilde{\varepsilon}_j$ $\varphi_i := \gamma_i^{-1}\tilde{\varphi}_j$. Additionally, if a $\mathfrak{g}_X$-crystal $\mathcal{B}$ is isomorphic to a virtual $\mathfrak{g}_X$-crystal $\mathcal{V} \subset \tilde{\mathcal{B}}$, we call the associated isomorphism $\Upsilon_\psi : \mathcal{B} \to \mathcal{V}$ the ***virtualization*** map.

X	**Y**	γ_i
C_n	A_{2n-1}	$\gamma_i = 1, 1 \leq i < n, \gamma_n = 2$
B_n	D_{n+1}	$\gamma_i = 2, 1 \leq i < n, \gamma_n = 1$
F_4	E_6	$\gamma_1 = \gamma_2 = 2, \gamma_3 = \gamma_4 = 1$
G_2	D_4	$\gamma_1 = 1, \gamma_2 = 3$
B_n	C_n	$\gamma_i = 2, 1 \leq i < n, \gamma_n = 1$
C_n	B_n	$\gamma_i = 1, 1 \leq i < n, \gamma_n = 2$
B_n	A_{2n-1}	$\gamma_i = 1, 1 \leq i \leq n$
G_2	A_5	$\gamma_1 = 1, \gamma_2 = 2$

Table 1. Explicit virtualization maps when $X = B_n, Y = C_n$, $X = C_n, Y = B_n$, and $X = B_n, C_n, Y = A_{2n-1}$ found in [16, 34, 9, 23, 38, 14].

In particular, virtualizations are closed under tensor products and unique (up to choice of embedding). A particularly important example of a virtualization map in the case when $X = Y$ is the following [23, Thm 3.1]:

Example 2.4. For any positive integer m, the m-***dilation*** map $\mathbb{D}_m : \mathcal{B}(\lambda) \hookrightarrow \mathcal{B}(m\lambda)$ is the unique embedding such that $\mathbb{D}_m(f_i b) = f_i^m \mathbb{D}_m(b)$, $\mathbb{D}_m(e_i b) = e_i^m \mathbb{D}_m(b)$ and $\varphi_i(\mathbb{D}_m(b)) = m\varphi_i(b)$, $\varepsilon_i(\mathbb{D}_m(b)) = m\varepsilon_i(b)$, $\mathrm{wt}(\mathbb{D}_m(b)) = m\mathrm{wt}(b)$.

Various explicit virtualizations have been studied in the literature (see Table 1). Baker described virtualization maps corresponding to the embeddings $B_n, C_n \hookrightarrow A_{2n-1}$ directly on Kashiwara-Nakashima tableaux [9], which were then extended by Schilling–Scrimshaw for $B_n, C_n \hookrightarrow D_{n+1}$ and $F_4 \hookrightarrow E_6$, $G_2 \hookrightarrow D_4$ [38]. Virtualizations from $B_n \hookrightarrow C_n$ and $C_n \hookrightarrow B_n$ were independently studied by Fujita [16] and Pappe–Pfannerer–Schilling–Simone [33]. More general constructions appear in [33] and [35].

3. Demazure crystals and keys under virtualization

Denote by $\mathfrak{b} \subset \mathfrak{g}$ any Borel subalgebra of $\mathfrak{g}$. The ***Demazure module*** $V_w(\lambda)$ is a $\mathfrak{b}$-module generated by the one dimensional weight space $V(\lambda)_{w\lambda}$ of weight $w\lambda$ [15].

Definition 3.1. Given $\lambda \in P^+$ and $w \in W$, the ***Demazure crystal*** $\mathcal{B}_w(\lambda)$ is the induced subset $\mathcal{B}_w(\lambda) := \mathcal{F}_w\{b_\lambda\} \subset \mathcal{B}(\lambda)$.

Now, consider the canonical embedding $\Theta_m := G_m \circ \mathbb{D}_m$ where $\mathbb{D}_m$ is the m-dilation map from Example 2.4 and $G_m : \mathcal{B}(m\lambda) \longrightarrow \mathcal{F}\{b_\lambda^{\otimes m}\}$ is the unique crystal isomorphism mapping $b_{m\lambda} \to b_\lambda^{\otimes m}$. It was shown in Proposition 8.3.2 in [20] that when m is large enough we have a decomposition $\Theta_m(b) = b_{w\lambda} \otimes b' \otimes b_{w'\lambda}$ for any $b \in \mathcal{B}(\lambda)$ with $b_{w\lambda}, b_{w'\lambda}$ extremal and $b' \in \mathcal{B}(\lambda)^{m-2}$. In particular, the pair $(b_{w\lambda}, b_{w'\lambda})$ is independent of the choice of any such m, hence the following is well-defined.

Definition 3.2. For a given $b \in \mathcal{B}(\lambda)$ the ***right key*** (resp. ***left key***) of b is the extremal vector $K^+(b) := b_{w\lambda}$ (resp. $K^-(b) := b_{w'\lambda}$).

Example 3.3. In Figure 1(left) the boxed vertices correspond to the right and left keys of the $\mathfrak{so}_5$-crystal $\mathcal{B}(\omega_1 + \omega_2)$.

The ***Demazure atom*** is then defined as the set $\mathring{\mathcal{B}}_w(\lambda) = \{b \in \mathcal{B}(\lambda) : K^+(b) = b_{w\lambda}\}$ of all vertices whose right key is $b_{w\lambda}$.

Remark 3.4. The related notions of ***opposite Demazure crystal*** and ***opposite Demazure atom*** can be defined analogously by exchanging $\mathcal{F}$ with $\mathcal{E}$ and right with left key. In particular, all theorems below hold analogously for the opposite setting. We refer the reader to [7] for details.

The following provides a tight bound for the values of m and thus refines [20, Prop. 8.3.2].

Theorem 3.5. *Let $m \in \mathbb{N}$. For all $b \in \mathcal{B}(\lambda)$, there exist $b' \in \mathcal{B}(\lambda)^{\otimes(m-2)}$ and fixed $w \geq w' \in W$ such that*

$\Theta_m(b) = b_{w\lambda} \otimes b' \otimes b_{w'\lambda}$ *if and only if* $m \geq \ell = \max\{\textit{length}(\rho) \mid \rho \textit{ is an } i\textit{-string for } i \in I\}$.

Remark 3.6. In particular, we note that the pair of keys $b_{w\lambda}$, $b_{w'\lambda}$ define the initial (resp. final) direction of the corresponding LS path in the isomorphic crystal of Lakshmibai-Seshadri paths.

Example 3.7. Consider the $\mathfrak{so}_5$-crystal $\mathcal{B}(\omega_1 + \omega_2)$ in Figure 1(left). Then its i-strings have lengths $1, 2$, and 3 so that $\ell = 3$. In Figure 2 we can see the 6-dilation of this crystal. Notice since $6 \geq 3$ then indeed every vertex has a decomposition of the form $b_{w\lambda} \otimes b' \otimes b_{w'\lambda}$ where $b_{w\lambda}$ and $b_{w'\lambda}$ arise as the right left keys of $\mathcal{B}(\omega_1 + \omega_2)$.

It was shown in [6, 4] that the Lusztig–Schützenberger involution commutes with Baker's virtualization from C_n into A_{2n-1}. We prove this holds for any virtualization between any classical Lie types. As a consequence, any virtualization map preserves right and left keys and thus embeds type X Demazure crystals and atoms into those of type Y.

Theorem 3.8. *Given a $\mathfrak{g}_X$-crystal $\mathcal{B}$ and Dynkin diagram embedding $\psi : X \to Y$ with virtualization map $\Upsilon : \mathcal{B} \to \mathcal{V} \subset \tilde{\mathcal{B}}$, with $\tilde{\mathcal{B}}$ a $\mathfrak{g}_Y$-crystal, the following holds: (1) $\Upsilon(\xi_{\mathcal{B}}(\mathcal{B})) = \xi_{\tilde{\mathcal{B}}}(\Upsilon(\mathcal{B}))$, (2) $K^+(\xi(b)) = \xi K^-(b)$, and (3) $\Upsilon(K^{\pm}(b)) = K^{\pm}(\Upsilon(b))$. Thus, virtualization embeds Demazure crystals and atoms correspondingly, so that for any $w \in W^X$ we have $\mathcal{B}_w(\lambda) \overset{\Upsilon}{\hookrightarrow} \tilde{\mathcal{B}}_{\psi(w)}(\psi(\lambda))$ and $\mathring{\mathcal{B}}_w(\lambda) \overset{\Upsilon}{\hookrightarrow} \mathring{\tilde{\mathcal{B}}}_{\psi(w)}(\psi(\lambda))$.*

As mentioned in the introduction, Theorem 3.8 provides an effective combinatorial framework for computing both the Lusztig–Schützenberger involution as well as the right and left key maps for any given vertex $b \in \mathcal{B}(\lambda)$ where $\mathfrak{g}$ is any classical Lie algebra. Namely, given any virtualization $\Upsilon : \mathcal{B} \to \tilde{\mathcal{B}}$ with a well defined inverse $\Upsilon^{-1} : \Upsilon(\mathcal{B}) \to \mathcal{B}$, then if $\tilde{\mathcal{B}}$ is endowed with a combinatorial model in which $\xi(b)$ as well as $K^+(b)$ can be computed then by Theorem 3.8, in order to compute $\xi(b)$ as well as $K^+(b)$ and $K^-(b)$ for any $b \in \mathcal{B}(\lambda)$, it suffices to compute the virtualization map Υ, perform the combinatorial computations in $\tilde{\mathcal{B}}$, and apply the inverse map Υ^{-1}. In particular, this implies that any such computation in non-simply laced Lie types or Lie-types with complicated combinatorial models, can be computed within simply-laced Lie algebras with easier combinatorial structures (provided such a virtualization exists). In the following section we exemplify this to derive a new algorithms to compute keys and evacuation in type B.

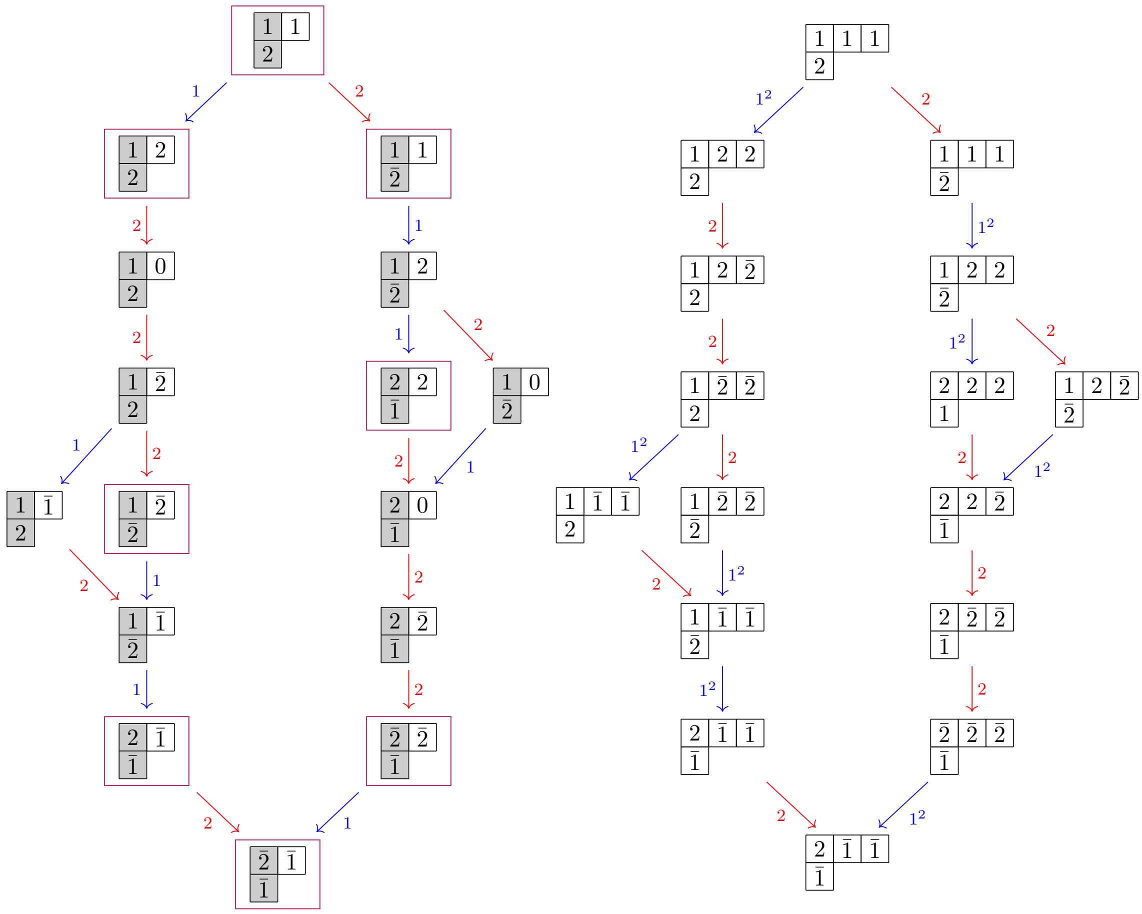

FIGURE 1. (Left) The $\mathfrak{so}_5$-crystal $\mathcal{B}(\omega_2+\omega_1)$ and (Right) the virtual $\mathfrak{sp}_4$-crystal $\Upsilon(\mathcal{B}(\omega_2+\omega_1))$ under the map $\Upsilon = \text{split}$.

4. KEYS AND EVACUATION IN TYPE B

Kashiwara and Nakashima [25] introduced the so-called *Kashiwara-Nakashima (KN) tableaux* and proved these provided a combinatorial model for crystal graphs in types B/C. We recall this construction but refer the reader to [7, 28, 18] for complete details.

The set of ***type*** B_n ***column KN tableau***, $\mathrm{KN}_n^B(1^k)$, consists of fillings C of shape (1^k) for some $1 \le k \le n$ with entries in $\{1 \prec \cdots \prec n \prec 0 \prec \bar{n} \cdots \prec \bar{1}\}$ such that (1) all entries in C are strictly increasing and non-repeating (except for 0) and (2) if both z and $\bar{z}$ appear in C with z in the p^{th} box from the top and $\bar{z}$ in the q^{th} box from the bottom, then $p+q \le z$. Similarly, the set of ***spin KN tableaux***, $s\mathrm{KN}_n^B$, consists of fillings S of shape (1^n) with entries in $\{1 \prec \cdots \prec n \prec \bar{n} \cdots \prec \bar{1}\}$ such that (1) all entries in S are strictly increasing and (2) S contains no pairs $(z, \bar{z})$ for any value of z. We denote spin columns as gray shaded tableaux.

Consider the set of elements $\{z_1, \ldots, z_s\} \subset \{1, \ldots n\}$ consisting of pairs $(z, \bar{z})$ where both $z, \bar{z}$ appear in C, indexed such that $z_{i+1} \prec z_i$. For each $1 \le i \le s$ recursively

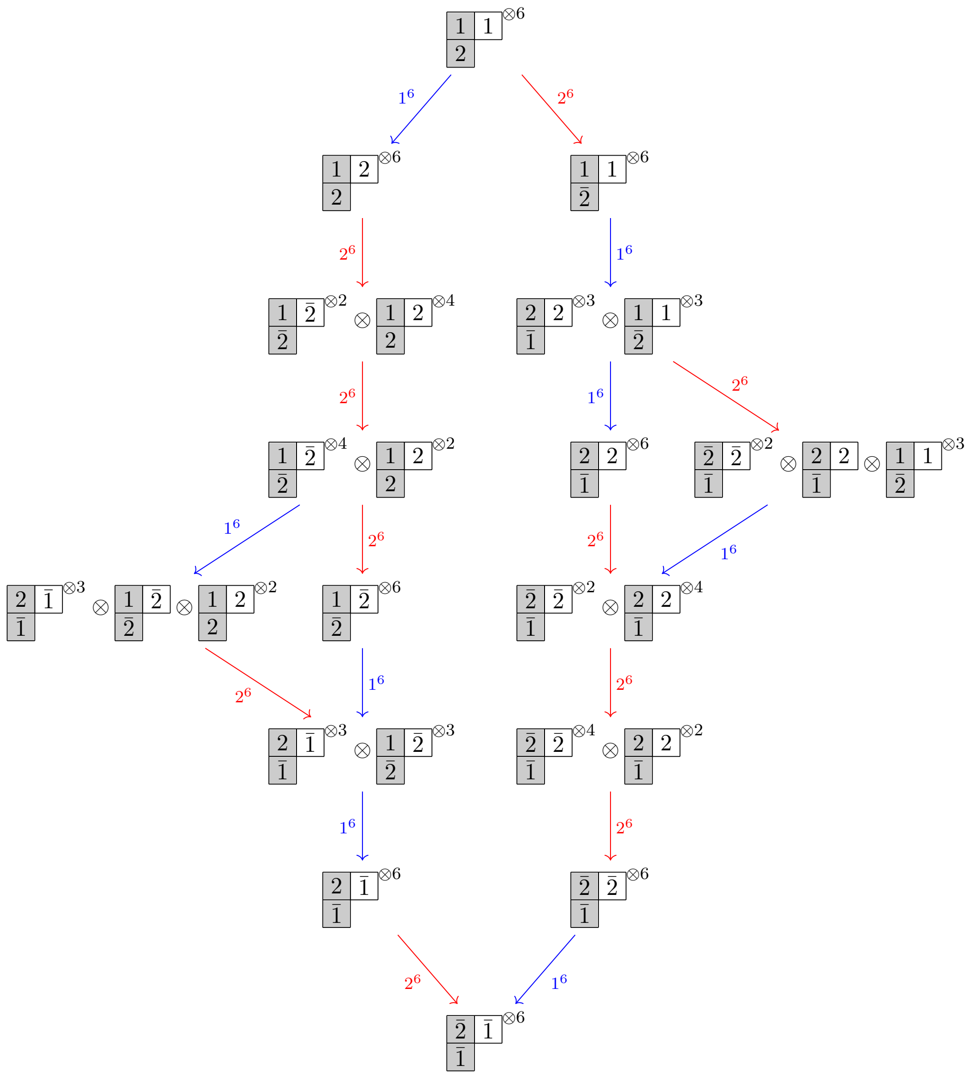

FIGURE 2. The 6-dilation of the $\mathfrak{so}_5$-crystal $\mathcal{B}(\omega_1+\omega_2)$ seen in Figure 1.

construct the set $J(C) := (t_1, \cdots, t_s)$ by setting $t_1 := \max\{t \in \mathsf{B}_n : t \prec z_1, t \notin C\}$, and thereafter $t_i := \max\{t \in \mathsf{B}_n : t \notin C, \bar{t} \notin C, t \prec t_{i-1}, t \prec z_i\}$.

Definition 4.1. Given $C \in \mathrm{KN}_n^B(1^k)$ let rC and lC be the columns obtained from C by replacing $\bar{z}_i$ with $\bar{t}_i$ and z_i with t_i, respectively. The ***splitting*** $\mathrm{split}(C) := lCrC$ of C is the tableau of shape (2^k) with entries lC in the left column and rC in the right column. If instead $C \in s\mathrm{KN}_n^B$, then $\mathrm{split}(C)$ is the tableau in $\mathrm{KN}_n^B(1^n)$ with the same entries as C.

Example 4.2. Let $n = 9$ and $C = (246900\bar{9}\bar{4}\bar{2})^t$. Then $(z_1, \dots, z_5) = (0, 0, 9, 4, 2)$, $J(C) = (8, 7, 5, 3, 1)$, $rC = (135678\bar{9}\bar{4}\bar{2})^t$ and $lC = (2469\bar{8}\bar{7}\bar{5}\bar{3}\bar{1})^t$, with $\mathrm{split}(C)$ obtained by concatenating them. If instead $C = \begin{array}{|c|}\hline 1\\\hline \bar{2}\\\hline\end{array} \in s\mathrm{KN}_2^B$, then $\mathrm{split}(C) = \begin{array}{|c|}\hline 1\\\hline \bar{2}\\\hline\end{array} \in \mathrm{KN}_2^B(1^2)$.

Definition 4.3. The set of ***type B_n KN tableaux of shape λ***, $\mathrm{KN}_n^B(\lambda)$, consists of fillings $\mathfrak{C}|T$ of partition shape $\lambda = (\mu_0|\mu)$ with μ_0 either empty or (1^n) and $\mu^t = (\mu_1, \dots, \mu_\ell)$, such that for C_i, the i^{th} column of T, we have: (1) $\mathfrak{C} \in s\mathrm{KN}_n^B$, (2) $C_i \in \mathrm{KN}_n^B|\mu_i|$, (3) every row of $\mathfrak{C}|T$ is weakly increasing with no repeated zeros, and (4) $\mathrm{split}(\mathfrak{C}|T) := \mathrm{split}(\mathfrak{C})\ \mathrm{split}(C_1) \cdots \mathrm{split}(C_\ell)$ is a semistandard tableau. ***Type C_n KN tableaux*** are be defined similarly by restricting all entries to nonzero values.

The key point to our construction is the observation that *the splitting map is a virtualization map from type B_n to type C_n crystals.*

Theorem 4.4. *For any $\mathfrak{C}|T \in \mathrm{KN}_n^B(\lambda)$ with $\lambda = (\mu_0|\mu)$, then:*

(1) $\mathrm{split}(\mathfrak{C}|T)$ *is a type C_n KN tableau, moreover* $\mathrm{split}(\mathfrak{C}|T) = \Theta_2(\mathfrak{C}|T)$.

(2) $\mathfrak{C}|T$ *is a type B_n key if and only if the columns of* $\mathfrak{C}|T$ *are nested and the letters i and $-i$, for any $i \in \{1 \prec \cdots \prec n \prec 0 \prec \bar{n} \cdots \prec \bar{1}\}$, do not appear simultaneously as entries in a given column.*

(3) $\xi_{\tilde{\mathcal{B}}}(\mathfrak{C}|T) = \mathrm{evac}^B(\mathfrak{C}|T) = \mathrm{split}^{-1} \circ \mathrm{evac}^C \circ \mathrm{split}(\mathfrak{C}|T)$, *where* evac *is the evacuation operator of the appropriate type.*

Example 4.5. In Figure 1 the $\mathfrak{so}_5$-crystal $\mathcal{B}(\omega_1 + \omega_2)$ and its virtualization under the splitting map can be seen.

In particular, if we apply Baker's virtualization of $C_n \hookrightarrow A_{2n-1}$ to the set of tableaux $\mathrm{split}(\mathfrak{C}|T)$, the resulting image coincides exactly with Baker's embedding of $B_n \hookrightarrow A_{2n-1}$ [9] on $\mathfrak{C}|T$.

Example 4.6. Let $\mathfrak{g} = \mathfrak{so}_5$ and suppose $\lambda = \omega_1 + \omega_2$. We can construct the Demazure atom for $w = s_2 s_1$ with vertices in $\mathrm{KN}_2^B(\lambda)$ as follows. From Figure 1(left) we have $b_{w\lambda} = \begin{array}{|c|c|}\hline 1 & \bar{2}\\\hline \bar{2} \\\cline{1-1}\end{array}$. Hence,

$$\mathring{\mathcal{B}}_w(\lambda) = \{\mathfrak{C}|T \in \mathcal{B}(\lambda) : K^+(\mathfrak{C}|T) = b_{w\lambda}\} = \left\{ \begin{array}{|c|c|}\hline 1 & 0\\\hline 2 \\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & \bar{2}\\\hline 2 \\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & \bar{2}\\\hline \bar{2} \\\cline{1-1}\end{array} \right\}. \tag{2}$$

Given any virtualization from B_n into C_n, the right key of $\mathfrak{C}|T$ can be computed by performing symplectic jeu de taquin or via Willis' direct way [36, 37, 6] on $\mathrm{split}(\mathfrak{C}|T)$ and then "unsplitting" (which is a well-defined operation). Alternatively, we can directly apply Baker's virtualization $C_n \hookrightarrow A_{2n-1}$ as executed in [6].

Now, from Figure 2 we can directly read the right or left key of any $\mathfrak{C}|T$. For instance,

$$K^-\left(\begin{array}{|c|c|}\hline 2 & 0\\\hline \bar{1} \\\cline{1-1}\end{array}\right), K^+\left(\begin{array}{|c|c|}\hline 2 & 0\\\hline \bar{1} \\\cline{1-1}\end{array}\right) = \begin{array}{|c|c|}\hline \bar{2} & \bar{2}\\\hline \bar{1} \\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 2 & \bar{2}\\\hline \bar{1} \\\cline{1-1}\end{array}$$

is the key pair of $\mathfrak{C}|T$.

We now use evacuation to relate right and left keys. Let $\mathfrak{T} = \begin{array}{|c|c|}\hline 2 & 0 \\ \hline \bar{1} & \\ \cline{1-1}\end{array} \in \mathrm{KN}_2^B(\lambda)$, one has

$$\xi(\mathfrak{T}) = \mathrm{evac}^B(\mathfrak{T}) = \begin{array}{|c|c|}\hline 1 & \bar{2} \\ \hline 2 & \\ \cline{1-1}\end{array} = \mathrm{split}^{-1} \circ \mathrm{evac}^C \circ \mathrm{split}(\mathfrak{T}) :$$

$$\begin{array}{|c|c|}\hline 2 & 0 \\ \hline \bar{1} & \\ \cline{1-1}\end{array} \xrightarrow{\text{split}} \begin{array}{|c|c|c|}\hline 2 & 2 & \bar{2} \\ \hline \bar{1} & & \\ \cline{1-1}\end{array} \xrightarrow{w_0^C} \begin{array}{|c|c|c|}\hline \bar{2} & \bar{2} & 2 \\ \hline 1 & & \\ \cline{1-1}\end{array} \xrightarrow{\pi\text{-rotation}} \begin{array}{ccc} & & \fbox{1} \\ \fbox{2} & \fbox{$\bar{2}$} & \fbox{$\bar{2}$}\end{array} \xrightarrow{\text{SJDT}} \begin{array}{|c|c|c|}\hline 1 & \bar{2} & \bar{2} \\ \hline 2 & & \\ \cline{1-1}\end{array} \xrightarrow{\text{split}^{-1}} \begin{array}{|c|c|}\hline 1 & \bar{2} \\ \hline 2 & \\ \cline{1-1}\end{array}. \quad (3)$$

Hence it follows from (3) and (2) that

$$K^-\left(\begin{array}{|c|c|}\hline 2 & 0 \\ \hline \bar{1} & \\ \cline{1-1}\end{array}\right) = \mathrm{evac}^{\mathrm{B}}\, K^+\left(\mathrm{evac}^B \begin{array}{|c|c|}\hline 2 & 0 \\ \hline \bar{1} & \\ \cline{1-1}\end{array}\right) = \mathrm{evac}^B\, K^+\left(\begin{array}{|c|c|}\hline 1 & \bar{2} \\ \hline 2 & \\ \cline{1-1}\end{array}\right) = \mathrm{evac}^B \begin{array}{|c|c|}\hline 1 & \bar{2} \\ \hline \bar{2} & \\ \cline{1-1}\end{array} = \begin{array}{|c|c|}\hline 2 & 2 \\ \hline \bar{1} & \\ \cline{1-1}\end{array}.$$

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