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A UNIFORM ACTION OF THE DIHEDRAL GROUP $\mathbb{Z}_2 \times D_3$ ON LITTLEWOOD–RICHARDSON COEFFICIENTS

OLGA AZENHAS, ALESSANDRO CONFLITTI, AND RICARDO MAMEDE

Abstract. We show that the dihedral group $\mathbb{Z}_2 \times D_3$ of order twelve acts faithfully on the set $\mathcal{LR}$, either consisting of Littlewood–Richardson (LR) tableaux, or their companion tableaux, or Knutson–Tao (KT) hives or puzzles, via involutions which simultaneously conjugate or shuffle a Littlewood-Richardson (LR) triple of partitions. The action of $\mathbb{Z}_2 \times D_3$ carries a linear time index two subgroup $H \simeq D_3$ action, where an involution which goes from H into the other coset of H is difficult in the sense that it is not manifest neither exhibited by simple means. Pak and Vallejo have earlier made this observation with respect to the subgroup of index two in the symmetric group $\mathfrak{S}_3$ consisting of cyclic permutations which H extends. The other half LR symmetries, not in the range of the H-action, are hidden and consist of commutativity and conjugation symmetries. Their exhibition is reduced to the action of a remaining generator of $\mathbb{Z}_2 \times D_3$, which belongs to the other coset of H, and enables to reduce in linear time all known LR commuters and transposers to each other, and to the Schützenberger–Luzstig involution. A hive is specified by superimposing the companion tableau pair of an LR tableau, and its $\mathbb{Z}_2 \times D_3$–symmetries are exhibited via the corresponding LR companion tableau pair. The action of $\mathbb{Z}_2 \times D_3$ on KT puzzles, naturally in bijection with Purbhoo mosaics, is consistent with the migration map on mosaics which translates to *jeu de taquin* slides or tableau-switching on LR tableaux. Their H–symmetries are reduced to simple procedures on a KT puzzle via label swapping together with simple reflections of an equilateral triangle, that is, puzzle dualities, and rotations on an equilateral triangle. Finally, the $\mathfrak{S}_3$–symmetries under this action, distributed in the two H-cosets, are consistent with the Thomas-Yong carton rule based on the infusion involution, a specific governance of *jeu de taquin* slides in the tableau switching.

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1. Introduction

This paper aims to fulfill the study by Pak and Vallejo with LR conjugation involutions (LR transposers), which have not been considered in [PV10], and thus to give a complete and uniform picture of the LR symmetries under the action of the dihedral group $\mathbb{Z}_2 \times D_3$. Namely, one shows that all LR transposers known up to date coincide. Furthermore, the LR transposers and LR commutors are linear time reducible to each other, in particular, to Schützenberger-Lusztig involution. This amounts to show the linear cost computational complexity of the involutions exhibiting the H–symmetries which in a KT puzzle consist of rotations and simple reflections on an equilateral triangle, the latter together with the label swapping of the puzzle. This is consistent with the coincidence of LR commutors known up to date [Az17].

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The Berenstein–Zelevinsky (BZ) triangles [BerZel92] give an interpretation of LR coefficients which manifest all $\mathfrak{S}_3$–symmetries except the commutativity, that is, the swapping of two entries in the LR triple (see [BerZel92, Remarks (a)]), and the conjugation symmetry is also hidden in BZ triangles (see [P24] for a recent and comprehensive account of combinatorial interpretations of LR coefficients). Pak and Vallejo have defined in [PV05] bijections, given by explicit *linear maps*, between LR tableaux, Knutson–Tao hives [KT99], and BZ triangles, which combined with the symmetries of BZ triangles give all the $\mathfrak{S}_3$–symmetries except the commutativity. The conjugation symmetry is not considered in their work. As pointed out in [PV05], [PV10, Section 7.6], regarding to the $\mathfrak{S}_3$–symmetries of LR triples, those defined by the index two subgroup R in $\mathfrak{S}_3$, consisting of cyclic permutations, can be given by easily computed involutions in every combinatorial model. The analysis of the symmetries of LR triples in BZ triangles, under the aforesaid action of $\mathbb{Z}_2 \times \mathfrak{S}_3$ [BerZel92, Remarks (a)], suggests that a larger subgroup $H \supseteq R$ of symmetries of LR triples, with index two, in $\mathbb{Z}_2 \times \mathfrak{S}_3$, can be given by easily computed involutions in every combinatorial model. The LR coefficients are preserved in linear time by the action of H. The H–symmetries are manifest in a KT puzzle. They are exhibited via rotations and reflections of the dihedral group D_3 on an equilateral triangle together with the label swapping of the puzzle. These involutions incur to simultaneously transpose and commute entries of the LR triple of partitions. We thus consider the isomorphic group $\mathbb{Z}_2 \times H \simeq \mathbb{Z}_2 \times D_3$ – action on the set $\mathcal{LR}$, either consisting of Littlewood-Richardson (LR) tableaux, or their companion tableaux, or hives, or Knutson-Tao (KT) puzzles.

The symmetries outside of H are linear time reducible either to the commutativity or to the transposition symmetries, and are given by the action of a remaining generator of $\mathbb{Z}_2 \times D_3$ in the other coset of H. More precisely, the other half LR symmetries, realized by the action of the elements in the other coset of H, are hidden and consist of commutativity and conjugation symmetries. They are given by the action of a remaining generator, realized by the *reversal involution* [BSS96] or the Schützenberger-Lusztig involution, which enables to reduce in linear time the bijections exhibiting commutativity and conjugation symmetries to each other. Since all known LR commutors are involutions and coincide, this incurs the coincidence and the involutory nature of all known LR transposers. The LR commutors and the LR transposers are linear time reducible to the Schützenberger-Lusztig involution or to the reversal involution. This amounts to show that the computational complexity of the involutions exhibiting the H-symmetries is of linear cost. This is consistent with the coincidence of LR commutors known up to date [Az17].

Explicit constructions of the corresponding symmetry involutions on companion tableaux of LR skew tableaux are provided as well. To pass from symmetries of LR (skew) tableaux to symmetries of companion tableaux we use the action of the longest element of a symmetric group on a crystal by sending an LR tableau to its reversal, and Lascoux's double crystal graph structure on biwords by relating the so called left and right strings of the double crystal graph [L03]. This analysis also incurs in an explicit relationship between two interlocking Gelfand–Tsetlin (GT) patterns in a hive, that is, the left and the right companion of an LR tableau. Finally, one takes the linear time index two subgroup H action on Knutson-Tao puzzles and Purbhoo mosaics and explains what operations on LR (skew) tableaux they translate to and back.

1.1. **LR coefficients as structure coefficients and symmetries.** Schur functions s_λ where λ runs over all Young shapes (partitions) form a linear $\mathbb{Z}$-basis for the ring Λ of symmetric functions in countably many variables $x_1, x_2, \dots$. The LR coefficients are the structure coefficients in the Schur function product with respect to this basis. The product $s_\mu s_\nu$ in Λ is therefore a non–negative integral linear combination of Schur functions s_λ,

$$s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\,\nu} s_\lambda, \tag{1.1}$$

where the structure constants $c^\lambda_{\mu\,\nu}$ in this Schur basis expansion, depending only on the three partitions μ, ν and λ, are called Littlewood–Richardson (LR) coefficients [LR34, Ma95, Sa01, St01].

Let $0 \le d \le n$ be fixed integers, and fix Young shapes μ, ν, λ contained in the ambient rectangle $D := d \times (n-d)$. Let $c_{\mu\,\nu\,\lambda^\vee} := c^\lambda_{\mu\,\nu}$ where $\lambda^\vee$ is the Young shape defined by the set complement of λ in that rectangle. Henceforth, $c_{\mu\,\nu\,\lambda^\vee}$ is the coefficient of $s_{d\times(n-d)}$ in the Schur expansion of $s_\mu s_\nu s_{\lambda^\vee}$ in Λ, and clearly $c_{\mu,\nu,\lambda}$ is invariant under the shuffling of the partition triple (μ, ν, λ):

$$c_{\mu,\nu,\lambda} = c_{\nu\,\mu\,\lambda} = c_{\mu\,\lambda\,\nu} = c_{\lambda\,\nu\,\mu} \tag{1.2}$$

$$c_{\mu,\nu,\lambda} = c_{\lambda\,\mu\,\nu} = c_{\nu\,\lambda\,\mu}. \tag{1.3}$$

Let λ^t denote the conjugate or transpose of the partition λ with ambient rectangle $(n-d) \times d$. While the $\mathfrak{S}_3$-symmetries (1.2), (1.3) are obvious from the Schur expansion of $s_\mu s_\nu s_{\lambda^\vee}$, it is not the case from the Schur expansion of $s_{\mu^t} s_{\nu^t} s_{\lambda^{\vee t}}$ where $c_{\mu^t\,\nu^t\,\lambda^{\vee t}}$ is the coefficient of $s_{(n-d)\times d}$ in that expansion, that the LR conjugation symmetry,

$$c_{\mu\,\nu\,\lambda} = c_{\mu^t\,\nu^t\,\lambda^t} \tag{1.4}$$

holds. This last symmetry is shown *via* the involutive $\mathbb{Z}$-automorphism ω of the $\mathbb{Z}$–algebra Λ of symmetric functions [Ma95, St01].

On the other hand, LR coefficients enumerate several combinatorial objects depending on three partitions μ, ν and λ, and the combinatorics of their symmetries is rather uniform in the sense that in all combinatorial means the commutativity (1.2) and the conjugation (1.4) symmetries are *hidden*, see [Az98, AzKiTe16, BerZel92, DK08, KT99, PV05, PV10, TeKiA18]. Interestingly, commuting and transposing simultaneously gives, $c^\lambda_{\mu\,\nu} = c^{\lambda^t}_{\nu^t\,\mu^t}$, a symmetry revealed by *simple* means. In Knutson-Tao (KT) puzzles [KTW04], it means the *puzzle duality*, that is, one gets this symmetry *via* the vertical reflection of a puzzle with label swapping. This *simple* involution, denoted ♠, Subsection 5.2, is in turn translated to LR tableaux through *simple* operations in Definition 5.1, as well as to LR companion tableaux (for the definition, see Subsection 2.9) in Algorithm 5.4. The vertical reflection with labelling swapping followed with a clockwise rotation of a KT puzzle by $\pi/3$ and $2\pi/3$ radians gives the two remaining *puzzle dualities* $c_{\mu\,\nu\,\lambda} = c_{\lambda^t\,\nu^t\,\mu^t}$ and $c_{\mu\,\nu\,\lambda} = c_{\mu^t\,\lambda^t\,\nu^t}$. They are again translated to LR tableaux or companion LR tableaux through simple involutions, denoted ♦ and ♣ respectively. (For the definitions, see algorithms 3.4, 3.6, Subsection 5.2, and Definition 5.2.) The three symmetries consisting of puzzle dualities and the three symmetries consisting of puzzle rotations $c_{\mu\,\nu\,\lambda} = c_{\lambda\,\mu\,\nu} = c_{\nu\,\lambda\,\mu}$ (1.3) are exhibited by the faithful action of a two index subgroup $H \simeq \mathcal{D}_3$ (5.2) of the dihedral group $\mathbb{Z}_2 \times D_3$ on KT puzzles.

1.2. **The set $\mathcal{LR}$ and a representation of $\mathbb{Z}_2 \times D_3$ in $Sym(\mathcal{LR})$.** Let $0 \le d \le n$ be fixed integers. Let $\binom{[n]}{d}$ denote the set of binary words consisting of d ones and $n-d$ zeroes. Throughout, a partition is identified with its Young diagram which fits inside, according to the French convention, a non empty rectangle $D := d \times (n-d)$. Our partitions in the ambient rectangle space D are identified with the 01–words in $\binom{[n]}{d}$ as follows: the positions of the zeroes and ones in a 01–word are respectively the positions of the horizontal and vertical steps along the boundary of the corresponding Young diagram, starting in the right lower corner of the rectangle and ending up at the upper left corner. In particular, the empty partition $\emptyset$ is identified with $0^{n-d}1^d$, and D with 1^d0^{n-d}.(See picture (2.1) in Section 2.1).

Given μ, ν, λ be partitions with at most d parts, $\mathrm{LR}^{\lambda}_{\mu,\nu}$ is the set of LR tableaux of shape λ/μ and content ν and its cardinality is $c^{\lambda}_{\mu\,\nu}$. Let $\mathcal{LR}_{d,n}$ be the set of all LR tableaux or KT puzzles of size n, that is,

$$\mathcal{LR}_{d,n} = \bigsqcup_{(\mu,\nu,\lambda)} \mathrm{LR}^{\lambda^\vee}_{\mu,\nu}, \tag{1.5}$$

where $(\mu, \nu, \lambda) \in \binom{[n]}{d}^3 \cup \binom{[n]}{n-d}^3$. (For some partition-triples (μ, ν, λ), we may have $\mathrm{LR}^{\lambda^\vee}_{\mu,\nu} = \emptyset$.) For simplicity we write $\mathcal{LR} := \mathcal{LR}_{d,n}$.

Given $(\mu, \nu, \lambda) \in \binom{[n]}{d}^3 \cup \binom{[n]}{n-d}^3$, the LR coefficients $c_{\mu\,\nu\,\lambda}$ are invariant under the following action of the dihedral group $\mathbb{Z}_2 \times D_3$: the non–identity element of $\mathbb{Z}_2$ transposes simultaneously μ, ν and λ, and the reflections in D_3 swap two entries in the triple (μ, ν, λ). Denoting by τ the non–identity element of the cyclic group $\mathbb{Z}_2$, and by ς_1, ς_2 two swaps or reflections of the dihedral group D_3 of order six, consider $\mathbb{Z}_2 \times D_3$, the dihedral group of order twelve, as the free group generated by the involutions τ, ς_1, ς_2 subject to the relations inherited from $\mathbb{Z}_2$ and D_3 as a Coxeter group, and such that τ commutes with both ς_1 and ς_2,

$$\mathbb{Z}_2 \times D_3 = \langle \tau, \varsigma_1, \varsigma_2 \mid \tau^2 = \varsigma_1^2 = \varsigma_2^2 = (\varsigma_1\varsigma_2)^3 = 1 = (\tau\varsigma_1)^2 = (\tau\varsigma_2)^2 \rangle. \tag{1.6}$$

Let H be the two index subgroup of $\mathbb{Z}_2 \times D_3$, defined by

$$\begin{aligned} H := \ & \langle \tau\varsigma_1, \tau\varsigma_2 \rangle &= \{1, \tau\varsigma_1, \tau\varsigma_2, \varsigma_1\varsigma_2, \varsigma_2\varsigma_1, \tau\varsigma_1\varsigma_2\varsigma_1\} \\ = \ & \langle \tau\varsigma_i, \tau\varsigma \rangle &= \{1, \tau\varsigma_i, \tau\varsigma, \varsigma_i\varsigma, \varsigma\varsigma_i, \tau\varsigma_i\varsigma\varsigma_i\},\ i = 1, 2, \end{aligned} \tag{1.7}$$

where $\varsigma = \varsigma_1\varsigma_2\varsigma_1 = \varsigma_2\varsigma_1\varsigma_2$. The subgroup $H \simeq D_3$ contains the two index cyclic group $R = \langle \varsigma_1\varsigma_2 \rangle = \{1, \varsigma_1\varsigma_2, \varsigma_2\varsigma_1\}$ of D_3. Since H is a subgroup of index 2 in $\mathbb{Z}_2 \times \mathfrak{S}_3$, H is normal, the quotient group $(\mathbb{Z}_2 \times \mathfrak{S}_3)/H$ is cyclic and every element different from the identity is a generator, *i.e.* $\zeta H = H\zeta \neq H$, for $\zeta = \varsigma_1, \varsigma_2, \tau, \varsigma_1\varsigma_2\varsigma_1, \tau\varsigma_2\varsigma_1, \tau\varsigma_1\varsigma_2 \notin H$. Therefore, as a set, $\mathbb{Z}_2 \times D_3 = H \sqcup \zeta H$, for any $\zeta \notin H$, and H affords the following presentations of $\mathbb{Z}_2 \times D_3 \simeq \mathbb{Z}_2 \times H$ useful for our purposes:

$$\langle \tau, \tau\varsigma_1, \tau\varsigma_2 : \tau^2 = (\tau\varsigma_1)^2 = (\tau\varsigma_2)^2 = (\tau\varsigma_1\tau\varsigma_2)^3 = (\tau\varsigma_1\tau)^2 = (\tau\varsigma_2\tau)^2 = 1 \rangle. \tag{1.8}$$

$$\langle \tau, \tau\varsigma_i, \tau\varsigma : \tau^2 = (\tau\varsigma_i)^2 = (\tau\varsigma)^2 = (\tau\varsigma_i\tau\varsigma)^3 = (\tau\varsigma_i\tau)^2 = (\tau\varsigma\tau)^2 = 1 \rangle,\ i = 1, 2. \tag{1.9}$$

$$\langle \varsigma_i, \tau\varsigma_1, \tau\varsigma_2 : \varsigma_i^2 = (\tau\varsigma_2)^2 = (\tau\varsigma_1)^2 = (\tau\varsigma_1\tau\varsigma_2)^3 = (\tau\varsigma_i\varsigma_i)^2 = (\tau\varsigma_j\tau\varsigma_i\varsigma_i)^2 = 1 \rangle,\ 1 \le j \neq i \le 2. \tag{1.10}$$

$$\langle \varsigma, \tau\varsigma_1, \tau\varsigma_2 : \varsigma^2 = (\tau\varsigma_2)^2 = (\tau\varsigma_1)^2 = (\tau\varsigma_1\tau\varsigma_2)^3 = 1 \rangle, \tag{1.11}$$

$$\langle \varsigma_i, \tau\varsigma_i, \tau\varsigma : \varsigma_i^2 = (\tau\varsigma_i)^2 = (\tau\varsigma)^2 = (\tau\varsigma_i\tau\varsigma)^3 = (\varsigma_i\tau\varsigma_i)^2 = (\tau\varsigma\tau)^2 = 1 \rangle,\ i = 1, 2. \tag{1.12}$$

We show that the group $\mathbb{Z}_2 \times D_3$ acts faithfully on the set $\mathcal{LR}$ trough involutions which conjugate or shuffle the entries of a partition-triple (μ, ν, λ). Using the presentation (1.9), the following is an injective group homomorphism where $\mathfrak{S}_{\mathcal{LR}}$ is the group of all bijections from $\mathcal{LR}$ to itself or permutations of $\mathcal{LR}$

$$\begin{aligned} \varpi : \mathbb{Z}_2 \times D_3 &\longrightarrow \mathfrak{S}_{\mathcal{LR}} \\ \tau &\mapsto \varrho \\ \tau\varsigma_1\varsigma_2\varsigma_1 &\mapsto \blacklozenge \\ \tau\varsigma_1 &\mapsto \spadesuit \end{aligned} \tag{1.13}$$

realized by the involutions $\blacklozenge$, $\spadesuit$ and ϱ on $\mathcal{LR}$. More precisely, $\blacklozenge$ is an involution on $\mathrm{LR}(\mu,\nu,\lambda)\sqcup\mathrm{LR}(\lambda^t,\nu^t,\mu^t)$ (3.1) and on the union of KTW puzzles of boundary (μ,ν,λ) and (λ^t,ν^t,μ^t), Subsection 5.2, which agrees with Tao's bijection between KT puzzles and LR tableaux, see Example 5.3; $\spadesuit$ denotes the vertical puzzle duality in Subsection 5.2, and the involution on $\mathrm{LR}(\mu,\nu,\lambda)\sqcup\mathrm{LR}(\nu^t,\mu^t,\lambda^t)$ given by the procedure in Definition 5.1; and $\varrho=\bullet\blacklozenge\,\eta$ in Theorem 4.3, with $\bullet$ the rotation map (2.1), and η the Schützenberger involution or reversal, is an involution in $\mathrm{LR}(\mu,\nu,\lambda)\sqcup\mathrm{LR}(\mu^t,\nu^t,\lambda^t)$.

Observe that for any nonempty partition γ, $\mathrm{LR}(\emptyset,\gamma,\gamma^\vee)$ consists of the sole straight LR tableau of shape and weight γ, $Y(\gamma)$, also called Yamanouchi tableau of shape γ. For any $g\neq h\in\mathbb{Z}_2\times D_3$, $\varpi(g)(T)\neq\varpi(h)(T)$ for some $T\in LR(\epsilon,\delta,\alpha)$ where (ϵ,δ,α) is a shuffle or a shuffle and a transposition of the entries of $(\emptyset,\gamma,\gamma^\vee)$ providing $\gamma\neq\gamma^\vee$. The monomorphism ϖ shows that $\mathbb{Z}_2\times D_3$ is a group of symmetries of LR coefficients regarding to the conjugation and the shuffling of the boundary partition-triple of an element in $\mathcal{LR}$, and thereby

$$\mathbb{Z}_2\times D_3\simeq\langle\spadesuit,\blacklozenge,\varrho\rangle:=\langle\spadesuit,\blacklozenge,\varrho:\varrho^2=\spadesuit^2=\blacklozenge^2=(\spadesuit\blacklozenge)^3=(\spadesuit\varrho)^2=(\blacklozenge\varrho)^2=1\rangle.\quad(1.14)$$

1.3. **The H-action and of a remaining generator of $\mathbb{Z}_2\times D_3$ on $\mathcal{LR}$.** The involutions exhibiting the H-symmetries of LR triples, define a faithfully group action of H on KT puzzles and LR tableaux. As for the computational complexity of our involutions, we study the invariance of LR coefficients, under the action of the two-index subgroup H on the set $\mathcal{LR}$, where

$$H=\langle\tau\varsigma_1,\tau\varsigma_2\rangle=\langle\tau\varsigma_1,\tau\varsigma_1\varsigma_2\varsigma_1\rangle\simeq\langle\spadesuit,\blacklozenge\rangle=\{\spadesuit,\blacklozenge:\spadesuit^2=\blacklozenge^2=\mathbf{1}=(\spadesuit\blacklozenge)^{\mathbf{3}}\}\simeq\mathcal{D}_{\mathbf{3}}.\quad(1.15)$$

The H-invariance of LR coefficients is proved through the exhibition of simple involutions $\spadesuit,\blacklozenge,\clubsuit$ on KT puzzles, called *puzzle dualities*, and simultaneously on LR tableaux. Tao's bijection shows how they do translate to each other. Example 5.3 clearly illustrates this translation for the involution $\blacklozenge$. Puzzle dualities on KT puzzles are the diagonal reflections (linear maps) together with 0 and 1 label swapping. Puzzle dualities $\spadesuit$ and $\clubsuit$ on a LR tableau T is obtained by an hybrid switching: a pair consisting of a row strict Yamanouchi tableau (the transpose of a Yamanouchi tableau) and T that T extends [BSS96, Section 2, p. 22]; and a pair consisting of T and the transpose of a Yamanouchi tableau in the anti-normal form that extends T. In Appendix A, one considers the index two subgroup H action on Knutson-Tao puzzles and Purbhoo mosaics and explains what operations on LR (skew) tableaux they translate to.

The action of the elements in the coset ζH exhibit the *hidden* symmetries consisting of the LR transposer (an involution exhibiting $c_{\mu\,\nu\,\lambda}=c_{\mu^t\,\nu^t\,\lambda^t}$) and the LR commutors (an involution exhibiting $c_{\mu\,\nu\,\lambda}=c_{\epsilon\,\delta\,\theta}$ with (ϵ,δ,θ) obtained by commuting two consecutive entries in (μ,ν,λ)) respectively. (See Section 5.5 for details.)

Recalling that BZ triangles, hives and LR tableaux are related through linear bijections [PV05], we prove that the involutions exhibiting the H-symmetries or equalities

$$c_{\mu\,\nu\,\lambda}=c_{\mu^t\,\lambda^t\,\nu^t},\ c_{\mu\,\nu\,\lambda}=c_{\nu^t\,\mu^t\,\lambda^t},\ c_{\mu\,\nu\,\lambda}=c_{\lambda\,\mu\,\nu}=c_{\nu\,\lambda\,\mu},\ c_{\mu\,\nu\,\lambda}=c_{\lambda^t\,\nu^t\,\mu^t},$$

are intrinsically easy to exhibit in every model under consideration.

While the H-symmetries are easy to exhibit the symmetries under the action of $\varsigma_1H=\varsigma_2H=\varsigma_1\varsigma_2\varsigma_1H=\tau H=\tau\varsigma_2\varsigma_1H=\tau\varsigma_1\varsigma_2H$, *i.e.* the commutativity and the conjugation symmetries, giving the equalities

$$c_{\mu\,\nu\,\lambda} = c_{\nu\,\mu\,\lambda},\ c_{\mu\,\nu\,\lambda} = c_{\mu\,\lambda\,\nu},\ c_{\mu\,\nu\,\lambda} = c_{\lambda\,\nu\,\mu}, \tag{1.16}$$

$$c_{\mu\,\nu\,\lambda} = c_{\mu^t\,\nu^t\,\lambda^t},\ c_{\mu\,\nu\,\lambda} = c_{\lambda^t\,\mu^t\,\nu^t},\ c_{\mu\,\nu\,\lambda} = c_{\nu^t\,\lambda^t\,\mu^t}, \tag{1.17}$$

are difficult to exhibit. The symmetries outside of H are reduced to the action of the elements in the coset ςH where ς is an LR commutor or an LR transposer and exhibit any of the six symmetries in (1.16). The LR commutors and transposers in ςH are therefore linear time equivalent to each other and can be reduced to the Schützenberger–Luzstig involution. The computational complexity analysis is uniform in the aforementioned combinatorial models.

In addition we also exhibit the LR symmetry involutions on LR companion tableaux [Nak05, Appendix, Theorem C]. For this aim we use the Lascoux's double crystal structure on biwords [L03] where for our purposes we adopt Burge correspondence, and observe that reversal involution [BSS96] on LR tableaux can be computed by the action of the longest element of the symmetric group on a type A crystal. In Lascoux's double crystal graph on biwords, one has a left and right structure. The left structure defines a Kashiwara crystal with crystal operators acting on the words defining the second row of the biword. Using the symmetry of Burge correspondence, reordering the billeters of the biword, we get the left structure which is isomorphic as a graph to the left structure: crystal operators on the left are transformed into *jeu de taquin* moves on consecutive rows on the right. Each left string has a right string and therefore one has a double action of the symmetric group through a reflection about the middle of each i-string. Each pair of left and right biwords can be seen as a pair of left and right skew-tableaux . This defines a natural bijection between skew-tableaux and companion tableaux. In particular, the bottom tableau pair of the double crystal graph consists of taking the top tableau pair, an LR tableau with its LR companion tableau, and the double action of the symmetric group on the left and the right respectively, calculates the reversal of the LR tableau by the action of the longest element of the symmetric group, and the anti normal form (*contre-tableau*) by the action of *jeu de taquin* on consecutive rows of the corresponding LR companion tableau.

The LR commutor on LR companion tableaux is computed by the so called combinatorial R-matrix, a crystal isomorphism between $B(\mu)\otimes B(\nu)$ and $B(\nu)\otimes B(\mu)$ where $B(\mu)$ and $B(\nu)$ are the crystals of all semistandard tableaux of shape μ and ν respectively, for a given finite alphabet $[d]$. In type A, the combinatorial R-matrix is realized by several involutions as discussed in [AzKiTe16, Section 12], [TeKiA18], and [Az17] (see also [LenLub15]). To each LR tableau T of shape λ/μ and content ν, we may associate a pair (L_μ, G_ν) of Gelfand-Tsetlin (GT) patterns of shapes μ and ν, and weights $rev(\lambda-\nu)$ and $\lambda-\mu$ respectively [GelZel86, BerZel89]. Gelfand-Tsetlin patterns are naturally in bijection with semi standard tableaux and so that pair is also called left and right LR companion tableau pair of T. (We refer to [AzKiTe16, Section 2.2] for details.) The crystal Littlewood-Richardson rule gives the decomposition $B_\mu\otimes B_\nu \cong \bigsqcup_{\substack{\lambda \\ T\in \mathrm{LR}^\lambda_{\mu,\nu}}} B_\lambda(T) \cong \bigsqcup_{\lambda} B(\lambda)^{c^\lambda_{\mu,\nu}}$, where λ is a partition with at most d parts. Each crystal connected component $B(\lambda)\times\{T\}\cong B_\lambda$ has highest weight element $Y_\mu\otimes G_\nu \cong Y(\lambda)$, and lowest weight element $L_\mu\otimes Y_{rev\nu}\cong Y(rev(\lambda)$, whenever (L_μ, G_ν) is the LR companion pair of T. It will be also convenient to identify a skew shape, the set difference of two nested partitions, with the nonnegative vector of the difference of those two partitions.

A type A version of the R combinatorial matrix is given by the Henriques-Kamnitzer commutor [HeKa06a, HeKa06] realized through the Schützenberger involution E on left companion tableaux L_μ of shape μ and weight $(\lambda/\nu)_{rev}$. That is, $L_\mu \otimes Y(rev\,\nu)$ is the lowest weight element of the crystal connected component with highest element $Y(\mu) \otimes G_\nu \cong Y(\lambda)$ in $B(\mu) \otimes B(\nu)$ if and only if $G_\nu^E \otimes Y(rev\mu)$ is the lowest weight element of the crystal connected component with highest element $Y(\nu) \otimes L_\mu^E \cong Y(\lambda)$ in $B(\nu) \otimes B(\mu)$. Equivalently, (L_μ, G_ν) is a LR companion pair for $\mathcal{LR}(\mu, \nu, \lambda^\vee)$ if and only if (G_ν^E, L_μ^E) is a LR companion pair of $\mathcal{LR}(\nu, \mu, \lambda^\vee)$.

We notice the pair of companion Gelfand-Tsetlin patterns or companion tableaux associated with an LR tableau and the linear map to pass from skew LR tableaux to their left and right companions of normal shape and recall its crystal characterization as highest weight and lowest elements of a crystal respectively.

We show that the LR commutor may be computed directly over the companion tableau G_ν of shape ν and weight λ/μ, to obtain the companion tableau $L_\mu^E = G_\nu^{E\blacklozenge\spadesuit}$ of shape μ and content λ/ν. (We convention that the composition of maps when acting on the right, written on the top right, is read left to right, for instance, $G_\nu^{E\blacklozenge\spadesuit} := \spadesuit\blacklozenge E(G_\nu)$.)

As a consequence of our analysis, we show that the involutions realising the LR symmetries determine, in fact, a faithful action of the dihedral group $\mathbb{Z}_2 \times \mathfrak{S}_3$ on the set of all LR tableaux and KTW puzzles of degree n.

1.4. **Organization and summary of our results.** The rest of this paper is structured into six sections as follows. In the next section we review operations on (skew) Young diagrams, fitting, according to the French convention, the lower left corner of a non empty rectangle D: complement, rotation, transposition and their compositions. Partitions, identified with Young diagrams, along with the mentioned operations, are also translated to $0-1$ words of length defined by the size of D. We recall the definitions of semi standard tableau its LR reading word, Yamanouchi word, the Littlewood-Richardson tableau as a skew semi standard tableau filled with a Yamanouchi word, and the bijection between Yamanouchi words and standard Young tableaux.

We notice the pair of companion Gelfand-Tsetlin patterns or companion tableaux associated with an LR tableau and the linear map to pass from skew LR tableaux to their left and right companions of normal shape and recall its crystal characterization as highest weight and lowest elements of a crystal respectively.

In Section 3 we introduce the linear time bijections rotation and orthogonal transpose (composition of rotation with transposition) on LR tableaux and in Section 4 we study LR transposers coincidence and linear equivalence to an LR commutor. More precisely, we consider only linear time reductions; since the bijections we consider require subquadratic time the reductions have to preserve that. Let $\mathcal{A}$ and $\mathcal{B}$ be two possibly infinite sets of finite integer arrays, and let $\delta : \mathcal{A} \longrightarrow \mathcal{B}$ be an explicit map between them. We say that δ has linear cost if δ computes $\delta(A) \in \mathcal{B}$ in linear time $O(\langle A\rangle)$ for all $A \in \mathcal{A}$, where $\langle A\rangle$ is the bit–size of A. The transposition of the recording matrix of a LR tableau is the recording matrix of a tableau of normal shape. We have then a linear map ι which defines a bijection between tableaux of normal shape and LR tableaux, see [Lee01, Ou05, PV10]. In Subsection 3.1, following the ideas of [PV10] we introduce the ideas of Relative Computational Complexity as linear equivalence of bijections, utilizing what is known as Relative Complexity, an approach based on reduction of combinatorial problems.

The conjugation symmetry map or a LR transposer is a bijection $\varrho : LR(\mu,\nu,\lambda) \longrightarrow LR(\mu^t,\nu^t,\lambda^t)$. Benkart, Sottile, and Stroomer [BSS96] have described conjugation symmetry map by Knuth and dual Knuth equivalence, denoted ϱ^{BSS},

$$\begin{array}{cccc} \varrho^{BSS} : LR(\mu,\nu,\lambda) & \longrightarrow & LR(\mu^t,\nu^t,\lambda^t) \\ T & \mapsto & \varrho^{BSS}(T) = [Y(\nu^t)]_K \cap [(\widehat{T})^t]_d \end{array},$$

where $\widehat{T}$ is the standardization of T, and $[Y(\nu^t)]_K$ is the Knuth class of all tableaux with rectification the Yamanouchi tableau $Y(\nu^t)$ of shape the conjugate of ν, and $[\widehat{T}^t]_d$ is the dual Knuth class of all tableaux of shape $(\lambda^t)^\vee/\mu^t$ with Q–symbol the transpose of $\widehat{T}$. The image of T by the ϱ^{BSS}–bijection is the unique tableau of shape $(\lambda^t)^\vee/\mu^t$ in the intersection of those two equivalence classes. Tableau–switching provides an algorithm to calculate it. In [BSS96], it is observed that the White and the Hanlon–Sundaram maps [Wh90, HaSu92] produce the same result, jointly denoted by ϱ^{WHS}. Thus $\varrho^{BSS}(T)$ can be obtained either by tableau–switching or by the White–Hanlon–Sundaram transformation ϱ^{WHS}.

We explicitly exhibit the Yamanouchi word produced by the conjugation symmetry map ϱ^{BSS} which in its turn leads to a new and very natural version of the same map already considered independently [Az99, Az98, Za96]. A consequence of this latter construction is that using notions of Relative Computational Complexity we are allowed to show that this conjugation symmetry map is linear time reducible to the Schützenberger involution and reciprocally. Thus the Benkart–Sottile–Stroomer conjugation symmetry map with the two other mentioned versions, the various versions of the commutative symmetry map, and Schützenberger involution, are linear time reducible to each other. This answers a question posed by Pak and Vallejo in [PV10]. The column reading word of $\varrho^{BSS}(T)$ is the Yamanouchi word of weight ν^t whose Q–symbol is given by the column reading word of $\widehat{T}^t$. The following transformation ϱ_3 studied in [Az99, Az98, AzCoMa09, AzKiTe16, Za96] makes clear the construction of that word and affords a simple way to construct $\varrho^{BSS}(T)$:

$$\begin{array}{ccc} \varrho_3 : LR(\mu,\nu,\lambda) & \longrightarrow & LR(\mu^t,\nu^t,\lambda^t) \\ T & \mapsto & \varrho_3(T) \\ \text{with word } w & & \text{with column word } (\sigma_0 w)^{*\blacklozenge}, \end{array}$$

where σ_0 is the crystal reflection operator corresponding to the longest permutation of $\mathfrak{S}_{\ell(\nu)}$, and thus $\sigma_0 w$ is the word of the lowest weight element of the crystal containing T; $*$ is the reverse complement (or dualization) map; and $\blacklozenge$ is the operator acting on highest and lowest weight words, that is, Yamanouchi and reverse Yamanouchi words which transforms a (reverse) Yamanouchi word of weight (reverse) ν, into a (reverse) Yamanouchi word of (reverse) weight ν^t, by replacing the subword $(i^{\nu_i - i+1})\, i^{\nu_i}$ with $(\nu_1\, \nu_1 - 1 \cdots \nu_1 - \nu_i + i\ )\ 12 \cdots \nu_i$, for all i. (See subsections 2.4 and 3.3 for definitions.)

More precisely, the $\blacklozenge$ operator is a bijection between the Knuth class of the Yamanouchi tableau $Y(\nu)$ and the Knuth class of the Yamanouchi tableau $Y(\nu^t)$ and similarly between their reverses. The reversal e of a LR tableau T, T^e, can be computed by the crystal reflection operator σ_0 on the skew LR tableau T of shape $\lambda^\vee/\mu$, it sends the highest weight element T of the crystal containing it to the lowest element T^e. Consider the set of skew-tableaux $B(\lambda^\vee/\mu)^\bullet$ of shape $\mu^\vee/\lambda$, as the image of $B(\lambda^\vee/\mu)$ under the map $\bullet : U \mapsto U^\bullet$ where $U^\bullet$ is obtained from U under rotation of the skew–diagram by π radians, with the dualization $*$ of its word. The map $\bullet : B(\lambda^\vee/\mu) \to B(\lambda^\vee/\mu)^\bullet$ is a set bijection preserving the connected components, and $B(\lambda^\vee/\mu)^\bullet$ has a crystal structure by flipping upside down each connected of the crystal $B(\lambda^\vee/\mu)$, reverting the arrows and applying the operation $\bullet$ to the vertices. (In fact these crystals are isomorphic because they have the same multiset of highest weights but the isomorphism is not canonical.)

Consider the set $\mathcal{U}$ of highest and lowest weight elements of $B(\lambda^\vee/\mu)$. The image of $U \in \mathcal{U}$ under the rotation and transposition of the skew–diagram together with the action of the operation $\blacklozenge$ on its word, is denoted by $U^\blacklozenge$. The maps $\bullet$ and $\blacklozenge$ are involutive. (Bijection $\blacklozenge\bullet$ appeared originally in [Za96] with a different formulation.) Then

$$\varrho_3(T) = T^{e\,\bullet\blacklozenge} = T^{\blacklozenge\bullet\,e} = T^{\bullet\blacklozenge\,e} \text{ and}$$

$(\sigma_0 w)^{*\,\blacklozenge} = (\sigma_0 w)^{\blacklozenge\,*} = \sigma_0(w^{\blacklozenge\,*})$ is the column word of $T^{e\bullet\blacklozenge} = [Y(\nu^t)]_K \cap [(\widehat{T})^t]_d$.

Following the ideas introduced in [PV10], we address in § 3.3.1 the problem of studying the computational cost of the conjugation symmetry map ϱ^{BSS} utilizing what is known as Relative Complexity, an approach based on reduction of combinatorial problems. To this aim we use the version ϱ_3.

Let ι be the linear map which sends a LR tableau to its companion (see Subsection 2.9). The maps $\bullet$, e and $\blacklozenge$ can also be translated to companion tableaux. Since the rotation map $\bullet$ is also a linear map, so maps of linear cost, the reversal T^e of a LR tableau T can be linearly reduced to the evacuation E of the corresponding tableau $\iota(T) = P$ of normal shape, *i.e.* $\iota(P^E) = T^{e\,\bullet}$. Additionally, in Algorithm 3.4, it is proved that the bijection $\blacklozenge$, exhibiting the symmetry $c_{\mu\,\nu\,\lambda} = c_{\lambda^t\,\nu^t\,\mu^t}$, is of linear cost. The following commutative scheme shows that the conjugation symmetry map ϱ_3, and therefore ϱ^{BSS} and ϱ^{WHS}, is linear equivalent to the Schützenberger involution or evacuation map on tableaux of normal shape. Let ϱ denote an LR transposer, then

Theorem 1.1. *The following commutative scheme holds*

$$\begin{array}{ccccccc}
T & \overset{e}{\longleftrightarrow} & T^e & \overset{\bullet}{\longleftrightarrow} & T^{e\bullet} & \overset{\blacklozenge}{\longleftrightarrow} & T^{e\bullet\blacklozenge} = \varrho T \\
\iota \updownarrow & & \iota \updownarrow & & \iota \updownarrow & & \iota \updownarrow \\
P & \underset{\mathrm{a}}{\longleftrightarrow} & P^{\mathrm{a}} & \underset{\bullet}{\longleftrightarrow} & P^{\mathrm{a}\bullet} = P^E & \longleftrightarrow & P^{E\blacklozenge} = \varrho P.
\end{array}$$

Theorem 1.2. *The conjugation symmetry maps ϱ^{BSS}, ϱ^{WHS} and ϱ_3 are identical, and linear time equivalent with the Schützenberger involution E and with the reversal map e.*

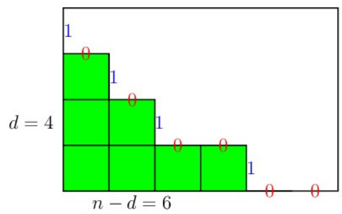
In Section 5 we study the $\mathbb{Z}_2 \times \mathfrak{S}_3$–symmetries and the subgroup $\mathcal{H}$ of KT puzzle dualities and rotations, and in Section 6 the action of $\mathbb{Z}_2 \times \mathfrak{S}_3$ on LR companion pairs. Finally in Appendix A we encompass our analysis with migration in Puhrboo mosaics.

Part of this work appears as an extended abstract [AzCoMa09] *in the FPSAC 09 proceedings and in the preprint* [AzCoMa09b]. *This paper was intended to be the full version.*

2. Preliminaries

2.1. **Young (skew) diagrams and linear transformations.** Throughout we fix a non empty rectangle D of size $d \times (n-d)$, $n > d > 0$ as an ambient space. A *partition* (or straight shape, normal shape or normal form) λ is a finite weakly decreasing sequence of non–negative integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d)$, with at most d parts (positive entries), $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d \geq 0$. We assume $\lambda_1 \leq n-d$. The number of parts is the *length* $\ell(\lambda) \leq d$, and the *weight* is $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_d \leq n$. We use lower case Greek letters such as λ to represent partitions. Often we drop the commas and parentheses when writing a partition. For instance, 2210 is the partition $(2, 2, 1, 0)$. We think of $\mathbb{Z} \times \mathbb{Z}$ as consisting of boxes, and number the rows and columns of $\mathbb{Z} \times \mathbb{Z}$ so that rows number increase bottom to top and columns number left to right. Compass directions are defined as usual according to the canonical basis of $\mathbb{R}^2$. The Young diagram of λ (normal shape) is the collection of boxes $\{(i,j) \in \mathbb{Z}^2 | \ 1 \le i \le d, 1 \le j \le \lambda_i\}$ in French convention. We do not make distinction between a partition λ and its Young diagram whose diagram fits inside, according to the French convention, the lower left corner of the rectangle D anchored at the origin of $\mathbb{Z} \times \mathbb{Z}$. In particular, we regard this rectangle D as a Young diagram with d parts of size $n-d$, and hence all our Young diagrams are subsets of the Young diagram D.

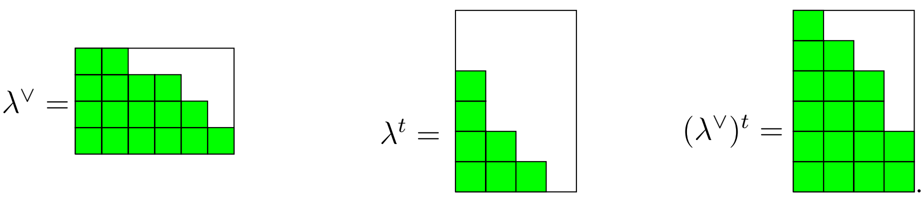
We let $\binom{[n]}{d}$ denote the set of 01–words of length n, with d 1's and $n-d$ 0's. There is a natural action of the symmetric group $\mathfrak{S}_n$ on this set. In particular, the longest permutation or *reverse permutation* rev acts on $\binom{[n]}{d}$ by reversing the words. Our partitions in D are identified with the 01–words in $\binom{[n]}{d}$ as follows: the positions of the zeroes and ones in a 01–word are respectively the positions of the horizontal and vertical steps along the boundary of the corresponding Young diagram, starting in the right lower corner of the rectangle and ending up at the upper left corner. In particular, the empty partition $\emptyset$ is identified with $0^{n-d}1^d$, and D with 1^d0^{n-d}. In the example below, with $d=4$, $n=10$, the partition $\lambda = 4210$, depicted in green, is identified with the 01–word $0010010101 \in \binom{[10]}{4}$,



(2.1)

Reverting each word in $\binom{[n]}{d}$ gives the complement of each partition in D. The *complement* of $\lambda = (\lambda_1, \dots, \lambda_d)$ is the partition $\lambda^\vee = (n-d-\lambda_d, \dots, n-d-\lambda_1)$. Equivalently, rotate by π radians, the set complement of λ in D, and put it against to the lower left corner to obtain the normal shape of $\lambda^\vee$. Since reversing is an involution on $\binom{[n]}{d}$, $(\lambda^\vee)^\vee = \lambda$. In particular, $\emptyset^\vee = D$ and $D^\vee = \emptyset$. Considering λ in (2.1), the left hand side of (2.2), gives the complement of λ in (2.1), $\lambda^\vee = 6542 = 1010100100$ the reverse of the 01–word of λ in (2.1).

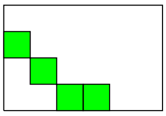
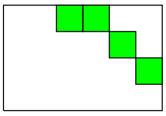
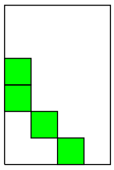
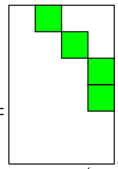
Indeed $\binom{[n]}{d}$ and $\binom{[n]}{n-d}$ are in bijection. The transposition operation on partitions realizes such a bijection. The *transpose* (or *conjugate*) of the partition λ is the partition λ^t obtained by reflecting λ about the line $y=x$. The *transpose* (or *conjugate*) of a partition λ in D, λ^t, is given by the 01–word with $n-d$ 1's and d zeroes, obtained from λ by reversing its 01 word and swapping zeroes and ones. In particular, D^t denotes the $(n-d) \times d$ rectangle, and λ^t is contained D^t. Since transposing is an involution on the set $\binom{[n]}{d} \cup \binom{[n]}{n-d}$, $(\lambda^t)^t = \lambda$. Considering λ in (2.1), the middle picture in (2.2), illustrates $\lambda^t = 321100 = 0101011011$. The *complement transpose* (or *transpose complement*) of λ, $\lambda^{\vee t} = \lambda^{t\vee}$, is the complement of the transpose (or the transpose of the complement) of λ, identified with the 01–word word obtained from λ by swapping zeroes and ones. The right hand side of (2.2) illustrates $\lambda^{t\vee} = 1101101010$ depicted in green.



(2.2)

A partition μ is said to be contained in a partition λ if the Young diagram of μ is contained in the Young diagram of λ. In this case, one defines the *skew shape* (or skew partition) λ/μ to be the set $\{(i,j) \in \mathbb{Z}^2 | \; (i,j) \in \lambda, (i,j) \notin \mu\}$ of boxes in the Young diagram of λ that remains after one removes those boxes corresponding to μ. It is convenient to identify λ/μ with the nonnegative vector $\lambda - \mu$. When μ is the null partition $\emptyset$, the skew–diagram λ/μ equals the normal shape λ. The number of boxes in λ/μ is $|\lambda/\mu| = |\lambda| - |\mu|$. The *antinormal shape* (or *antinormal form*) of λ is the skew shape $D/\lambda^\vee$. Equivalently, λ is π radians-rotated and placed against the upper right corner of D. We also think of it as the reverse of λ fitting the upper right corner of D.

The *transpose* (conjugate) of λ/μ is defined to be $(\lambda/\mu)^t := \lambda^t/\mu^t$, or as the image of the image of λ/μ under the linear transformation $(i,j) \mapsto (j,i)$. The *rotate* of λ/μ, $(\lambda/\mu)^\bullet$, is the image of λ/μ under the linear transformation $(i,j) \mapsto (d-i+1, n-d-j+1)$. Equivalently $(\lambda/\mu)^\bullet = \mu^\vee/\lambda^\vee$. In particular, $(D/\mu)^\bullet = \mu^\vee$, and $\lambda^\bullet := (\lambda/\emptyset)^\bullet = D/\lambda^\vee$ is the anti-normal shape of λ and we think of $\lambda^\bullet$ as the reverse of λ. The *orthogonal transpose* or the *rotate transpose* is the composition of the transposition and the rotation maps $\bullet t = t \bullet$. The *rotate transpose shape* $(\lambda/\mu)^{\bullet t} = (\mu^\vee)^t/(\lambda^\vee)^t = (\mu^t)^\vee/(\lambda^t)^\vee = (\lambda/\mu)^{t\bullet}$ is then the image of λ/μ under the linear transformation $(i,j) \mapsto (n-d-j+1, d-i+1)$. In particular, $(D/\mu)^{\bullet t} = \mu^{\vee t}$ and $\lambda/\emptyset)^{\bullet t} = D^t/(\lambda^t)^\vee$ is the anti-normal shape of λ^t. For instance, if $\mu = 21$ and $\lambda = 4210$ as above, we have

$\lambda/\mu =$  $(\lambda/\mu)^\bullet =$  $(\lambda/\mu)^t =$  $(\lambda/\mu)^{t\bullet} =$  . (2.3)

2.2. **Tableaux and Littlewood-Richardson tableaux.** A (*semistandard*) *Young tableau* T of shape λ/μ on the alphabet $[n]$, is a filling of the boxes of the skew diagram λ/μ with positive integers such that the entries are strictly increasing in each column from bottom to top, and weakly increasing in each row from left to right. When μ is the empty partition we say that T has normal shape λ. The (*row reading*) *word* $w(T)$ of a Young tableau T is the sequence of positive integers obtained by reading the entries of T right–to–left, the rows of T, from bottom to top. The *column word* $w_{col}(T)$ is the word obtained by reading the entries of T, from right to left along each column, starting in the rightmost column and moving upwards. The nonnegative vector $m = (m_1, \ldots, m_n)$ is said to be the *content* (or *weight*) of T if it is the content (weight) of its word, that is, m_k is the number of k's in T. Denote by $YT(\lambda/\mu, m)$ the set of Young tableaux of shape λ/μ and content m. If $\mu = \emptyset$ then we write $YT(\lambda, m)$ to indicate the Young tableaux of normal shape λ and weight m.

Let δ_r denote the *staircase shape* partition $\delta_r = (r, r-1, \ldots, 1)$. A word w of length r can be identified with the *diagonally-shaped* tableau with word w, a semistandard tableau of shape δ_r/δ_{r-1} with row reading word w.

For any skew diagram λ/μ a *Littlewood–Richardson* (LR) tableau is a Young tableau of shape λ/μ such that any prefix of its word contains at least as many letters i as letters $i+1$, for all i. A word such that every prefix satisfies this property is called a *lattice permutation, ballot* or *Yamanouchi.* Its content is always a partition. The column word of an LR–tableau is also a Yamanouchi word of the same content. When $\mu = 0$ we get the the LR tableau of straight shape ν or the *Yamanouchi tableau* $Y(\nu)$, the unique tableau of shape and weight ν, that is, the tableau of shape ν where each row i

is filled with ν_i i's. from right to left along each column, starting in the rightmost column and moving upwards See Example 2.1.

For a given finite alphabet, say $[r]$, a word is said to be *opposite* or *anti-Yamanouchi* (*ballot*) if every suffix contains at least as many letters i as letters $i-1$, for all $i \leq r$. Its content it is always the reverse of a partition. A Young tableau of shape λ/μ whose word is anti-ballot is called an *opposite* (*anti-*) *LR tableau.* When $\mu = 0$, we get the the opposite LR tableau of straight shape ν or the *opposite Yamanouchi tableau* $Y(\nu^\bullet)$, the unique tableau of shape ν and reverse weight $\nu^\bullet$. See Example 2.2.

Given the rectangle D with $\mu, \nu, \lambda \subseteq D$, the *boundary data* of a LR tableau of shape λ/μ and weight ν is $(\mu, \nu, \lambda^\vee)$, and $\mathrm{LR}(\mu, \nu, \lambda^\vee)$ denotes the set of LR tableaux with that boundary data. Let $c_{\mu,\nu,\lambda^\vee}$ denote the cardinal of $\mathrm{LR}(\mu, \nu, \lambda^\vee)$. We define $\mathrm{LR}(\mu, \nu^\bullet, \lambda)$ to be the set of *opposite LR tableaux* of shape $\lambda^\vee/\mu$ and weight $\nu^\bullet$, and $c_{\mu,\nu^\bullet,\lambda}$ denotes its cardinality. In Example 2.1, for $n = 9$ and $d = 3$, T is an LR tableau in $\mathrm{LR}(210, 532, 320)$ with Yamanouchi word $w(T) = 1111221332$.

Example 2.1. Let $n = 9$ and $d = 3$. $T = \begin{array}{|c|c|c|c|c|c|}\hline 2&3&3& & & \\ \hline &1&2&2& & \\ \hline & &1&1&1&1\\ \hline\end{array}$ is an LR tableau with boundary data $(\mu, \nu, \lambda^\vee)$, where $\mu = 210$, $\nu = 532$ and $\lambda^\vee = 320$. Its word is $w(T) = 1111221332$ and its column word is $w_{col}(T) = 1112123132$ both of content the partition $\nu = 532$. The Yamanouchi tableau $Y(\nu) = \begin{array}{|c|c|c|c|c|c|}\hline 3&3& & & & \\ \hline 2&2&2& & & \\ \hline 1&1&1&1&1& \\ \hline\end{array}$.

Example 2.2. An *opposite* LR tableau $S = \begin{array}{|c|c|c|c|c|c|}\hline 3&3&3& & & \\ \hline &2&2&2& & \\ \hline & &1&1&3&3\\ \hline\end{array}$ on the alphabet $[3]$, with the same boundary data, and opposite ballot word $w(S) = 3311222333$ of content $\nu^\bullet = 235$, the reverse partition ν. The opposite Yamanouchi tableau $Y(\nu^\bullet) = \begin{array}{|c|c|c|c|c|c|}\hline 3&3& & & & \\ \hline 2&2&3& & & \\ \hline 1&1&2&3&3& \\ \hline\end{array}$.

The *standard order* of the boxes on a semistandard Young tableau is given by the numerical ordering of the labels with priority, in the case of equality, given by rule northwest=smaller, southeast=larger. A Young tableau with s boxes is *standard* if it is filled with the numbers 1 through s without repetitions. The *standardization* of a semistandard tableau T of content m, denoted by $\widehat{T}$, is the enumeration of the labeled boxes according to the standard order of the boxes in T. The standardization $\widehat{w}$ of a word w is defined accordingly, from right to left. For instance, the standardization of the tableau T above in Example 2.1 is

$$\widehat{T} = \begin{array}{|c|c|c|c|c|c|}\hline 6&9&10& & & \\ \hline &1&7&8& & \\ \hline & &2&3&4&5\\ \hline\end{array},$$

and $\widehat{w(T)} := w(\widehat{T}) = 5432871(10)96$. If $w = w_1 w_2 \dots w_s$ is a word and α is a permutation in the symmetric group $\mathfrak{S}_s$, define $\alpha w = w_{\alpha(1)} \dots w_{\alpha(s)}$. In the case T is standard we have $w_{col}(\widehat{T}) = rev\, w(\widehat{T}^t)$, with rev the longest permutation in $\mathfrak{S}_s$. The transposition of a standard tableau T is still a standard tableau written T^t. If T is a semistandard then T^t means a tableau strictly increasing eastward and weakly increasing northward.

2.3. **The recording matrix of a tableau.** Given the tableau $T \in YT(\lambda/\mu, m)$, where $m = (m_1, \ldots, m_n)$ and $\ell(\lambda) \leq n$, let $M = (M_{ij})$ be the $n \times n$ matrix with non–negative integer entries such that M_{ij} is the number of $j's$ in the ith row of T, called the *recording matrix* of T [Lee01, PV10]. The recording matrix of a Young tableau of normal shape is an upper triangular matrix. Observe that in the case of a non straight shape the recording matrix determines the skew tableau only up to a parallel shift on the skew shape λ/μ, see [Lee01, PV10].

2.4. **The linear involution rotation map $\bullet$ on SSYT's and LR tableaux.** Given the alphabet $[d]$, and an integer i in $[d]$, let $i^\bullet := d-i+1$ the *complement* of i with respect to $[d]$. Given the word $w = w_1 w_2 \cdots w_s$, over the alphabet $[d]$, of weight $m = (m_1, \ldots, m_d)$, $w^\bullet := w_s^\bullet \cdots w_2^\bullet w_1^\bullet$ is the *dual or reverse complement word* of w and $\operatorname{rev} m = (m_d, \ldots, m_1)$, the reverse of m, its weight. Indeed $w^{\bullet\bullet} = w$. We next extend the map $\bullet$ on words to skew tableaux recalling that a word is identified with a diagonally shaped tableau.

Given a Young tableau T of shape λ/μ and weight m and an ambient rectangle, the *rotate* or *dual* of T, $\bullet(T)$, is defined to be the Young tableau of shape $(\lambda/\mu)^\bullet$ and reverse weight $\operatorname{rev} m$, obtained from T by rotating π radians the shape λ/μ while complementing each entry, that is, replacing each entry i with $i^\bullet$. The word of $T^\bullet$ satisfies $w(\bullet(T)) = w(T)^\bullet$, and $\bullet\bullet(T) =$. The *rotation map* is involutive and commutes with standardization $\bullet(\widehat{T}) = \widehat{\bullet(T)}$. It is also a linear map since M is the recording matrix of T if and only if $M^\bullet := P_{\rm rev} M P_{\rm rev}$ is the recording matrix of $\bullet(T)$. Given the finite alphabet $[d]$, the rotation map $\bullet : T \to \bullet(T),\ w(T) \mapsto w(T)^\bullet$ is a linear involution on the set of semistandard Young tableaux over $[d]$, which swaps the inner border with the outer border and reverses the weight.

If w is a Yamanouchi word of weight $\nu = (\nu_1, \ldots, \nu_d)$, write $\nu^t = (\nu_1^t, \ldots, \nu_{\nu_1}^t)$ and observe that w is a shuffle of the words $12\cdots\nu_i^t$ for $i = 1, \ldots, \nu_1$. Similarly, if w is an opposite Yamanouchi word of weight $\nu^\bullet$, w is a shuffle of the words $d - \nu_i^t + 1 \cdots d-1d$, for $i = 1, \ldots, \nu_1$. Thus, fixing such a shuffle for w, we may obtain $w^\bullet$, in the Yamanouchi case, by first replacing, for each $i = 1, \ldots, \nu_1$, the word $12\cdots\nu_i^t$ in the shuffle of w with $dd{-}1\cdots d{-}\nu_i^t{+}1$ and then reversing the resulting word; and, in the opposite Yamanouchi case, by first replacing, for each $i = 1, \ldots, \nu_1$, the word $d - \nu_i^t + 1 \cdots d-1d$ in the shuffle of w with $\nu_i^t \cdots 21$ and then reversing the resulting word. (The result does not depend on the chosen shuffle.) The dual Yamanouchi word $w^\bullet$, in the former case is a shuffle of the words $d - \nu_i^t + 1 \cdots d-1d$, for $i = 1, \ldots, \nu_1$, that is, an opposite Yamanouchi word, and a Yamanouchi word in the latter. Thus, a word is *opposite Yamanouchi* if and only its *dual word is Yamanouchi*. Therefore, for a skew diagram λ/μ, an *opposite Littlewood–Richardson* (LR) tableau is a Young tableau of shape λ/μ such that its dual is an LR tableau or its word is the dual of a Yamanouchi word. When $\mu = 0$, one obtains the rotate Yamanouchi tableau $\bullet(Y(\nu))$.

The rotation $\bullet$ of $\mathrm{LR}(\mu, \nu^\bullet, \lambda)$, the set of opposite LR tableaux of shape λ/μ and content $\nu^\bullet$, gives $\mathrm{LR}(\lambda, \nu, \mu)$ the set of LR tableaux of shape $(\lambda^\vee/\mu)^\bullet = \mu^\vee/\lambda$ and content ν, and *vice-versa*.

Proposition 2.1. *The rotation map*

$$\bullet : \mathrm{LR}(\mu,\nu,\lambda) \cup \mathrm{LR}(\lambda,\nu^\bullet,\mu) \to \mathrm{LR}(\lambda,\nu^\bullet,\mu) \cup \mathrm{LR}(\mu,\nu,\lambda),\ \mathrm{T} \mapsto \bullet(\mathrm{T}), \tag{2.4}$$

is a linear involution on $\mathrm{LR}(\mu,\nu,\lambda) \cup \mathrm{LR}(\lambda,\nu^\bullet,\mu)$ *that transforms the LR tableau T into its dual $\bullet(T)$, the opposite LR tableau of shape $(\lambda^\vee/\mu)^\bullet$ and content $\nu^\bullet$, and* vice versa. *It exhibits the symmetry $c_{\mu,\nu,\lambda} = c_{\lambda,\nu^\bullet,\mu}$.*

Example 2.3. For $d = 3$ and $n = 7$. Given $\nu = 421$, the Yamanouchi $Y(\nu)$ and the rotate Yamanouchi tableau $\bullet(Y(\nu))$ are displayed below

$$Y(\nu) = \begin{array}{|c|c|c|c|}\hline 3 & & & \\ \hline 2 & 2 & & \\ \hline 1 & 1 & 1 & 1 \\ \hline \end{array}, \quad w = 1111223, \quad \bullet(Y(\nu)) = \begin{array}{|c|c|c|c|}\hline 3 & 3 & 3 & 3 \\ \hline & & 2 & 2 \\ \hline & & & 1 \\ \hline \end{array}, \ w^{\bullet} = 1223333.$$

For $d = 4$ and $n = 8$, $\nu = 4210$,
$$\bullet(Y(\nu)) = \begin{array}{|c|c|c|c|}\hline 4 & 4 & 4 & 4 \\ \hline & & 3 & 3 \\ \hline & & & 2 \\ \hline & & & \\ \hline \end{array}, \ w^{\bullet} = 2334444.$$

2.5. **Jeu de taquin and Burge correspondence.**

2.5.1. *Jeu de taquin.* Whenever partitions $\nu \subseteq \mu \subseteq \lambda$, we say that the shape λ/μ extends the shape μ/ν. An *inside corner* of λ/μ is a box in the diagram μ such that the boxes above and to the right (if any) are not in μ. When a box extends λ/μ, this box is called an *outside corner*. Let T be a (semi standard) Young tableau of shape λ/μ, and let b be an inside corner for T. A *contracting slide*, see [BSS96, Sch63] of T into the box b is performed by moving the empty box at b through T, successively interchanging it with the neighbouring integers to the north and east according to the following rules: (i) if the empty box has only one neighbour, interchange with that neighbour; (ii) if it has two unequal neighbours, interchange with the smaller one; and (iii) if it has two equal neighbours, interchange with that one to the north. The empty box moves in this fashion until it becomes an outside corner. This contracting slide can be reversed by performing an analogous procedure over the outside corner, called an *expanding slide.* This procedure is known as Schützenberger *jeu de taquin.* Performing contracting slides over successive inside corners in μ reduces T to a tableau $\mathrm{rect}(T)$ of normal shape, called the *rectification* or *the normal form* of T. The rectification of T is independent of the particular sequence of inside corners used, [Th77], and so $\mathrm{rect}(T)$ is well defined. When $T \in \mathrm{LR}(\mu, \nu, \lambda)$, $\mathrm{rect}T = Y(\nu)$, and if $T \in \mathrm{LR}(\mu, \nu^{\bullet}, \lambda)$, $\mathrm{rect}T = Y(\nu^{\bullet})$.

Similarly, inside the rectangle D, there exists exactly one tableau of anti-normal shape $T^{\mathrm{a}} :=$ `arectT` produced by the *reverse jeu de taquin* by performing expanding slides over each successive outside corner in D/λ [BSS96], called the *anti-normal form* (or the *contre-tableau* or *anti-rectification*) of T. In particular, if $\mu = 0$, the anti-rectification of T produces $\lambda^{\vee}$. Applying reverse *jeu de taquin* slides to the canonical LR tableau or Yamanouchi tableau $Y(\nu)$ of shape ν inside the D rectangle, we obtain its anti-normal form $\mathrm{arect}Y(\nu)$. As $\mathrm{arect}Y(\nu)$ fits the upper right corner of D, $\mathrm{arect}Y(\nu)$ is the LR tableau of anti-normal shape $D/\nu^{\vee}$ and content ν. For instance,

$$Y(\lambda) = \begin{array}{|c|c|c|c|c|}\hline 3 & & & & \\ \hline 2 & 2 & & & \\ \hline 1 & 1 & 1 & 1 & \\ \hline \end{array} \longleftrightarrow \mathrm{arect}Y(\lambda) = \begin{array}{|c|c|c|c|c|}\hline & 1 & 1 & 2 & 3 \\ \hline & & & 1 & 2 \\ \hline & & & & 1 \\ \hline \end{array}. \qquad (2.5)$$

The (anti) rectification of a word means the (anti) rectification of the diagonally shaped tableau with that word.

2.5.2. *Burge correspondence a variation of RSK.* We consider a *variation of the RSK–correspondence* on a two-line array known as the *Burge correspondence*, see [Bur74], and [Fu97, Appendix A.4.1], where the ordering on the two-line array $W = \left(\begin{smallmatrix} a_1 & a_2 & \dots & a_N \\ b_1 & b_2 & \dots & b_N \end{smallmatrix}\right)$ of positive integers is such that $a_i < a_{i+1}$, or $a_i = a_{i+1}$ and $b_i \geq b_{i+1}$, called *Burge array.* The procedure to transform the biword W into the semistandard tableau pair $(P(W), Q(W))$ of the same shape is the column bump $(\cdots(b_1 \leftarrow b_2)\cdots) \leftarrow b_N$ and place in $a_1, \dots, a_N$

respectively. That is, $Q(W)$ is defined to be the semistandard tableau of shape λ such that if $P(W)$ has shape λ and b_i is inserted in $(\cdots(b_1 \leftarrow b_2)\cdots) \leftarrow b_{i-1}$ to create a node in λ then we fill the node with a_i. As usual, when the first line is the permutation $12\cdots N$ (the standardization of the first row of W), we identify W with the second line word $w := b_1 b_2 \cdots b_N$. In this case, $P(w) = P(W)$ and $Q(w) = \widehat{Q(W)}$. The insertion tableau in the Burge and in the RSK correspondence coincide as follows: the column bumping $(\cdots(b_1 \leftarrow b_2)\cdots) \leftarrow b_N$ is equal to the row bumping $(\cdots(b_N \leftarrow b_{N-1})\cdots) \leftarrow b_1$ giving $P(w)$. Instead of biwords we may consider matrices of nonnegative integers. The biword W may also be described by the $m \times t$ matrix whose (i, j) entry is the number of times $\binom{i}{j}$, $i \in [m]$ and $j \in [t]$, occurs in the array. The Burge correspondence is then a correspondence between matrices A with nonnegative entries and ordered pairs (P, Q) of tableaux of the same shape.

From now on when we refer to the RSK-correspondence we mean the Burge correspondence. Thanks to RSK–correspondence a word w is uniquely determined by a tableau pair $(P(w), Q(w))$ of the same normal shape, where $P(w)$ is the *insertion tableau* obtained by *column insertion* of the letters of w from left to right, and $Q(w)$ standard, called the *Q–symbol* or *recording tableau* of w. Reciprocally, every tableau pair (P, Q) of the same shape with Q standard determines a unique word on the alphabet of P and with same weight. Given the alphabet $[d]$, the RSK correspondence gives a bijection between $\mathsf{words(k)}$ the set of words of length $k \geq 0$ and pairs of tableaux

$$\mathsf{RSK} : \mathsf{words}(k) \longrightarrow \bigsqcup_{|\lambda|=k} \mathrm{SSYT}(\lambda) \times \mathrm{SYT}(\lambda),\ \mathsf{RSK}(\mathrm{w}) = (\mathrm{P}(\mathrm{w}), \mathrm{Q}(\mathrm{w})). \tag{2.6}$$

Insertion can be translated into the language of Knuth elementary transformations on a word [Fu97]. Two words w and v are said *Knuth equivalent* if one can be transformed into another by a sequence of Knuth moves. Two words are Knuth equivalent if and only if they have the same insertion tableau. Each Knuth class is in bijection with the set of all standard tableaux with normal shape given by the unique tableau of normal shape in that Knuth class.

Two tableaux T and R of arbitrary shape are *Knuth equivalent*, written $T \equiv R$, if and only if $P(w(T)) = P(w(R))$. Since row and column words of T are Knuth equivalent, one also has $P(w(T)) = P(w_{col}(T))$ [Fu97]. Schützenberger sliding in a skew tableau T preserves the Knuth class of its word. Thereby, $P(w(T)) = T^{\mathrm{n}}$, and $T \equiv R$ if and only if $T^{\mathrm{n}} = R^{\mathrm{n}}$, i.e. one of them can be transformed into the other with a sequence of contracting and expanding *jeu de taquin* slides. In particular, inside D, there exists exactly one tableau T^{a} of anti-normal shape Knuth equivalent to T. Hence $T \equiv R$ if and only if $T^{\mathrm{n}} = R^{\mathrm{n}}$, equivalently, $T \equiv R$ if and only if $T^{\mathrm{a}} = R^{\mathrm{a}}$. Recall that $T \equiv R$ if and only if $\widehat{T} \equiv \widehat{R}$, and $P(w(\widehat{T})) = \widehat{P(w(T))}$, $Q(w(\widehat{T})) = Q(w(T))$.

2.5.3. *The recording tableau in Burge correspondence and LR tableaux.* Under Burge correspondence, there is a bijection between Burge arrays $\binom{\mathbf{y}}{w}$, where w is a Yamanouchi word of weight ν, and tableau pairs $(Y(\nu), G)$ where G is of shape ν and weight $\mathrm{wt}(\mathbf{y})$. Let $w = w_1 w_2 \cdots w_s$ be a Yamanouchi word of content ν such that $\mathsf{RSK}(\binom{\mathbf{y}}{w}) = (Y(\nu), G)$ for some Burge array $\binom{\mathbf{y}}{w}$, and put the number k in the w_kth row of the diagram ν. The labels of the ith row are the k's such that $w_k = i$, thus its length is ν_i and its shape is ν. We denote this standard tableau of shape ν by $U(w)$. Hence, $\mathsf{RSK}(w) = (Y(\nu), U(w))$ where $U(w) = \widehat{G}$.

Definition 2.1. Given the partition ν, $\mathbb{Y}(\nu)$ *denotes the set of Yamanouchi words of weight* ν.

Any tableau pair $(Y(\nu), P)$ with P standard of shape ν produces a Yamanouchi word of weight ν, and thus $P = U(w)$. Then the map $w \mapsto U(w)$ defines a bijection between Yamanouchi words of content ν and standard Young tableaux with shape ν. Hence $|\mathbb{Y}(\nu)| = |\mathsf{SYT}|$. In Example 2.1, $w = 1111221332$, a Yamanouchi word of content $\nu = 532$, gives

$$U(w) = \begin{array}{|c|c|c|c|c|}\hline 8 & 9 & & & \\ \hline 5 & 6 & 10 & & \\ \hline 1 & 2 & 3 & 4 & 7 \\ \hline \end{array}, \tag{2.7}$$

where the entries of the ith row are the positions of the i's in the word of T (according to the LR numbering).

Let $T \in \mathrm{LR}(\mu, \nu, \lambda^\vee)$. We may associate to T two biwords (or matrices) $W^{\lambda/\mu}$ and W^ν consisting of the same biletters but ordered differently. Consider the words $\mathbf{y} = 1^{\lambda_1-\mu_1} 2^{\lambda_2-\mu_2} \ldots \ell(\lambda)^{\lambda_{\ell(\lambda)}-\mu_{\ell(\lambda)}}$ of weight λ/μ and $\mathbf{x} = 1^{\nu_1}2^{\nu_2} \ldots \ell(\nu)^{\nu_{\ell(\nu)}}$ of weight ν, both of length $|\nu| = |\lambda| - |\mu|$, and put

$$W^{\lambda/\mu} := \left(\begin{smallmatrix} \mathbf{y} \\ w(T) \end{smallmatrix} \right), \qquad W_\nu := \left(\begin{smallmatrix} \mathbf{g} \\ \mathbf{x} \end{smallmatrix} \right) \tag{2.8}$$

where W_ν is a reordering of the biletters of $W^{\lambda/\mu}$ such that $\mathbf{g} = g_1 g_2 \ldots g_{|\nu|}$ satisfies $g_i \geq g_{i+1}$ whenever $x_i = x_{i+1}$. The first, as a matrix, is a lower triangular matrix, and the second, as a matrix, is the transpose of the former thus an upper triangular matrix. See Example 2.5.2. The symmetry of Burge correspondence [Fu97, LLThi02, L03] gives the following result.

Proposition 2.2. *Let* $T \in \mathrm{LR}(\mu, \nu, \lambda^\vee)$ *with the Burge arrays* (2.8). *Then under RSK correspondence one has:*

(a) $W^{\lambda/\mu} \mapsto (Y(\nu), G)$ *and* $W_\nu \mapsto (G, Y(\nu))$ *where* $Q(W_\nu) = Y(\nu) = P(w(T))$ *and* $Q(W^{\lambda/\mu}) = G = P(\mathbf{g})$.

(b) $w(G) = \mathbf{g}$, *that is,* $W_\nu = \left(\begin{smallmatrix} w(G) \\ \mathbf{x} \end{smallmatrix} \right)$.

(c) $\widehat{G} = \widehat{Q(W^{\lambda/\mu})} = Q(w(T)) = U(w(T))$, *that is, the Q-symbol of a Yamanouchi word* w, *with respect to Burge correspondence, is* $U(w)$. *That is,* G *is a semistandard Young tableau of shape* ν *and content* λ/μ, *such that each row* i *tell us in which rows of* T *the* i*'s are filled in.*

Example 2.4. Let T be the LR tableau in Example 2.1. The Burge correspondence gives: $W^{\lambda/\mu} = \left(\begin{smallmatrix} 1^4 & 2^3 & 3^3 \\ & w(T) & \end{smallmatrix} \right) = \left(\begin{smallmatrix} 1&1&1&1&2&2&2&3&3&3 \\ 1&1&1&1&2&2&1&3&3&2 \end{smallmatrix} \right) = \left(\begin{smallmatrix} 4&0&0 \\ 1&2&0 \\ 0&1&2 \end{smallmatrix} \right) \to (Y(\nu), G)$, and $W_\nu = \left(\begin{smallmatrix} & w(G) & \\ 1^5 & 2^3 & 3^2 \end{smallmatrix} \right) = \left(\begin{smallmatrix} 2&1&1&1&1&3&2&2&3&3 \\ 1&1&1&1&1&2&2&2&3&3 \end{smallmatrix} \right) = \left(\begin{smallmatrix} 4&1&0 \\ 0&2&1 \\ 0&0&2 \end{smallmatrix} \right) \to (G, Y(\nu))$ where $Y(\nu) = P(w(T)) = Q(W^\nu)$ where

$$G = \begin{array}{|c|c|c|c|c|c|}\hline 3 & 3 & & & & \\ \hline 2 & 2 & 3 & & & \\ \hline 1 & 1 & 1 & 1 & 2 & \\ \hline \end{array} \tag{2.9}$$

of normal shape $\nu = 532$ and weight $\lambda/\mu = 643 - 210 = 433$. The standardization of G gives $\widehat{G} = U(w(T))$ in (2.7).

2.6. **Evacuation, reverse complementation, rotation and RSK.** Given a tableau T of normal shape, the *Schützenberger evacuation* of T, evacT, is a tableau with the shape and reverse weight of T that can be characterized in different ways: the normal form of the rotation of T, $T^{\bullet\mathrm{n}}$; the insertion tableau of the word $w(T^\bullet) = w(\mathrm{T})^\bullet$ [Fu97];

or the rotation of the anti–normal form $T^{\rm a}$. Thus $\rm evacT = T^{a\bullet} = T^{\bullet n} = P(w(T)^\bullet)$ and $T^{\rm a} := \texttt{arect}(T) = \bullet\texttt{evac}(T)$. Indeed $\rm evac\, evacT = T$. Given the Yamanouchi tableau $Y(\nu)$, its opposite satisfies $Y(\nu^\bullet) = Y(\nu)^{\rm a\bullet} = Y(\nu)^{\bullet\rm n} = \rm evacY(\nu)$. For instance, using Example 2.3

$$Y(\nu)=\begin{array}{|c|c|c|c|}\hline 3 & & & \\ \hline 2 & 2 & & \\ \hline 1 & 1 & 1 & 1 \\ \hline\end{array}\ w = 1111223,\quad Y(\nu)^\bullet=\begin{array}{|c|c|c|c|}\hline 3 & 3 & 3 & 3 \\ \hline & & 2 & 2 \\ \hline & & & 1 \\ \hline\end{array},\ w^\bullet = 1223333,$$

$$Y(\nu)^{\rm a}=\begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 3 \\ \hline & & 1 & 2 \\ \hline & & & 1 \\ \hline\end{array},\quad Y(\nu^\bullet)=\begin{array}{|c|c|c|c|}\hline 3 & & & \\ \hline 2 & 3 & & \\ \hline 1 & 2 & 3 & 3 \\ \hline\end{array} = Y(\nu)^{\rm a\bullet} = Y(\nu)^{\bullet\rm n} = \rm evacY(\nu) = P(w^\bullet).$$

Proposition 2.3. [Duality of Burge correspondence [Fu97, Appendix A 4.1].] *Let w and u be two words. Under Burge correspondence*

(*a*) *The word w corresponds to the tableau-pair (P, Q) if and only if $w^\bullet$ corresponds to* $(\rm evac\, P, evac\, Q)$.

(*b*) *For any tableau T, $w(T)$ corresponds to the tableau-pair $(T^{\rm n}, Q)$ and $w(T)^\bullet = w(T^\bullet)$ to the pair $(T^{\bullet\rm n} = \rm evacT^n, evac\, Q)$.*

(*c*) *$w \equiv u$ if and only if $w^\bullet \equiv u^\bullet$, and $Q(u) = Q(v)$ if and only if $Q(u^\bullet) = Q(v^\bullet)$. Similarly,* $\rm rev\, w$ *corresponds to* (P^t, Q^{Et}).

2.7. **Tableau switching and reversal involution.** In this subsection we recall Haiman's result [Hai92, Theorem 2.13]: a skew tableau is uniquely determined by the skew shape, dual Knuth class and Knuth class (rectification or anti normal form).

Two tableaux T and R of the same shape are said to be *dual Knuth equivalent*, written $T \overset{d}{\equiv} R$, if for some sequence of contracting slides or/and expanding slides that can be applied to one of them, can also be applied to the other, and the sequence of shape changes is the same for both, see [Hai92]. In fact if two tableaux on the same shape have the same shape changes for some sequence of *jeu de taquin* slides they also have the same shape changes for any other. Hence, dual Knuth equivalent tableaux have the same (skew) shape as well as the same shape of their normal forms and the same anti–normal shape of their anti–normal forms. Moreover, two tableaux of the same normal shape or anti–normal shape are dual Knuth equivalent [Hai92, Proposition 2.14]. *Dual Knuth equivalence on tableaux of the same shape* may also be characterized by the Q symbols or recording tableaux of their words in the RSK correspondence: $T \overset{d}{\equiv} R$ if and only if $Q(w(T)) = Q(w(R))$. In addition, either row or column words may be used, $Q(w(T)) = Q(w(R))$ if and only if $Q(w_{col}(T)) = Q(w_{col}(R))$. *Dual Knuth equivalence on words* of the *same length* is defined by identifying two words of the same length with the same Q-symbol. Alternatively, we may identify a word of length r with the the diagonal shape tableau δ_r/δ_{r-1} with that word, where $\delta_i = (i, i-1, \dots, 2, 1)$, for $i = r-1, r$, and apply the definition of dual Knuth equivalence on tableaux of the same shape.

Let S and T be tableaux such that T extends S, that is, the outer border of S is the inner border of T, and consider the set union $\rm S \cup T$. The *tableau switching*, see [BSS96], may be presented as a procedure based on *jeu de taquin* elementary moves, for moving two tableaux past one another, transforming $\rm S \cup T$ into $\rm A \cup B$, where B is a tableau Knuth equivalent to T which extends A, and A is a tableau Knuth equivalent to S. We write $\rm S \cup T \overset{s}{\longrightarrow} A \cup B$. In particular, if S is of normal shape, $\rm A = T^n$, and $S = B^{\rm n}$. Switching of S with T may be described as follows: $\widehat{T}$ is a set of instructions telling where expanding slides can be applied to S. (Similarly, $\widehat{S}$ is a set of instructions telling where contracting slides can be applied to T.) Moreover, switching commutes with standardization. Switching and dual Knuth equivalence are related as in the theorem below. It combines tableau switching [BSS96] with Haiman dual equivalence [Hai92, Corollaries 2.8, 2.9].

Theorem 2.4. [BSS96, Theorem 4.3], [Hai92, Corollaries 2.8, 2.9]. *Let T and U be tableaux with the same shape and dual equivalent and let W be a tableau which extends T (or T extends). If $T \cup W \xrightarrow{s} Z \cup X$ and $U \cup W \xrightarrow{s} Z \cup Y$, then $X \overset{d}{\equiv} Y$. If $W \cup T \xrightarrow{s} Z \cup X$ and $W \cup U \xrightarrow{s} Z \cup Y$, then $X \overset{d}{\equiv} Y$.*

Theorem 2.5. [Hai92, Theorem 2.13]. *Let $\mathcal{D}$ be a dual Knuth equivalence class and $\mathcal{K}$ be a Knuth equivalence class, both corresponding to the same normal shape (that is, the elements of $\mathcal{D}$ rectify to the normal shape of the unique tableau of normal shape in $\mathcal{K}$). Then, there is a unique tableau in $\mathcal{D} \cap \mathcal{K}$ which is the unique tableau in $\mathcal{D}$ that rectifies to the normal shape of the unique tableau of normal shape in $\mathcal{K}$. Tableau switching* $\mathbf{s}$ *allows to compute $\mathcal{D} \cap \mathcal{K}$.*

Algorithm 2.6. [BSS96] Computation of $\mathcal{D} \cap \mathcal{K}$ with $\mathcal{D}$ and $\mathcal{K}$ corresponding to the same normal shape. Let $Q \in \mathcal{D}$ and let $V \in \mathcal{K}$ be the unique tableau with normal shape in this Knuth class (V and Q^{n} have the same normal shape), and W any tableau of normal shape that Q extends:

Step 1. Compute

$$\begin{array}{ccc} W \cup Q & & W \cup X \\ \mathbf{s}\downarrow & & \uparrow\mathbf{s} \\ Q^{\mathrm{n}} \cup Z & \rightarrow & V \cup Z. \end{array} \tag{2.10}$$

Step 2. $X \overset{d}{\equiv} Q$, $X \equiv V$, and $\mathcal{D} \cap \mathcal{K} = \{X\}$ where X is the only tableau in $\mathcal{D}$ that rectifies to V.

In particular, if $\mathcal{K}$ is the Yamanouchi Knuth class given by the normal shape corresponding to $\mathcal{D}$, X is the only LR tableau in $\mathcal{D}$ whose content is the normal shape corresponding to $\mathcal{D}$.

Remark 2.1. In Algorithm 2.6, $\mathcal{D}$ and $\mathcal{K}$ also correspond to the same anti normal shape. If in (2.10) we consider $V^{\mathrm{a}} \in \mathcal{K}$, the anti-normal form of V, and W any tableau with anti-normal shape that extends $Q \in \mathcal{D}$, then $Q \cup W \overset{\mathbf{s}}{\to} Z \cup Q^{\mathrm{a}} \to Z \cup V^{\mathrm{a}} \overset{\mathbf{s}}{\to} X \cup W$ (Q^{a} and V^{a} have the same anti-normal shape) to obtain $\mathcal{D} \cap \mathcal{K} = \{X\}$. Note that $X \overset{d}{\equiv} Q^{\mathrm{a}} \overset{d}{\equiv} Q$ and $X \equiv V^{\mathrm{a}} \equiv V$. Note also that $Q^{\mathrm{a}} = \bullet\mathtt{evac}Q^{\mathrm{n}}$.

Corollary 2.7. *LR tableaux form a complete transversal for the set of dual Knuth equivalence classes. Moreover, the LR coefficient $c_{\mu\nu}^{\lambda}$ counts the number of dual Knuth equivalence classes of tableaux of shape λ/μ whose rectification has shape ν.*

2.7.1. *The reversal involution.*

Definition 2.2. [BSS96] Given a tableau T of any shape, the *reversal* of T, T^e, is defined to be the unique tableau Knuth equivalent to $\mathrm{T}^{\bullet}$ ($\mathrm{T}^{\bullet} \equiv T^{\bullet\mathrm{n}} = \mathrm{evac}\mathrm{T}^{\mathrm{n}}$), and dual Knuth equivalent to T. In other words, T^e is the unique tableau dual equivalent to T that rectifies to the evacuation of T^{n}, that is, $T^{e\,\mathrm{n}} = \mathrm{evac}\mathrm{T}^{\mathrm{n}}$. If T has normal shape, $\mathrm{evacT} = \mathrm{T}^{\mathrm{e}}$.

Algorithm 2.6 calculates $T^e = [\mathrm{evac}\mathrm{T}^{\mathrm{n}}]_{\mathrm{K}} \cap [\mathrm{T}]_{\mathrm{dK}}$ (by abuse of notation we omit the brackets in $\{T^e\}$), where $[\ ]_K$ denotes Knuth class and $[\ ]_{dK}$ dual Knuth class. That is, we choose any W of straight shape that T extends, to form $W \cup T$, we rectify T, using W as a set of *jeu taquin* instructions to obtain $T^{\rm n} \cup Z$, replace $T^{\rm n}$ with evacT to obtain evacT $\cup$ Z. Then by reverse *jeu de taquin* instructed by Z, we obtain $W \cup T^e$. Alternatively, if $\mathsf{RSK}(\mathsf{w}(\mathsf{T})) = (T^{\rm n}, Q))$ and since $\mathsf{RSK}(\mathsf{w}(T^e)) = (\mathrm{evacT^n}, \mathrm{Q})$ then T^e can be calculated as $\mathsf{RSK}^{-1}(\mathrm{evacT^n}, \mathrm{Q}) = \mathrm{w}(\mathrm{T^e})$.

The mapping $T \to T^e$ is called *reversal* and is an involution on the set of SSYT which preserves the shape and reverses the weight. Observe that $T^{e\,e} = [T^{e\,\mathrm{n\,E}}]_K \cap [T^e]_{dK} = [T^{\bullet\mathrm{n\,E}}]_K \cap [T]_{dK} = [T^{\mathrm{n\,EE}}]_K \cap [T]_{dK} = T$.

Corollary 2.8. *Let $T \in \mathrm{LR}(\mu,\nu,\lambda^\vee)$. The reversal of T, $T^e = [Y(\nu^\bullet)]_K \cap [T]_{dK}$ is the only opposite LR tableau in $\mathrm{LR}(\mu,\nu^\bullet,\lambda^\vee)$ dual Knuth equivalent to T. Opposite LR tableaux form another complete transversal for the set of dual Knuth equivalence classes. The LR coefficient $c^\lambda_{\mu\nu} = c^\lambda_{\mu\nu^\bullet}$ also counts the number of dual Knuth equivalence classes of tableaux of shape λ/μ whose anti–normal form has shape $\nu^\bullet$.*

2.8. **Crystals of tableaux and Luzstig involution.** Kashiwara and Nakashima has shown that semistandard tableaux can be arranged into crystals. We recall briefly.

A $\mathfrak{gl}_d$-crystal is a finite set B along with maps $\mathrm{wt} : \mathrm{B} \to \mathbb{Z}^{\mathrm{r}}$ $e_i, f_i : B \to B \cup \{0\}$ for $i = 1,\dots,d$ obeying the following axioms for any $b, b' \in B$,

(i) if $e_i(b) \neq 0$ then $\mathrm{wt}(\mathrm{e_i}(\mathrm{b})) = \mathrm{wt}(\mathrm{b}) + \alpha_{\mathrm{i}}$,

(ii) if $f_i(b) \neq 0$ then $\mathrm{wt}(\mathrm{f_i}(\mathrm{b})) = \mathrm{wt}(\mathrm{b}) - \alpha_{\mathrm{i}}$,

(iii) $b' = e_i(b)$ if and only if $b = f_i(b')$, and

(iv) if $b, b' \in B$ such that $e_i(b) = f_i(b') = 0$ and $f_i^k(b) = b'$ for some $k \geq 0$, then $\mathrm{wt}(\mathrm{b}') = \mathrm{s_i}\mathrm{wt}(\mathrm{b})$

where $\alpha_i = e_i - e_{i+1}$, and s_i is the simple transposition of $\mathfrak{S}_d$, $i = 1,\dots,d-1$. The crystals that we deal with also allow to define length functions $\varepsilon_i, \varphi_i : B \to \mathbb{Z}$ $i = 1,\dots,d-1$,

$$\varepsilon_i(b) = \max\{a : e_i^a(b) \neq 0\}, \quad \varphi_i(b) = \max\{a : f_i^a(b) \neq 0\}.$$

Let $B_d = \{1,\dots,d\}$ be the standard $\mathfrak{gl}_d$-crystal consisting of the words of a sole letter on the alphabet $[d]$ whose coloured crystal graph is

$$1 \xrightarrow{1} 2 \xrightarrow{2} \cdots \xrightarrow{d-1} d-1 \xrightarrow{d-1} d.$$

The Kashiwara raising operators f_i and lowering operators e_i are defined for $i \in I = [d-1]$ as follows: $f_i(i) = i+1$, $f_{d-1}(d) = 0$, and $e_i(i+1) = i$, $e_1(1) = 0$, otherwise, the letters are unchanged. The weight $\mathrm{wt}(\mathrm{b}) = \epsilon_{\mathrm{b}}$, for $b = 1,\dots,d$, the canonical basis of $\mathbb{R}^d$. The highest (lowest) weight element of B_d is the word 1 (d), and the highest (lowest) weight is ϵ_1 (ϵ_d).

The tensor product of crystals allows us to define the crystal $\mathcal{W}_d = \bigsqcup_{k>0} {B_d}^{\otimes k} \sqcup \{\emptyset\}$ of all finite words on $[d]$ where $\emptyset$ is the empty word and the vertex $w_1 \otimes \cdots \otimes w_k \in {B_d}^{\otimes k}$ is identified with the word $w = w_1 \cdots w_k$ of length k on $[d]$. We describe the crystal of ${B_d}^{\otimes k}$ as the crystal structure on the set $\mathsf{word(k)}$ of all words of length k on $[d]$. Following the tensor product rule, the action of the Kashiwara *raising* and *lowering operators* e_i and f_i on w, for $i \in [d-1]$, is given by the *i-signature rule* [KasNak94, Kwo09, BumSch16] which is induced from those operators on B_d. We substitute each i by $+$ and each $i+1$ by $-$, and erase the letter in any other case. Then successively erase any pair $+-$ until all the remaining letters form a word $\mathsf{sign(w)}_i = -^a+^b$. We define $\varphi_i(w) := b$ and $\varepsilon_i(w) := a$. If $a = 0$, $e_i(w) = 0$, and if $a > 0$, e_i changes to i the letter $i+1$ of w corresponding to the rightmost unbracketed $-$ (i.e., not erased), whereas if $b = 0$, $f_i(w) = 0$, and if $b > 0$, f_i changes to $i+1$ the letter i corresponding to the leftmost unbracketed $+$.

The crystal ${B_d}^{\otimes k}$, as a graph, is the union of connected components. The connected components of ${B_d}^{\otimes k}$ are the coplactic classes or dual Knuth classes in the RSK correspondence that identify words with the same recording tableau in $SYT(\lambda)$ for some λ. For each standard tableau $Q \in SYT(\lambda)$ there is an embedding of $SSYT(\lambda)$ in ${B_d}^{\otimes k}$,

$$\mathsf{read}_Q = \mathsf{RSK}^{-1}(.,Q) : SSYT(\lambda) \longrightarrow {B_d}^{\otimes k}.$$

Furthermore, given $w \in {B_d}^{\otimes k}$ there exists some λ and $R \in SYT(\lambda)$ such that $\mathsf{read}_S = w$.

The crystal $B_d(\lambda)$ is defined as the crystal structure on the set $\mathsf{SSYT}(\lambda)$ on the alphabet $d]$ induced by the map read_Q for any $Q \in \mathsf{SYT}(\lambda)$ which does not depend on the choice of Q. We have a crystal isomorphism afforded by RSK correspondence

$${B_d}^{\otimes k} \longrightarrow \bigsqcup_{|\lambda|=k} B(\lambda) \times \mathsf{SYT}(\lambda),\ w \mapsto (P(w), Q)$$

Choose a word w on $[d]$ such that the shape of $P(w)$ is λ. If we replace every word of its coplactic class with its insertion tableau we obtain the crystal of tableaux $B_d(\lambda)$ that has all semistandard tableaux of shape λ on the alphabet $[d]$,

$${B_d}^{\otimes k} \approx \bigsqcup_{|\lambda|=k} B(\lambda)^{|\mathsf{SYT}(\lambda)|} \approx \bigsqcup_{|\lambda|=k} B(\lambda)^{|\mathbb{Y}(\lambda)|}, \tag{2.11}$$

where $\mathbb{Y}(\lambda)$ denotes the set of Yamanouchi words of weight λ. Each connected component of ${B_d}^{\otimes k}$ has a unique highest weight word which is a Yamanouch word and a unique lowest weight word which is the reversal of that Yamanouchi word. The highest weight element of $B(\lambda)$ is the Yamanouchi tableau $Y(\lambda)$, and the lowest weight element $Y(\lambda^\bullet) = \mathrm{evacY}(\lambda)$. Two connected components are isomorphic if and only if they have the same highest (lowest) weight [Kas95]. Two words on $[d]$ belong to the same connected component of $\mathcal{W}_d$ if and only if they are dual equivalent. This means that both words are obtained from the same highest weight word, through a sequence of crystal operators f_i, or one is obtained from another by some sequence of crystal operators f_i and e_j, $i, j \in [d-1]$. Also, two words w_1, w_2 on $[d]$ are Knuth equivalent if and only if they occur in the same place in two isomorphic connected components of $\mathcal{W}_d$, that is, they are obtained from two highest words with the same weight through a same sequence of crystal operators. Crystal operators preserve dual Knuth classes and commute with any admissible sequence of *jeu de taquin* moves.

2.8.1. *Crystal of a skew-tableau.* For $\mu \subseteq \lambda \subseteq D$, let $B(\lambda/\mu)$ be the set of all semistandard tableaux of shape λ/μ on the alphabet $[d]$. The column reading of each tableau in $B(\lambda/\mu)$ embeds it in $B^{\otimes|\lambda|-|\mu|}$ the crystal of words of length $|\lambda| - |\mu|$ on the alphabet $[d]$. From (2.11) it decomposes

$$B(\lambda/\mu) \cong \bigsqcup_{\substack{\nu,\ \ell(\nu)\le d \\ T\in \mathrm{LR}^{\lambda}_{\mu,\nu}}} B(T) \cong \bigsqcup_{\nu,\ \ell(\nu)\le d} B(\nu)^{c_{\mu,\nu,\lambda}},$$

where $B(T)$ is the crystal connected component of $B(\lambda/\mu)$ containing the LR tableau $T \in \mathrm{LR}(\mu, \nu, \lambda^\vee)$ for some partition $\nu \subseteq \lambda$. Each crystal $B(T)$ with $T \in \mathrm{LR}(\mu, \nu, \lambda^\vee)$ is $\mathfrak{gl}_d$-crystal isomorphic to $B(\nu)$. It consists of all tableaux with the same shape as T on the alphabet $[d]$ whose normal forms define the set $B(\nu)$, with highest weight element $Y(\nu) \equiv T$. That is, for each $T \in LR(\mu, \nu, \lambda^\vee)$, $B(T)$ is a dual Knuth class with highest weight element T. Let T^{low} be the lowest weight element of $B(T)$, the unique opposite LR tableau in $B(T)$ Knuth equivalent to $Y(\nu^\bullet)$.

2.8.2. *The rotated crystal graph and the reversal.* Consider the set of skew-tableaux $B(\lambda/\mu)^{\texttt{rotate}}$ of shape $\mu^\vee/\lambda^\vee$ as the image of $B(\lambda/\mu)$ under the map $\texttt{rotate}$, $U \mapsto \texttt{rotate}(U)$, where $\texttt{rotate}(U)$ is obtained from $U \in B(\lambda/\mu$ under rotation by π radians while dualizing its word. The map $\texttt{rotate} : B(\lambda/\mu) \to B(\lambda/\mu)^{\texttt{rotate}}$ is a set bijection preserving the connected components but not a crystal isomorphism. The set $B(\lambda/\mu)^{\texttt{rotate}} = B(\mu^\vee/\lambda^\vee)$ has a crystal structure by flipping upside down each connected component of the crystal $B(\lambda/\mu)$, reverting the arrows and applying the operation $\texttt{rotate}$ to the vertices. If T is the highest weight of a connected component of $B(\lambda/\mu)$ then $\texttt{rotate}(T^{\mathrm{low}}) \equiv Y(\nu)$ and $\texttt{rotate}(T) \equiv \texttt{rotate}(T)^{\mathrm{n}} = Y(\texttt{rotate}(\nu))$ are respectively the highest and the lowest weights elements of a same connected component of $B(\lambda/\mu)^{\texttt{rotate}}$. The crystals are isomorphic $B(\lambda/\mu) \approxeq B(\lambda/\mu)^\bullet$ are isomorphic because they have the same multiset of highest weights but the isomorphism is not canonical. Reversal is a set involution on each connected component of $B(\lambda/\mu)$,

$$e : B(\lambda/\mu) \to B(\lambda/\mu),\ T \mapsto e(T) \equiv \texttt{rotate}(T),$$

is the unique element of the connected component $B(T)$ Knuth equivalent to $\texttt{rotate}(T)$. In particular, evacuation is a set involution on $B(\lambda)$

$$\mathrm{evac} : B(\lambda) \to B(\lambda),\ T \mapsto \mathrm{evac}(T) = \texttt{rotate}(T)^{\mathrm{n}},$$

that is, is the unique element of $B(\lambda)$ Knuth equivalent to $T^\bullet$. Reversal interchanges the lowest and highest weight elements in each connected component.

2.8.3. *The action of the symmetric group on a crystal and partial Schützenberger involutions.* Given the word $w = w_1 \cdots w_k \in {B_d}^{\otimes k}$ with i-signature $-^a+^b = x_{j_1} \cdots x_{j_a} \cdots x_{j_{a+b}}$, we define the crystal (Kashiwara) reflection operator σ_i on w by putting $\sigma_i(i^a\ i+1^b) = i^b\ i+1^a = x'_{j_1} \cdots x'_{j_a} \cdots x'_{j_{a+b}}$ and $\sigma_i(w) = y_1 \cdots y_k$ where $y_j = x'_j$ if $j \in \{j_1, \ldots, j_{a+b}\}$ and $y_j = x_j$ otherwise. The operators σ_i, $i \in [d-1]$ are involutions and define an action of the symmetric group $\mathfrak{S}_d$ on $\mathcal{W}_d$ by acting on its connected components isomorphic to $B(\lambda)$, for some λ. They commute with any meaningful sequence of *jeu de taquin* moves. The subgraph obtained from $B_d(\lambda)$ by erasing all edges of colour $\neq i$ is a disjoint union of i-strings of various lengths

$$\bullet \overset{i}{\dashrightarrow} \bullet \overset{i}{\dashrightarrow} \cdots \bullet \overset{i}{\dashrightarrow} \bullet$$

The operator σ_i is the involution on $B(\lambda)$ which reverses each i-string, that is, $\sigma_i(w)$ is the vertex on the i string of w such that $\varepsilon_i(\sigma_i(w)) = \varphi_i(w)$. It coincides with the action of the partial Schützenberger involution on the alphabet $\{i, i+1\}$ on the i-strings. The i-string is itself a crystal graph with highest weight the top and lowest weight the bottom of the i-string, thus σ_i interchanges the highest with the lowest weight. Each simple transposition $s_i \in \mathfrak{S}_d$ acts in a i-string so that $\mathrm{wt}(\sigma_i \mathrm{w}) = \mathrm{s_i wt(w)}$. Therefore, the action of the longest Weyl group element (in type A, the reverse permutation) on a connected component of $\mathcal{W}_d$ isomorphic to $B(\lambda)$ agrees with the action of the Schützenberger's involution (or evacuation) on Yamanouchi or opposite Yamanouchi tableaux and with the reversal on Yamanouchi or opposite Yamanouchi words. The highest weight element is the Yamanouchi tableau $Y(\nu)$, and the lowest weight element $\sigma_0 Y(\nu) = Y(\nu^\bullet)$, with $\sigma_0 \in \mathfrak{S}_d$. Crystal operators and crystal reflection operators acting on words (for definitions see [LLThi02, LS81, Kwo09, BumSch16]) preserve Knuth equivalence and the Q–symbol, and, henceforth, also the dual Knuth class, when acting on the word of a tableau. Let w be a Yamanouchi word of weight ν, with $\ell(\nu) \le d$, and let σ_i denote the crystal reflection operator (for definitions see [LLThi02, Section 5.5] or [BumSch16, Definition 2.35]) acting on the subword over the alphabet $\{i, i+1\}$, for $1 \le i < d$. The operators σ_i

satisfy the Coxeter relations of the symmetric group. If $\omega_0 := s_{i_N} \cdots s_{i_1}$ is the $\mathfrak{S}_d$ long element, put $\sigma_0 := \sigma_{i_N} \cdots \sigma_{i_1}$. Then $\sigma_0 w$ is the opposite Yamanouchi word of weight $\nu^\bullet$, $\sigma_0 w \equiv w^\bullet \equiv Y(\nu^\bullet)$, and $Q(\sigma_0 w) = Q(w) = U(w)$ (whereas $Q(w^\bullet) = \mathrm{evacQ(w)}$).

For $\mu \subseteq \lambda \subseteq D$, let $B(\lambda/\mu)$ be the set of all semi-standard tableaux of shape λ/μ on the alphabet $[d]$.

$$B(\lambda/\mu) \simeq \bigoplus_{\substack{\nu,\, \ell(\nu)\le d \\ T\in \mathrm{LR}^{\lambda}_{\mu,\nu}}} B(T),$$

where $B(T)$ is the crystal connected component of $B(\lambda/\mu)$ containing the LR tableau $T \in \mathrm{LR}(\mu,\nu,\lambda^\vee)$ for some partition $\nu \subseteq \lambda$. Each crystal $B(T)$ with $T \in \mathrm{LR}(\mu,\nu,\lambda^\vee)$ is $\mathfrak{gl}_d$-crystal isomorphic to $B(\nu)$. That is, $B(T)$ is a dual Knuth class. Since $\sigma_0 T$ is dual Knuth equivalent to T, the lowest weight element of $B(T)$ is the reversal LR tableau $eT = \sigma_0 T \equiv Y(\nu^\bullet)$ in $\mathrm{LR}(\mu,\nu^\bullet,\lambda)$ and $T^{e\,e} = \sigma_0\sigma_0 T = T$.

For LR tableaux we may provide another procedure used in [Az99] to calculate the reversal. This procedure is illuminated by the action of the longest permutation of the symmetric group $\mathfrak{S}_d$ on the crystal $B_d(T)$ of a skew tableau T of shape λ/μ and content ν [Kwo09, BumSch16] over the alphabet $\{1,\dots,d\}$, with $\ell(\nu) \le d$.

2.9. **Left and right LR companion tableaux, crystals and hives.** The *recording matrix M of an LR tableau* $T \in \mathrm{LR}(\mu,\nu,\lambda^\vee)$ is the $d\times d$ lower triangular matrix identified with the LHS of (2.8). Its transposition M^t is the upper triangular matrix identified with the RHS of (2.8). M^t is the recording matrix of the semistandard Young tableau G of shape ν and weight λ/μ, the recording tableau of $\mathsf{RSK}(M) = (Y(\nu), G)$ in Proposition 2.2. The semistandard Young tableau G of shape ν and content λ/μ is such that each row i tell us in which rows of T the i's are filled in, Proposition 2.2, (c), and is called the *right Gelfand-Tsetlin* (GT) pattern, or the *right LR companion tableau* of T. It satisfies $\widehat{G} = U(w(T))$.

A GT pattern, G, is a triangular array of non-negative integers $G = (\nu_j^{(i)})_{1\le j\le i\le d}$ displayed as below (for more details and references therein we refer to [AzKiTe16, TeKiA18]):

$$\begin{array}{ccccccccc}
\nu_1^{(d)} & & \nu_2^{(d)} & & \cdots & & \nu_{d-1}^{(d)} & & \nu_n^{(d)} \\
& \nu_1^{(d-1)} & & \nu_2^{(d-1)} & & \cdots & & \nu_{d-1}^{(d-1)} & \\
& & \cdots & & \cdots & & \cdots & & \\
& & & \nu_1^{(2)} & & \nu_2^{(2)} & & & \\
& & & & \nu_1^{(1)} & & & &
\end{array} \tag{2.12}$$

where entries satisfy the betweenness conditions $\nu_j^{(i+1)} \ge \nu_j^{(i)} \ge \nu_{j+1}^{(i+1)}$ for $1 \le j \le i < d$. The ith row, enumerated from bottom to top, necessarily constitutes a partition $\nu^{(i)}$ of length $\le i$. Such a GT pattern is said to be of *shape* (or *type*) $\nu^{(d)}$ and of weight $\gamma = (\gamma_1,\gamma_2,\dots,\gamma_n)$ where $\gamma_i = |\nu^{(i)}| - |\nu^{(i-1)}|$ for $i = 1,2,\dots,d$, with $|\nu^{(0)}| = 0$. There is a natural bijection between GT patterns and semistandard tableaux of the same shape and weight: a semistandard tableau of shape ν is a nested sequence of partitions $\nu^{(1)} \subseteq \nu^{(2)} \subseteq \cdots \subseteq \nu^{(d)}$ where $\nu^{(i)}$ specify the shape of that part of the semistandard tableau consisting of entries $\le i$, that is to say having $\nu_j^{(i)} - \nu_j^{(i-1)}$ entries i in row j for $1 \le j \le i \le d$.

Definition 2.3. Denote by $\mathrm{LR}_{\nu,\lambda/\mu}$ the set of right LR companion tableaux of $\mathrm{LR}(\mu,\nu,\lambda^\vee)$. The elements of $\mathrm{LR}_{\nu,\lambda/\mu}$ are called right LR companion tableaux of shape ν and content λ/μ.

Thanks to Proposition 2.2, the linear transformation $M \mapsto M^t$, where M is the recording matrix of an LR tableau, defines a linear bijection ι between LR tableaux of boundary data (μ, ν, λ) and their right companions. Let

$$\iota : \mathrm{LR}(\mu, \nu, \lambda^\vee) \to \mathrm{LR}_{\nu,\lambda/\mu},\ T \mapsto \iota(T) = G, \tag{2.13}$$

where G is the semistandard tableau of shape ν and content λ/μ with recording matrix M^t given that T has recording matrix M.

The *left LR companion tableau* (or *left Gelfand-Tsetlin pattern*) of $T \in \mathrm{LR}(\mu, \nu, \lambda^\vee)$ is defined to be the semistandard tableau L of shape μ and content $\mathrm{rev}(\lambda/\nu)$ which records the sequence of partitions giving the shapes occupied by the entries $< r$ in rows $r, r+1, \ldots, d$ of T, for $r = 1, 2, \ldots, d$ (for more details we refer to [AzKiTe16, TeKiA18] and references therein). Given the recording matrix $M = (M_{ij})_{1\le i,j\le d}$ of T, the semistandard tableau L is given by the nested sequence of partitions

$$(\mu_d + \sum_{j=1}^{d-1} m_{dj}) \subseteq \cdots \subseteq (\mu_r + \sum_{j=1}^{r-1} m_{rj}, \ldots, \mu_{d-1} + \sum_{j=1}^{r-1} m_{d-1,j}, \mu_d + \sum_{j=1}^{r-1} m_{d,j}) \subseteq \cdots \subseteq \mu, \tag{2.14}$$

where $\mu = (\mu_1, \ldots, \mu_d)$ and $1 \le r \le d$. This is equivalent to the linear transformation given in [PV10, Proposition 12] to obtain the recording matrix of the LR left companion L of T given its recording matrix.

Definition 2.4. Denote by $\mathrm{LR}^-_{\nu,\lambda/\mu}$ the *set of left LR companion tableaux* of $\mathrm{LR}(\mu, \nu, \lambda^\vee)$.

Clearly the sets $\mathrm{LR}^-_{\mu,\lambda/\nu}$, $\mathrm{LR}_{\nu,\lambda/\mu}$ and $\mathrm{LR}(\mu, \nu, \lambda^\vee)$ are mutually in linear bijection.

Given a semistandard tableau G of shape ν and content $\gamma = \lambda - \mu$ the transpose of its recording matrix determines an LR skew tableau only up to a parallel shift of the skew shape. Given a semistandard tableau G of shape ν and content λ/μ how do we check whether G is in $\mathrm{LR}_{\nu,\lambda/\mu}$? The same question for L of shape μ and content $\mathrm{rev}(\lambda - \nu)$ to be in $\mathrm{LR}^-_{\mu,\lambda/\nu}$.

The elements of $\mathrm{LR}_{\nu,\lambda/\mu}$ are those tableaux (GT patterns) $G = \nu^1 \supseteq \nu^2 \supseteq \cdots \supseteq \nu^d = (\nu_1, \ldots, \nu_d)$ and content λ/μ [GelZel86, Theorem 1] (we refer to [AzKiTe16, Section 2.4] for more details) such that

$$\sum_{k=1}^{i} (\nu_k^{(j)} - \nu_k^{(j-1)}) - \sum_{k=1}^{i-1} (\nu_k^{j-1} - \nu_k^{j-2}) \le \mu_{j-1} - \mu_j,\ 1 \le i < j \le d. \tag{2.15}$$

Alternatively, let $B(\mu)$ and $B(\nu)$ be the crystals of all semistandard tableaux of shape μ and ν, on the alphabet $[d]$, respectively. Let us consider $B(\mu) \otimes B(\nu)$ and their highest weight elements. Given $G \in B(\nu)$ of weight λ/μ, $Y(\mu) \otimes G$ is the highest weight element of weight λ of a connected component, isomorphic to $B(\lambda)$, of $B(\mu) \otimes B(\nu)$ if and only if $\varepsilon_j(G) \le \mu_{j-1} - \mu_j$, for all $1 < j \le d$, where $\varepsilon_j(G) = \max\{k \in \mathbb{Z}_{\ge 0} : e_i^k(G) \ne 0\}$ and e_i is a raising operator [Nak05, Appendix], [Kwo09]. The only if part of the former statement is equivalent to (2.15) holds. Therefore $G \in \mathrm{LR}_{\nu,\lambda/\mu}$ if and only if $Y(\mu) \otimes G$ is a highest weight element of $B(\lambda$ if and only if G is a vertex of $B(\nu)$ of weight λ/μ such that $\varepsilon_j(G) \le \mu_{j-1} - \mu_j$, for all $1 < j \le d$ if and only if (2.15) holds. Equivalently, $\mathrm{LR}_{\nu,\lambda/\mu}$ is the set of semistandard tableaux G of shape ν and content λ/μ satisfying the equation $Y(\mu).G = Y(\lambda)$ where "." refers to the column insertion of G in $Y(\mu)$ [Th78, Kwo09].

The elements of $\mathrm{LR}^-_{\mu,\lambda/\nu}$ are those tableaux (GT patterns) $L = \mu^1 \supseteq \mu^2 \supseteq \cdots \supseteq \mu^d = (\mu_1, \ldots, \mu_d)$ and content $\mathrm{rev}(\lambda/\nu)$ such that [BerZel89] (see also [AzKiTe16, Section

2.4] for details)

$$\sum_{k=j}^{d}(\mu_{k-i}^{(d-i)}-\mu_{k-i+1}^{(d-i+1)})-\sum_{k=j+1}^{d}(\mu_{k-i-1}^{(d-i-1)}-\mu_{k-i}^{(d-i)})\leq\nu_i-\nu_{i+1},\ 1\leq i<j\leq d. \tag{2.16}$$

Similarly, under considerations of the lowest weight elements of $B(\mu)\otimes B(\nu)$, given $L\in B(\mu)$ of weight $\mathrm{rev}(\lambda/\nu)$, $L\otimes Y(\mathrm{rev}\nu)$ is the lowest weight element of weight $\mathrm{rev}\lambda$ of a connected component, isomorphic to $B(\lambda)$, of $B(\mu)\otimes B(\nu)$ if and only if $\varphi_{d-i}(L)\leq\nu_i-\nu_{i+1}$, for $1\leq i<d$, where $\varphi_j(L)=\max\{k\in\mathbb{Z}_{\geq 0}:f_j^k(L)\neq 0\}$ and f_j is a lowering operator [Kwo09]. The only if part of the former statement is equivalent to (2.16). Therefore $L\in\mathrm{LR}^-_{\nu,\lambda/\mu}$ if and only if $L\otimes Y(\mathrm{rev}\nu)$ is a lowest weight element of $B(\lambda$ if and only if L is a vertex of $B(\mu)$ of weight $\mathrm{rev}\lambda/\nu$ such that $\varphi_{d-i}(L)\leq\nu_i-\nu_{i+1}$, for all $1<i<d$ if and only if (2.16) holds. Equivalently, L of shape μ and content $\mathrm{rev}(\lambda/\nu)$ satisfies the equation $L.Y(\mathrm{rev}\nu)=Y(\mathrm{rev}\lambda)$ where "." refers to the column insertion of $Y(\mathrm{rev}\nu)$ in L [Kwo09].

2.9.1. *LR companion pairs and hives.* Thanks to [HeKa06], a pair (L_μ,G_ν) of semistandard tableaux of shapes μ and ν and weights $rev(\lambda/\nu)$ and λ/ν respectively, is said to be a *LR companion pair* of $\mathrm{LR}(\mu,\nu,\lambda^\vee)$ if and only if $L_\mu\otimes Y(\mathrm{rev}\nu)$ and $Y(\mu)\otimes G_\nu$ are the lowest and the highest weight elements of a connected component, isomorphic to $B(\lambda)$, of $B(\mu)\otimes B(\nu)$ (see [AzKiTe16, Subsection 12.1] for more details). That is $Y(\mu).G$ and $L.Y(\mathrm{rev}\nu)$ have the same recording tableau Q (an LR tableau of shape λ/μ and content ν) [Th78, Kwo09]. Alternatively, (L,G) satisfy certain linear equalities which can be expressed by the triangle condition in a hive [AzKiTe16, Section 2.4].

In [PV10, Proposition 12], given $T\in\mathrm{LR}(\mu,\nu,\lambda^\vee)$, it is given a linear transformation [PV10, Proposition 12] to transform recording matrix of T into the left companion. Thereby given a right companion the corresponding left companion can be obtained from that by a linear transformation. Later we give another relation using the action of $\mathfrak{S}_3\times\mathbb{Z}_2$ on $\mathcal{LR}$.

Example 2.5. Consider $T\in\mathrm{LR}(210,532,320)$ in Example 2.1. The recording matrix of T is $M=\begin{pmatrix}4&0&0\\1&2&0\\0&1&2\end{pmatrix}$. Its transposition $M^t=\begin{pmatrix}4&1&0\\0&2&1\\0&0&2\end{pmatrix}$ encodes the right companion tableau

$$G=\begin{array}{|c|c|c|c|c|c|}\hline 3&3&&&&\\\hline 2&2&3&&&\\\hline 1&1&1&1&2&\\\hline\end{array} \tag{2.17}$$

of normal shape $\nu=532$ and weight $\lambda/\mu=643-210=433$. The standardization of G gives $\widehat{G}=U(w(T))$ in (2.7). The left companion of T is given by the sequence of shapes $(1)\subseteq(20)\subseteq(210)$, equivalently,

$$L=\begin{array}{|c|c|c|c|c|c|}\hline &&&&&\\\hline 3&&&&&\\\hline 1&2&&&&\\\hline\end{array} \tag{2.18}$$

of normal shape $\mu=210$ and weight $\mathrm{rev}\lambda/\nu=\mathrm{rev}(643-532)=111$.

2.9.2. *Companion tableau of an opposite LR tableau.* The *recording matrix* M of an *opposite LR tableau* $T\in\mathrm{LR}(\mu,\nu^\bullet,\lambda^\vee)$ is a $d\times d$ upper triangular matrix M. Its transposition M^t, a lower triangular matrix, is the recording matrix of the antitableau H of shape $\nu^\bullet$ and weight λ/μ such that the row i entries of H tell us which rows of T are filled with

i's. The antitableau H is the companion of T. In Example 2.2 the recording matrix of S is $M(S) = \begin{pmatrix} 2&0&2\\0&3&0\\0&0&3 \end{pmatrix}$ and $M(S)^t = \begin{pmatrix} 2&0&0\\0&3&0\\2&0&3 \end{pmatrix}$ with companion tableau $H = \begin{array}{|c|c|c|c|c|}\hline 1&1&3&3&3\\\hline &&2&2&2\\\hline &&&1&1\\\hline\end{array}$

3. The linear rotation and orthogonal transpose maps on LR tableaux and companions

3.1. **Linear reduction and linear equivalence of bijections.** We follow closely [PV10] for this section. Using ideas and techniques of Theoretical Computer Science, see [AHU75, CLRS01], each bijection can be seen as an algorithm having one type of combinatorial objects as *input*, and another as *output*. We define a *correspondence* as a one–to–one map established by a bijection; therefore, obviously several different defined bijections can produce the same correspondence. In this way one can think of a correspondence as a function which is computed by the algorithm, viz. the bijection. The computational complexity is, roughly, the number of steps in the bijection. Two bijections are *identical* if and only if they define the same correspondence. Obviously one task can be performed by several different algorithms, each one having its own computational complexity, see [AHU75, CLRS01]. For example we recall that there are several ways to multiply large integers, from naive algorithms, e.g. the Russian peasant algorithm, to that ones using FFT (Fast Fourier Transform), e.g. Schönhage–Strassen algorithm; see e.g. [GG03] for a comprehensive and update reference. Formally, a function f reduces linearly to g, if it is possible to compute f in time linear in the time it takes to compute g; f and g are linearly equivalent if f reduces linearly to g and vice versa. This defines an equivalence relation on functions, which can be translated into a linear equivalence on bijections.

Let $D = (d_1, \ldots, d_n)$ be an array of integers, and let $m = m(D) := \max_i d_i$. The *bit–size* of D, denoted by $\langle D\rangle$, is the amount of space required to store D; for simplicity from now on we assume that $\langle D\rangle = n \lceil \log_2 m + 1\rceil$. We view a bijection $\delta : \mathcal{A} \longrightarrow \mathcal{B}$ as an algorithm which inputs $A \in \mathcal{A}$ and outputs $B = \delta(A) \in \mathcal{B}$. We need to present Young tableaux as arrays of integers so that we can store them and compute their bit–size. Suppose $A \in YT(\mu, m, \lambda)$: a way to encode A is through its recording matrix $(c_{i,j})$, which is defined by $c_{i,j} = a_{i,j} - a_{i,j-1}$; in other words, $c_{i,j}$ is the number of j's in the i–th row of A; this is the way Young tableaux will be presented in the input and output of the algorithms. Finally, we say that a map $\gamma : \mathcal{A} \longrightarrow \mathcal{B}$ is *size–neutral* if the ratio $\frac{\langle\gamma(A)\rangle}{\langle A\rangle}$ is bounded for all $A \in \mathcal{A}$. Throughout the paper we consider only size–neutral maps, so we can investigate the linear equivalence of maps comparing them by the number of times other maps are used, without be bothered by the timing. In fact, if we drop the condition of being size–neutral, it can happen that a map increases the bit–size of combinatorial objects, when it transforms the input into the output, and this affects the timing of its subsequent applications. Let $\mathcal{A}$ and $\mathcal{B}$ be two possibly infinite sets of finite integer arrays, and let $\delta : \mathcal{A} \longrightarrow \mathcal{B}$ be an explicit map between them. We say that δ has *linear cost* if δ computes $\delta(A) \in \mathcal{B}$ in linear time $O(\langle A\rangle)$ for all $A \in \mathcal{A}$. There are many ways to construct new bijections out of existing ones: we call such algorithms *circuits* and we define below several of them that we need.

: ◦ Suppose $\delta_1 : \mathcal{A}_1 \longrightarrow \mathcal{X}_1$, $\gamma : \mathcal{X}_1 \longrightarrow \mathcal{X}_2$ and $\delta_2 : \mathcal{X}_2 \longrightarrow \mathcal{B}$, such that δ_1 and δ_2 have linear cost, and consider $\chi = \delta_2 \circ \gamma \circ \delta_1 : \mathcal{A} \longrightarrow \mathcal{B}$. We call this circuit *trivial* and denote it by $I(\delta_1, \gamma, \delta_2)$.

: ∘ Suppose $\gamma_1 : \mathcal{A} \longrightarrow \mathcal{X}$ and $\gamma_2 : \mathcal{X} \longrightarrow \mathcal{B}$, and let $\chi = \gamma_2 \circ \gamma_1 : \mathcal{A} \longrightarrow \mathcal{B}$. We call this circuit *sequential* and denote it by $S(\gamma_1, \gamma_2)$.
: ∘ Suppose $\delta_1 : \mathcal{A} \longrightarrow \mathcal{X}_1 \times \mathcal{X}_2$, $\gamma_1 : \mathcal{X}_1 \longrightarrow \mathcal{Y}_1$, $\gamma_2 : \mathcal{X}_2 \longrightarrow \mathcal{Y}_2$, and $\delta_2 : \mathcal{Y}_1 \times \mathcal{Y}_2 \longrightarrow \mathcal{B}$, such that δ_1 and δ_1 have linear cost. Consider $\chi = \delta_2 \circ (\gamma_1 \times \gamma_2) \circ \delta_1 : \mathcal{A} \longrightarrow \mathcal{B}$: we call this circuit *parallel* and denote it by $P(\delta_1, \gamma_1, \gamma_2, \delta_2)$.

For a fixed bijection α, we say that $\beth$ is an *α–based ps–circuit* if one of the following holds:

: • $\beth = \delta$, where δ is a bijection having linear cost.
: • $\beth = I(\delta_1, \alpha, \delta_2)$, where δ_1, δ_2 are bijections having linear cost.
: • $\beth = P(\delta_1, \gamma_1, \gamma_2, \delta_2)$, where γ_1, γ_2 are α–based ps–circuits and δ_1, δ_2 are bijections having linear cost.
: • $\beth = S(\gamma_1, \gamma_2)$, where γ_1, γ_2 are α–based ps–circuits.

In other words, $\beth$ is an α–based ps–circuit if there is a parallel–sequential algorithm which uses only a finite number of linear cost maps and a finite number of application of map α. The α–cost of $\beth$ is the number of times the map α is used; we denote it by $s(\beth)$.

Let $\gamma : \mathcal{A} \longrightarrow \mathcal{B}$ be a map produced by the α–based ps–circuit $\beth$. We say that $\beth$ computes γ at cost $s(\beth)$ of α. A map β is *linearly reducible* to α, write $\beta \hookrightarrow \alpha$, if there exist a finite α–based ps–circuit $\beth$ which computes β. In this case we say that β can be computed in at most $s(\beth)$ cost of α. We say that maps α and β are linearly equivalent, write $\alpha \sim \beta$, if α is linearly reducible to β, and β is linearly reducible to α. We recall, gluing together, results proved in Section 4.2 of [PV10].

Proposition 3.1. *Suppose $\alpha_1 \hookrightarrow \alpha_2$ and $\alpha_2 \hookrightarrow \alpha_3$, then $\alpha_1 \hookrightarrow \alpha_3$. Moreover, if α_1 can be computed in at most s_1 cost of α_2, and α_2 can be computed in at most s_2 cost of α_3, then α_1 can be computed in at most $s_1 s_2$ cost of α_3. Suppose $\alpha_1 \sim \alpha_2$ and $\alpha_2 \sim \alpha_3$, then $\alpha_1 \sim \alpha_3$ Suppose $\alpha_1 \hookrightarrow \alpha_2 \hookrightarrow \ldots \hookrightarrow \alpha_n \hookrightarrow \alpha_1$, then $\alpha_1 \sim \alpha_2 \sim \ldots \sim \alpha_n \sim \alpha_1$.*

3.2. **The linear involution rotation on LR and companion tableaux.** Given a Yamanouchi word $w = w_1 \cdots w_s$ of weight ν, define the standard tableau $U(w^\bullet)$ of anti-normal shape $\nu^\bullet$ such that the label k is in row i if and only if $w^\bullet_{s-i+1} = k$ where $w^\bullet = w^\bullet_s \cdots w^\bullet_1$. Thus $U(w^\bullet) = U(w)^\bullet$ and $w \mapsto U(w)^\bullet$ defines a bijection between Yamanouchi words and standard tableaux of anti-normal shape given by the reverse content of the Yamanouchi word.

If $T \in \mathrm{LR}(\mu, \nu, \lambda)$ and M is the recording matrix of T, $M^\bullet := P_{rev} M P_{rev}$, the π radians rotation of M, is the recording matrix of $\bullet T \in LR(\lambda, \nu^\bullet, \mu)$. Since $M^{\bullet\, t} = M^{t \bullet}$, if G is the companion of T then $M^{\bullet\, t}$ is the recording matrix of $\bullet G$ which we define to be the companion of $\bullet T$. Therefore $U(w(T)^\bullet) = U(w(T))^\bullet = \bullet\widehat{G} = \widehat{\bullet G}$. Let $\mathrm{LR}_{\nu^\bullet,(\lambda/\mu)^\bullet}$ be the set of companion tableaux of the opposite LR tableaux of shape λ/μ and content $\nu^\bullet$. Then $\mathrm{LR}_{\nu^\bullet,(\lambda/\mu)^\bullet} := \bullet \mathrm{LR}_{\nu,\lambda/\mu}$. The map $\bullet$ is an involution on the set $\mathrm{LR}(\mu,\nu,\lambda) \sqcup \mathrm{LR}(\lambda,\nu^\bullet,\mu)$ and on the set of LR companion tableaux $\mathrm{LR}_{\nu,\lambda/\mu} \sqcup \mathrm{LR}_{\nu^\bullet,(\lambda/\mu)^\bullet}$.

Proposition 3.2. *If G is the right companion tableau of T, the* companion tableau *of $\bullet T$ is the semistandard tableau $\bullet G$ of anti-normal shape $\nu^\bullet$, the rotated of G, whose i-th row tell us in which rows of $\bullet T$ the i's are filled in.*

Example 3.1. Let $T \in LR(\mu, \nu, \lambda^\vee)$, in Example 2.1, with recording matrix $M = \begin{pmatrix} 4 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}$,

and companion tableau $G =$

3	3				
2	2	3			
1	1	1	1	2	

of shape $\nu = 532$ and weight $\lambda/\mu = 433$

encoded by M^t. Then $\bullet T = \begin{array}{|c|c|c|c|c|c|}\hline 3&3&3&3&&\\ \hline &&2&2&3&\\ \hline &&&1&1&2\\ \hline\end{array}$ is in $LR(\lambda^\vee, \nu^\bullet, \mu)$ with dual Yamanouchi word $w^\bullet = 2113223333$, and has recording matrix $M^\bullet := P_{rev} M P_{rev} = \begin{pmatrix} 2&1&0\\ 0&2&1\\ 0&0&4 \end{pmatrix}$, with P_{rev} the rev permutation matrix. The matrix $M^{\bullet t} = M^{t\bullet} = \begin{pmatrix} 2&0&0\\ 1&2&0\\ 0&1&4 \end{pmatrix}$ encodes $\bullet G = \begin{array}{|c|c|c|c|c|c|}\hline &2&3&3&3&3\\ \hline &&&1&2&2\\ \hline &&&&1&1\\ \hline\end{array}$ of anti normal shape $\nu^\bullet = 235$ and weight $(\lambda/\mu)^\bullet = \mu^\vee/\lambda^\vee = 334 = (\lambda - \mu)_{rev}$. The antitableau $\bullet G$ gives explicit information of $\bullet T$ namely the row i entries of $\bullet G$ tell us which rows of $\bullet T$ are filled with i's. One has

$$\widehat{\bullet G} = \bullet\widehat{G} = U(w)^\bullet = U(w^\bullet) = \begin{array}{|c|c|c|c|c|c|}\hline &4&7&8&9&10\\ \hline &&&1&5&6\\ \hline &&&&2&3\\ \hline\end{array}.$$

3.3. **The linear cost orthogonal transpose involution $\blacklozenge$.** There is another simple bijection, denoted $\blacklozenge$, between LR tableaux of conjugate weights and rotate conjugate shapes [Az99, Az98, Za96]. Given a *Yamanouchi* word w on the alphabet $[d]$ of weight $\nu = (\nu_1, \dots, \nu_d)$, let $\nu^t = (\nu_1^t, \dots, \nu_{\nu_1}^t)$. We define $w^\blacklozenge$ to be the Yamanouchi word, on the alphabet $[\nu_1]$, of weight ν^t obtained by replacing in w the subword of length ν_i, consisting of all letters i, with the subword $12\cdots\nu_i$, for each $i = 1, \dots, d$. When w is *opposite Yamanouchi* word, $w^\blacklozenge$ is defined to be the *opposite Yamanouchi* word, on the alphabet $[\nu_1]$, of weight $\nu^{t\bullet}$ obtained by replacing in w for each i, for each $i = 1, \dots, d$, the subword of length ν_i, consisting of all letters $d-i+1$ with the subword $(\nu_1 - \nu_i + 1)\cdots(\nu_1 - 1)\,\nu_1$.

If w is a *Yamanouchi* word, the word $w^{\blacklozenge\bullet}$ is calculated by first replacing in w each string i^{ν_i} with $12\cdots\nu_i$ in w, for $i = 1, \dots, d$, to obtain $w^\blacklozenge$. Then, after reversing $w^\blacklozenge$, for $i = 1, \dots, d$, our string $12\cdots\nu_i$ is transformed into $\nu_i\cdots 21$ which we replace with $(\nu_1 - \nu_i + 1)\cdots(\nu_1 - 1)\,\nu_1$. On the other hand, the word $w^{\bullet\blacklozenge}$ is calculated by first replacing in w each string i^{ν_i} with $(d-i+1)^{\nu_i}$, for $i = 1, \dots, d$, and then reversing the word to obtain $w^\bullet$. Then, for $i = 1, \dots, d$, our string $(d-i+1)^{\nu_i}$ in $w^\bullet$ is transformed into $(\nu_1 - \nu_i + 1)\cdots(\nu_1 - 1)\,\nu_1$. Thereby, $w^{\bullet\blacklozenge} = w^{\blacklozenge\bullet}$ is an *opposite Yamanouchi* word of weight $\nu^{t\bullet} = \nu^{\bullet t}$ (2.3). Henceforth, if w is *Yamanouchi*, the word $w^{\blacklozenge\bullet}$ can be obtained in just one single step by replacing in w, for each i, the subword of length ν_i, consisting of all letters i with the subword $\nu_1\,(\nu_1 - 1)\cdots(\nu_1 - \nu_i + 1)$, and then reversing the resulting word. See Example 3.2. If w is *opposite Yamanouchi* word, we also have $w^{\bullet\blacklozenge} = w^{\blacklozenge\bullet}$ by reducing to the previous case because every *opposite Yamanouchi* word is the dual of some *Yamanouchi* word and $\bullet$ is an involution. The Yamanouchi word $w^{\blacklozenge\bullet}$ is obtained by replacing each string i^{ν_i} with $\nu_i\cdots 21$ and then reverse the resulting word.

The map $w \mapsto w^\blacklozenge$ defines a bijection between (opposite) Yamanouchi words of conjugate content. Clearly, $U(w^\blacklozenge) = U(w)^t$ has shape ν^t, and $U(w^{\bullet\blacklozenge}) = U(w^{\blacklozenge\bullet}) = U(w)^{\bullet t} = U(w)^{t\bullet}$ has shape $\nu^\bullet$.

The operation $\blacklozenge$ on Yamanouchi or opposite Yamanouchi words is now extended to LR or opposite LR tableaux in the sense that $\blacklozenge$ can be seen as defined on diagonally-shaped LR or opposite LR tableaux. The orthogonal transpose map $\blacklozenge$ is defined on LR and opposite LR tableaux as follows. Given T $\in$ LR(μ, ν, λ) (respectively LR$(\mu, \nu^\bullet, \lambda)$) with (opposite) Yamanouchi word w , the *orthogonal transpose* of T, $\blacklozenge$T, is the (opposite) LR tableau of shape $(\lambda/\mu)^{t\bullet} = (\mu^\vee)^t/(\lambda^\vee)^t$ and (opposite) Yamanouchi column word $w^\blacklozenge$ of weight ν^t $(\nu^{\bullet t})$. It is obtained from T by replacing the word w with $w^\blacklozenge$, and then transpose and rotate the shape λ/μ by π radians,

$$\begin{array}{rcl} \blacklozenge : \mathrm{LR}(\mu,\nu,\lambda)\cup\mathrm{LR}(\mu,\nu^{\bullet},\lambda)) & \longrightarrow & \mathrm{LR}(\lambda^t,\nu^t,\mu^t)\cup\mathrm{LR}(\lambda^t,\nu^{\bullet t},\mu^t), \\ T & \mapsto & T^{\blacklozenge},\ \ w_{col}(T^{\blacklozenge}) = w(T)^{\blacklozenge}. \end{array} \tag{3.1}$$

The map $\blacklozenge$ is an involution on $\mathrm{LR}(\mu,\nu,\lambda)\cup\mathrm{LR}(\lambda^t,\nu^t,\mu^t)$ and on $\mathrm{LR}(\mu,\nu^{\bullet},\lambda)\cup\mathrm{LR}(\lambda^t,\nu^{\bullet t},\mu^t)$ that transposes inner shape, outer shape and weight, and simultaneously swaps the inner and the outer shapes. In subsection 3.3.1, one proves that $\blacklozenge$ is a linear time involution. It exhibits the symmetries $c_{\mu,\nu,\lambda}=c_{\lambda^t,\nu^t,\mu^t}$, $c_{\mu,\nu^{\bullet},\lambda}=c_{\lambda^t,\nu^{\bullet t},\mu^t}$ in linear time.

Let $T\in\mathrm{LR}(\mu,\nu,\lambda)\cup\mathrm{LR}(\mu^t,\nu^{\bullet t},\lambda^t)$. Because $\bullet\blacklozenge T$ and $\blacklozenge\bullet T$ have the same column word $w(T)^{\blacklozenge\bullet}=w(T)^{\bullet\blacklozenge}$, it follows that

$$\begin{array}{rcl} \bullet\blacklozenge=\blacklozenge\bullet : \mathrm{LR}(\mu,\nu,\lambda)\cup\mathrm{LR}(\mu,\nu^{\bullet},\lambda) & \longrightarrow & \mathrm{LR}(\mu^t,\nu^{\bullet t},\lambda^t)\cup\mathrm{LR}(\mu^t,\nu^t,\lambda^t) \\ w(T) & \mapsto & w_{col}(T^{\blacklozenge\bullet}) = w(T)^{\blacklozenge\bullet}. \end{array} \tag{3.2}$$

is an involution which transposes the inner shape, outer shape and reverses and transposes the weight.

Remark 3.1. If T is an LR tableau, $(\bullet\blacklozenge T)^t$ is obtained from T by replacing the horizontal strip i^{ν_i}, from SE to NW, with $\nu_1\nu_1-1\cdots\nu_1-\nu_i+1$, for all i.

If T is a opposite LR tableau, $(\bullet\blacklozenge T)^t$ is obtained from T by replacing the horizontal strip $(d-i+1)^{\nu_i}$, from NW to SE, with $12\cdots\nu_i$, for all i.

From the discussion above, it follows the next proposition.

Proposition 3.3. *The rotation $\bullet$ and the orthogonal transpose $\blacklozenge$ maps commute on the set of LR and opposite LR tableaux, $\blacklozenge\bullet=\bullet\blacklozenge$, that is, $(\blacklozenge\bullet)^2=1$.*

Example 3.2. Let $n=7$ and $d=3$. Let $\mathrm{T}=\begin{array}{|c|c|c|c|}\hline 1&3&&\\\hline &1&2&2\\\hline &&1&1\\\hline\end{array}$ be an LR tableau of shape $\lambda^\vee/\mu$, weight $\nu=421$, where $\lambda=200$, $\lambda^\vee=442$, $\mu=210$, and with word $w=1122131$. Then $w^{\bullet}=3132233$, $w^{\blacklozenge}=1212314$, column word of $T^{\blacklozenge}$, and $w^{\blacklozenge\bullet}=1423434=w^{\bullet\blacklozenge}$, $\nu^t=3211$, column word of $T^{\bullet\blacklozenge}$,

$$\mathrm{T}=\begin{array}{|c|c|c|c|}\hline 1&3&&\\\hline &1&2&2\\\hline &&1&1\\\hline\end{array}\longleftrightarrow\begin{array}{|c|c|c|c|}\hline 4&1&&\\\hline &3&2&1\\\hline &&2&1\\\hline\end{array}\ \underset{\longleftrightarrow}{\text{rotate \& transpose diagram}}\ \begin{array}{|c|c|c|}\hline 4&&\\\hline 1&3&\\\hline &2&2\\\hline &1&1\\\hline\end{array}=\mathrm{T}^{\blacklozenge}\underset{\bullet}{\longleftrightarrow}\begin{array}{|c|c|c|}\hline 4&4&\\\hline 3&3&\\\hline &2&4\\\hline &&1\\\hline\end{array}=\bullet\blacklozenge\mathrm{T}\ .$$

$$T\longleftrightarrow T^{\bullet}=\begin{array}{|c|c|c|c|}\hline 3&3&&\\\hline 2&2&3&\\\hline &&1&3\\\hline\end{array}\longleftrightarrow\begin{array}{|c|c|c|c|}\hline 4&3&&\\\hline 4&3&2&\\\hline &&4&1\\\hline\end{array}\ \underset{\longleftrightarrow}{\text{rotate \& transpose diagram}}\ \begin{array}{|c|c|c|}\hline 4&4&\\\hline 3&3&\\\hline &2&4\\\hline &&1\\\hline\end{array}=T^{\bullet\blacklozenge}.$$

$\mathrm{T}^{\blacklozenge}$ is an LR tableau with shape $(\lambda^\vee/\mu)^{\bullet\, t}$ and column word $w^{\blacklozenge}=1212314$ of weight ν^t, while $\mathrm{T}^{\blacklozenge\bullet}=\mathrm{T}^{\bullet\blacklozenge}$ is a opposite LR tableau with shape $(\lambda/\mu)^t$ and column word $w^{\blacklozenge\bullet}=1423434$ of weight $\nu^{t\bullet}$, where $U(w)=\begin{array}{|c|c|c|c|}\hline 6&&&\\\hline 3&4&&\\\hline 1&2&5&7\\\hline\end{array}$, $U(w^{\blacklozenge})=\begin{array}{|c|c|c|}\hline 7&&\\\hline 5&&\\\hline 2&4&\\\hline 1&3&6\\\hline\end{array}=U(w)^t$, and $U(w^{\blacklozenge\bullet})=\begin{array}{|c|c|c|}\hline 2&5&7\\\hline &4&6\\\hline &&3\\\hline &&1\\\hline\end{array}=U(w)^{\bullet t}$.

Let $T = \begin{array}{|c|c|c|c|}\hline 3 & 3 & & \\ \hline & 2 & 2 & 3 \\ \hline & & 1 & 2 \\ \hline \end{array}$ be a opposite LR tableau with word $w = 2132233$. Following the remark above, $(\blacklozenge \bullet \mathrm{T})^{\mathrm{t}} = \begin{array}{|c|c|c|c|}\hline 1 & 2 & & \\ \hline & 1 & 2 & 3 \\ \hline & & 1 & 3 \\ \hline \end{array}$ and $\blacklozenge \bullet \mathrm{T}$ is a LR tableau with column word $w^{\blacklozenge\bullet} = 1212313$.

3.3.1. *Computational complexity of bijection* $\blacklozenge$ *on LR tableaux.* We now show that the computational complexity of bijection $\blacklozenge$ is linear on the input where we use skew LR tableaux. Hence, recalling the definition of a linear cost bijection in Subsection 3.1, the bijection $\blacklozenge$ is of linear cost.

Algorithm 3.4. [Bijection $\blacklozenge$.]
Input: LR tableau T of skew shape λ/μ, with $\lambda = (\lambda_1 \geq \ldots \geq \lambda_n)$, $\mu = (\mu_1 \geq \ldots \geq \mu_n)$, and filling $\nu = (\nu_1 \geq \ldots \geq \nu_n)$, having $A = (a_{i,j}) \in M_{n\times n}(\mathbb{N})$ ($a_{i,j} = 0$ if $j > i$) as (lower triangular) recording matrix.

```
Write Ã, a copy of the matrix A.
For j := n down to 2 do
   For i := 1 to n do
      Begin
         If i = j then ã_{i,i} := ã_{i,i} + λ_1 − λ_i
                 else
                       If j > i then ã_{i,j} = 0 else ã_{i,j} := ã_{i,j} + ã_{i,j+1}.
      End
```

So far the computational cost is $O(n^2) = O(\langle A\rangle)$.

```
Set a matrix B = (b_{i,j}) ∈ M_{λ_1×λ_1}(N) such that b_{i,j} = 0 for all i, j.
For i := 1 to n do
   Begin
      Set c := 0.
      For j := 0 to n do
         Begin
            r := ã_{i+j,i} − a_{i+j,i},     see Remark 3.2.
            For t := 1 to a_{i+j,i} do b_{r+t,c+t} := b_{r+t,c+t} + 1.
            c := c + a_{i+j,i}.
         End
   End
```

This part has total computational cost at most equal to

$$O\left(\sum_{1\leq i.j\leq n} a_{i,j}\right) = O\left(|\lambda \setminus \mu|\right) = O\left(|\lambda| - |\mu|\right) = O\left(\langle T\rangle\right).$$

Output: B recording matrix of the output tableau $T^{\blacklozenge}$.

Remark 3.2. For all $1 \leq i \leq n$ and $0 \leq j \leq n - i + 1$, we have

$$\widetilde{a}_{i+j+1,i} - \widetilde{a}_{i+j,i} \geq a_{i+j+1,i}.$$

Corollary 3.5. *The composition* $\blacklozenge\bullet = \bullet\blacklozenge$ *is a linear cost involution on* $\mathrm{LR}(\mu,\nu,\lambda) \cup \mathrm{LR}(\mu^t,\nu^{\bullet t},\lambda^t)$*. It exhibits the symmetry* $c_{\mu,\nu,\lambda} = c_{\mu^t,\nu^{\bullet t},\lambda^t}$ *in linear time.*

3.4. **The linear cost involution orthogonal transpose on LR companion tableaux.** If G is the right companion of $T \in \mathrm{LR}(\mu,\nu,\lambda)$ and $\iota(T^{\blacklozenge}) =: G^{\blacklozenge}$, where ι is the linear map bijection (2.13), then if B is the recording matrix of $T^{\blacklozenge}$, Algorithm 3.4, B^t is the recording matrix of $G^{\blacklozenge}$ and $G \overset{\iota^{-1}}{\to} T \overset{\blacklozenge}{\to} T^{\blacklozenge} \overset{\iota}{\to} G^{\blacklozenge}$ is a linear cost bijection. It is then not difficult to define a linear cost algorithm to directly calculate $G^{\blacklozenge}$ from G without making recourse of the recording matrix. We now recall the involution on LR companion tableaux, also denoted $\blacklozenge$, as described in Steps 2 and 3 in Section 6.1 of [LecLen17]

$$\blacklozenge : \mathrm{LR}_{\nu,\lambda^\vee/\mu} \to \mathrm{LR}_{\nu^t,(\lambda^\vee/\mu)^{t\bullet}},\ \mathrm{G} \mapsto \mathrm{G}^{\blacklozenge} \tag{3.3}$$

such that $\iota(T^{\blacklozenge}) = \iota(T)^{\blacklozenge}$ whenever $T \in \mathrm{LR}(\mu,\nu,\lambda)$. We reproduce it below with slightly different notation.

Algorithm 3.6. [Construction of $G^{\blacklozenge}$.] Let $G \in \mathrm{LR}_{\nu,\lambda^\vee/\mu}$. The construction of $G^{\blacklozenge}$ has the following two steps.

Step 1. Transpose the tableau G and denote the resulting filling of shape ν^t by G^t.

Step 2. For each $i = 1,\ldots,d$, consider in G^t the vertical strip of $i's$ with size $\lambda_i^\vee - \mu_i$, and replace these entries, from southeast to northwest, with $(\lambda^\bullet)_i + 1, (\lambda^\bullet)_i + 2, \ldots, (\lambda^\bullet)_i + \lambda_i^\vee - \mu_i$ respectively. The resulting tableau is $G^{\blacklozenge}$ of shape ν^t and weight $(\lambda^{\vee t}/\mu^t)^\bullet$.

Example 3.3. In the previous example, $T \in LR(\mu,\nu,\lambda)$ has recording matrix $A = \begin{pmatrix} 2&0&0&0\\ 1&2&0&0\\ 1&0&1&0\\ 0&0&0&0 \end{pmatrix}$ and A^t is the recording matrix of the companion tableau $G = \begin{array}{|c|c|c|c|} \hline 3 \\ \cline{1-1} 2&2 \\ \cline{1-2} 1&1&2&3 \\ \hline \end{array}$ of shape $\nu = 421$ and weight $\lambda^\vee/\mu = 232$, where $\lambda = 200$, $\lambda^\vee = 442$, $\mu = 210$. One has $G^t = \begin{array}{|c|c|c|} \hline 3 \\ \cline{1-1} 2 \\ \cline{1-1} 1&2 \\ \cline{1-2} 1&2&3 \\ \hline \end{array}$, $\lambda^\bullet = 002$ and $(\lambda^\bullet)_1{+}1 = 1, (\lambda^\bullet)_1{+}2 = 2, (\lambda^\bullet)_2{+}1 = 1, (\lambda^\bullet)_2{+}2 = 2, (\lambda^\bullet)_2{+}3 = 3,$ $(\lambda^\bullet)_3 + 1 = 3, (\lambda^\bullet)_3 + 2 = 4$. Then $B = A^{\blacklozenge} = \begin{pmatrix} 2&0&0&0\\ 0&2&0&0\\ 1&0&1&0\\ 0&0&0&1 \end{pmatrix}$ is the recording matrix of $T^{\blacklozenge}$ and B^t is the recording matrix of $G^{\blacklozenge} = \begin{array}{|c|c|c|} \hline 4 \\ \cline{1-1} 3 \\ \cline{1-1} 2&2 \\ \cline{1-2} 1&1&3 \\ \hline \end{array} = \iota(T^{\blacklozenge})$ with shape $\nu^t = 3211$ and weight $(\lambda^{\vee t}/\mu^t)^\bullet = (3322 - 2100)_{rev} = 2221$ is the companion tableau of $T^{\blacklozenge}$.

From the second part of Proposition 2.2 and duality of Burge correspondence, it follows

Proposition 3.7. *Let w and u be two Yamanouchi words such that $w \equiv Y(\nu)$. Then*
(a) $w^{\blacklozenge} \equiv Y(\nu^t)$, *and* $w \equiv u$ *if and only if* $w^{\blacklozenge} \equiv u^{\blacklozenge}$.
(b) $w^\bullet \equiv Y(\nu^\bullet)$, *and* $w^{\bullet\blacklozenge} = w^{\blacklozenge\bullet} \equiv Y(\nu^{t\bullet})$.
(c) $Q(w^\bullet) = \operatorname{evac} Q(w) = \operatorname{evac} U(w) = U(w)^{\bullet \mathrm{n}} = U(w^\bullet)^{\mathrm{n}} = U(w)^{\mathrm{a}\bullet}$.
(d) $Q(w^{\blacklozenge}) = Q(w)^t = U(w)^t$.
(e) $Q(w^{\bullet\blacklozenge}) = Q(w^{\blacklozenge\bullet}) = \operatorname{evac} Q(w)^t = (\operatorname{evac} Q(w))^t = \operatorname{evac} U(w)^t = (\operatorname{evac} U(w))^t$.

4. LR transposers coincidence and linear equivalence to an LR commutor

In this section we work in $\mathcal{LR}$ as a set of LR tableaux or their companions.

4.1. **An LR commutor for the symmetry** $c_{\mu,\nu,\lambda} = c_{\lambda,\nu,\mu}$**.** Let $T \in \mathrm{LR}(\mu,\nu,\lambda)$ and $B(T)$ the crystal connected component of $B(\lambda/\mu)$ containing T. The highest and lowest weight elements of $B(T)$ are T and $e(T)$ respectively, and henceforth the highest and lowest weight elements of $B(T)^\bullet = B(\bullet T)$ are $\bullet e(T)$ and $\bullet(T)$ respectively. Since highest and lowest weight elements in a connected crystal component are related by reversal e,

$$e \bullet T = \bullet eT. \tag{4.1}$$

Theorem 4.1. *Let* $\rho := \bullet\, e = e\bullet$*. Then the involution*

$$\rho : \mathrm{LR}(\mu,\nu,\lambda) \longrightarrow \mathrm{LR}(\lambda,\nu,\mu), T \mapsto \rho(T) = e \bullet T$$

is an LR commutor that exhibits the symmetry $c_{\mu,\nu,\lambda} = c_{\lambda,\nu,\mu}$*.*

Remark 4.1. We may also use Knuth and dual Knuth equivalence to characterize $\bullet e(T)$ and $e \bullet (T)$. That is we may use tableau switching as in Algorithm 2.6 to calculate $e \bullet (T)$. In fact $T^\bullet \in \mathrm{LR}(\lambda,\nu^\bullet,\mu)$, and, from Corollary 2.8, $T^{\bullet e} = [Y(\nu)]_K \cap [T^\bullet]_{dK}$, is the unique LR tableau in $\mathrm{LR}(\lambda,\nu,\mu)$ of the crystal connected component $B(T^\bullet)$ in $B(\mu^\vee/\lambda)$. On the other hand, $T^{e\bullet} = [Y(\nu^\bullet)^\bullet]_K \cap [T^\bullet]_{dK} = [Y(\nu)^{\mathrm{a}}]_K \cap [T^\bullet]_{dK} = T^{\bullet e}$.

4.2. **Coincidence of LR transposers and linear equivalence to an LR commuter.** Recall that σ_0 coincides with the reversal on the T^{high} and on the T^{low} of $B(T^{\mathsf{high}}, d)$. The $\texttt{reversal}(T^{\mathsf{high}})$ may be computed by the action of σ_0 on $B(T^{\mathsf{high}}, d)$. Recall that $\sigma_0 w$ coincides with the reversal e on the LR or opposite LR diagonally-shaped tableau with word w. Thus the column word of $\bullet\blacklozenge eT$ is $(\sigma_0 w)^{\blacklozenge\bullet}$.

Proposition 4.2. *The following holds on* $\mathcal{LR}$ *as a set of LR tableaux:*
(*a*) $\sigma_0(w^\blacklozenge) = (\sigma_0 w)^\blacklozenge$*, where* w *is a Yamanouchi or opposite Yamanouchi word.*
(*b*) $e\blacklozenge = \blacklozenge e$*, that is,* $(e\blacklozenge)^2 = 1$*.*
(*c*) *the involutions* $\blacklozenge$*,* e*,* $\bullet$ *pairwise commute.*

Proof. (*a*) Suppose that $w \equiv Y(\nu)$. Then, from Proposition 3.7, $w^\blacklozenge \equiv Y(\nu^t)$ and Subsection 2.8.3, $\sigma_0 w \equiv w^\bullet \equiv Y(\nu^\bullet)$. Thereby, $(\sigma_0 w)^\diamond \equiv w^{\bullet\blacklozenge} = w^{\blacklozenge\bullet} \equiv \sigma_0(w^\blacklozenge) \equiv Y(\nu^{t\bullet})$. On the other hand, $\sigma_0(w^\blacklozenge)$ and $(\sigma_0 w)^\blacklozenge$ are dual equivalent because their Q-symbol is $Q(w)^t$. In fact $Q(w)^t = Q(w^\blacklozenge) = Q(\sigma_0(w^\blacklozenge))$ and $Q(w)^t = Q(\sigma_0 w)^t = Q((\sigma_0 w)^\blacklozenge)$. Therefore $\sigma_0(w^\blacklozenge) = (\sigma_0 w)^\blacklozenge$.

(*b*) Let T be an LR tableau where $T \in B_d(\lambda/\mu)$. The row and the column reading of the tableaux in $B(T)$ embeds $B(T)$ into usually different subcrystals of $B^{|\lambda|-|\mu|}$ but isomorphic. Henceforth, either we consider the row reading or the column reading of T, one always has, $\sigma_0(w_{col}(T)) = w_{col}(T^e)$, and $\sigma_0(w(T)) = w(T^e)$, with $\omega_0 \in \mathfrak{S}_d$. Then the (row reading) word of T^e is $\sigma_0 w(T)$, the column reading word of $T^\blacklozenge \in B_{\nu_1}(\mu^{t\vee}/\lambda^{t\vee}) \subseteq B_{n-d}(\mu^{t\vee}/\lambda^{t\vee})$ is $w(T)^\blacklozenge$ on the alphabet $[\nu_1]$, and the column reading word of $T^{\blacklozenge e}$ is $\sigma_0(w(T)^\blacklozenge)$ with with $\omega_0 \in \mathfrak{S}_{\nu_1}$. Henceforth, from (*a*), $w_{col}(T^{e\blacklozenge}) = (\sigma_0 w)^\blacklozenge = \sigma_0(w^\blacklozenge) = \sigma_0(w_{col}(T^\blacklozenge)) = w_{col}(T^{\blacklozenge e})$.

(*c*) It follows from (*b*), Theorem 4.1 and Proposition 3.3. □

The following is an illustration of the previous result.

Example 4.1. Let $w = 1111221332$ on the alphabet $[3]$ with weight $\nu = 532$, and $w^\blacklozenge = 1234125123$ on the alphabet $[\nu_1 = 5]$ of weight $\nu^t = 33211$. Then $\sigma_0 w = \sigma_1\sigma_2\sigma_1 w = \sigma_1\sigma_2(2211221332) = \sigma_1(3311221333) = 3311222333$ and $(\sigma_0 w)^\blacklozenge = 1245345345$. On the other hand,

$$\sigma_0(w^\blacklozenge) = \sigma_1.\sigma_2\sigma_1.\sigma_3\sigma_2\sigma_1.\sigma_4\sigma_3\sigma_2\sigma_1(w^\blacklozenge) = \sigma_1.\sigma_2\sigma_1.\sigma_3\sigma_2\sigma_1.\sigma_4\sigma_3 1234135123$$
$$= \sigma_1.\sigma_2\sigma_1.\sigma_3\sigma_2\sigma_1.\sigma_4 1234145124 = \sigma_1.\sigma_2\sigma_1.\sigma_3\sigma_2\sigma_1 1235145125$$

$= \sigma_1.\sigma_2\sigma_1.\sigma_3\sigma_2 1235245125 = \sigma_1.\sigma_2\sigma_1.\sigma_3 1235345135 = \sigma_1.\sigma_2\sigma_1 1245345145$
$= \sigma_1.\sigma_2 1245345245 = \sigma_1 1245345345 = 1245345345 = (\sigma_0 w)^{\blacklozenge}$.

If one considers the alphabet [4], one has $\sigma_0 w = \sigma_1\sigma_2\sigma_1\sigma_3\sigma_2\sigma_1 w = 4422333444$ but still $(\sigma_0 w)^{\blacklozenge} = \sigma_0 w^{\blacklozenge}$.

From this discussion, Section 3, and the computational complexity of the reversal involution e [PV10], it then follows

Theorem 4.3. *Let* T *be a LR tableau with shape* λ/μ *and word* w. *Let* $\varrho := \blacklozenge\rho = \blacklozenge \bullet e = \bullet\blacklozenge\, e = \bullet\, e\blacklozenge = \rho\blacklozenge$ *where* $\rho = \bullet\, e$. *Then*

$$e : LR(\mu,\nu,\lambda) \to LR(\mu,\nu^\bullet,\lambda),\ T \mapsto eT,\ w(eT) = \sigma_0 w;$$

$$\rho : LR(\mu,\nu,\lambda) \to LR(\lambda,\nu,\mu),\ T \mapsto \rho(T) = \bullet eT,\ w(\rho(T)) = (\sigma_0 w)^\bullet;$$

and

$$\varrho : LR(\mu,\nu,\lambda) \to LR(\mu^t,\nu^t,\lambda^t),\ T \mapsto \varrho(T) = \blacklozenge\rho(T) = \blacklozenge \bullet eT,\ w_{col}(\varrho(T)) = (\sigma_0 w)^{\blacklozenge\bullet}$$

are involutions exhibiting the symmetries $c_{\mu\,\nu\,\lambda} = c_{\mu\,\nu^\bullet\,\lambda}$, $c_{\mu\,\nu\,\lambda} = c_{\lambda\,\nu\,\mu}$ *and* $c_{\mu\,\nu\,\lambda} = c_{\mu^t\,\nu^t\,\lambda^t}$ *respectively. The three involutions* e, ρ, *and* ϱ *are linear time equivalent to each other and in particular to the reversal* e.

Remark 4.2. From the identity $\varrho = \blacklozenge\rho = (\blacklozenge\bullet)e$ we conclude that $\varrho(T)$ can be obtained from T^e by replacing, for $i = 1, \ldots, \ell(\nu)$, from NW to SE, the entries of the i-horizontal strip of length $\ell(\nu) - i + 1$ in T^e, with $1, \ldots, \nu_i - i + 1$. This gives $(\varrho(T))^t = (\blacklozenge\rho(T))^t$ a row semistandard tableau.

We may also use Knuth and dual Knuth equivalence to characterize the bijection ϱ. This shows that $\varrho(T)$ can also be calculated using tableau switching as in Algorithm 2.6. This is the procedure offered in [BSS96].

Corollary 4.4. *Let* T *be a LR tableau with shape* λ/μ *and weight* ν. *Then* $\varrho(T) = T^{e\blacklozenge\bullet}$ *is the unique tableau Knuth equivalent to* $Y(\nu^t)$ *and dual Knuth equivalent to* $\widehat{T}^t$. *That is,* $\varrho(T) = T^{e\blacklozenge\bullet} = [Y(\nu^t)]_K \cap [\widehat{T}^t]_{dK}$.

Proof. Let w be the word of T with weight ν. One has $\sigma_0 w \equiv w^\bullet$, and $(\sigma_0 w)^\bullet \equiv w \equiv Y(\nu)$. Then $(\sigma_0 w)^{\blacklozenge\bullet} = ((\sigma_0 w)^\bullet)^\blacklozenge = (\sigma_0(w^\bullet))^\blacklozenge \equiv w^\blacklozenge \equiv Y(\nu^t)$. Recall that dual Knuth equivalence between tableaux can be checked either by using row or column words.

From Haiman's theorem, Theorem 2.5, $[Y(\nu^t)]_K \cap [\widehat{T}^t]_{dK}$ has a sole tableau dual Knuth equivalent to $\widehat{T}^t$ and Knuth equivalent to $Y(\nu^t)$. Since $T^{e\blacklozenge\bullet} \equiv Y(\nu^t)$, it is enough to see that the column words of $T^{e\blacklozenge\bullet}$ and $\widehat{T}^t$ have the same Q–symbol, that is, $T^{e\blacklozenge\bullet}$ is the highest weight element of the connected component $B(\widehat{T}^t)$ in $B((\lambda/\mu)^t)$, on the alphabet $|\lambda| - |\mu|$. Let $\widehat{w}$ be the word of $\widehat{T}$. As $rev\,\widehat{w}$, the reverse word of $\widehat{T}$, is the column word of $\widehat{T}^t$, we want to show that $Q(rev\widehat{w}) = Q((\sigma_0 w)^{\blacklozenge\bullet})$.

We know that any word u is dual Knuth equivalent to $\sigma_0 u$, $Q(u) = Q(\sigma_0 u)$ and, from Proposition 3.7, $Q(u^\bullet) = Q(u)^E$ and $Q(rev\,u) = Q(u)^{Et}$ [St01]. Recalling Proposition 3.7, $Q(rev\,\widehat{w}) = Q(\widehat{w})^{E\,t} = Q(w)^{E\,t} = Q(\sigma_0 w)^{Et} = Q((\sigma_0 w)^{\blacklozenge\bullet})$. □

In [BSS96], it is observed that the White and the Hanlon–Sundaram maps [Wh90, HaSu92] produce the same result, denoted by ϱ^{WHS}. Thus $\varrho^{BSS}(T)$ can be obtained either by tableau–switching or by the White–Hanlon–Sundaram transformation ϱ^{WHS} or by ϱ.

Theorem 4.5. *The LR transposers* ϱ^{BSS}, ϱ^{WHS} *and* ϱ *are identical, and linear time equivalent to the reversal involution* e.

Remark 4.3. The Schützenberger's *jeu de taquin* formulation of the LR rule says: Fix $\mathrm{T}_\nu \in \mathrm{SYT}(\nu)$. The number of $\mathrm{T} \in \mathrm{SYT}(\lambda^\vee/\mu)$ such that $\mathsf{rectification}(\mathrm{T}) = \mathrm{T}_\nu$ equals $c_{\mu,\nu,\lambda}$ [St01, Appendix 1]. Equivalently, since T_ν is Knuth equivalent to $\mathrm{T}^{\mathrm{a}}_\nu$, the number of $\mathrm{T} \in \mathrm{SYT}(\lambda^\vee/\mu)$ such that $\mathsf{arectification}(\mathrm{T}) = \mathrm{T}^{\mathrm{a}}_\nu$ equals $c_{\mu,\nu,\lambda}$. The definition only depends on the shapes μ, ν, λ and not on a particular choice of the filling of the Young diagram of ν. However, choosing a certain filling of $\mathrm{T}_\nu \in \mathrm{SYT}(\nu)$ allows to relate this definition with LR tableaux [St01, Appendix 1]. That is, choosing T_ν to be a standardization of the Yamanouchi tableau $\mathrm{Y}(\nu)$ incurs that $\mathsf{rectification}(\mathrm{T}) = \mathrm{T}_\nu$, with $\mathrm{T} \in \mathrm{SYT}(\lambda^\vee/\mu)$, holds only if the ν-semistandardization of T is an LR tableau of shape $\lambda^\vee/\mu$ and weight ν [St01, Lemma A1.3.6, Lemma A1.3.7].

Fix the tableaux $\mathrm{T}_\mu \in \mathrm{SYT}(\mu)$, $\mathrm{T}_\nu \in \mathrm{SYT}(\nu)$ and $\mathrm{T}_\lambda \in \mathrm{SYT}(\lambda)$ to be the standardizations of the Yamanouchi tableaux $\mathrm{Y}(\mu)$, $\mathrm{Y}(\nu)$, and $\mathrm{Y}(\lambda)$ respectively. Choose $\mathrm{T} \in \mathrm{SYT}(\lambda^\vee/\mu)$ to be the standardization of an LR tableau that rectifies to T_ν, to initialize a Thomas-Yong *carton filling* [TY08], built upon Fomin's *jeu de taquin* growth-diagrams and the *infusion involution* [TY16], a particular case of Benkart-Sottille-Stroomer tableau-switching on pairs of standard tableaux. Let $\mathtt{CARTONS}_{\mu,\nu,\lambda}$ be the set of all carton fillings built in this way with initial data $\mathrm{T}_\mu, \mathrm{T}_\nu, \mathrm{T}_\lambda$. The number of carton fillings is equal to the number of standard tableaux of shape $\lambda^\vee/\nu$ which rectify to T_ν, that is, $c_{\mu,\nu,\lambda}$. For this particular choice of T_μ, T_ν and T_λ the carton filling besides to showing the $\mathfrak{S}_3$-symmetries of LR coefficients $c_{\mu,\nu,\lambda} = c_{\epsilon,\delta,\gamma}$, where $(\epsilon, \delta, \gamma)$ is any permutation of (μ, ν, λ), also gives tableau-switching bijections on LR tableaux exhibiting such symmetries. More precisely, composing the infusion involution with the semistandardization of those standard tableaux in the carton filling , one obtains the tableau-switching on LR tableaux and thus the carton filling gives tableau-switching bijections on LR tableaux exhibiting such symmetries.

Let $\mathrm{R}(\lambda^\vee/\mu, \mathrm{T}_\nu)$ be the set of standard tableaux of shape $\lambda^\vee/\nu$ which rectify to T_ν. Taking transposes there is an obvious bijection between $\mathrm{R}(\lambda^\vee/\mu, \mathrm{T}_\nu)$ and $\mathrm{R}(\lambda^{\mathrm{t}\vee}/\mu^{\mathrm{t}}, \mathrm{T}^{\mathrm{t}}_\nu)$, that is, $\mathrm{T} \in \mathrm{R}(\lambda^\vee/\mu, \mathrm{T}_\nu)$ if and only if $\mathrm{T}^{\mathrm{t}} \in \mathrm{R}(\lambda^{\mathrm{t}\vee}/\mu^{\mathrm{t}}, \mathrm{T}^{\mathrm{t}}_\nu)$ [Fu97, Section.5.1], and thus that bijection exhibits $c_{\mu\nu\lambda} = c_{\mu^t\nu^t\lambda^t}$. Transposing all the Young diagrams defining a carton filling in $\mathtt{CARTONS}_{\mu,\nu,\lambda}$ with initial data $\mathrm{T}_\mu, \mathrm{T}_\nu, \mathrm{T}_\lambda$, one obtains another carton filling in $\mathtt{CARTONS}_{\mu^t,\nu^t,\lambda^t}$ with initial data $\mathrm{T}^{\mathrm{t}}_\mu, \mathrm{T}^{\mathrm{t}}_\nu, \mathrm{T}^{\mathrm{t}}_\lambda$, showing the identities $c_{\mu,\nu,\lambda} = c_{\epsilon,\delta,\gamma} = c_{\epsilon^t,\delta^t,\gamma^t}$, where $(\epsilon, \delta, \gamma)$ is any permutation of (μ, ν, λ). However, in this case, the initial data $\mathrm{T}^{\mathrm{t}}_\mu, \mathrm{T}^{\mathrm{t}}_\nu, \mathrm{T}^{\mathrm{t}}_\lambda$ is not given by the standardization of Yamanouchi tableaux, and applying semistandardization after transposing does not give semistandard tableau growth diagrams and thereby such a procedure bijection between $\mathtt{CARTONS}_{\mu,\nu,\lambda}$ and $\mathtt{CARTONS}_{\mu^t,\nu^t,\lambda^t}$, with the aforesaid initial data, does not provide LR transposers.

Any carton filling gives a growth diagram on the face $\emptyset - \nu - \lambda^\vee - \mu$ for which the edge $\mu - \lambda^\vee$ is a standard tableau $\mathrm{T}_{\lambda^\vee/\mu}$ of shape $\lambda^\vee/\mu$ rectifying to T_ν. By the *jeu de taquin* Littlewood-Richardson rule, fillings of this face count $c_{\mu,\nu,\lambda}$. Any such growth-diagram of this face extends uniquely to a filling of the entire carton. The carton initialized with $\mathrm{T}_\mu, \mathrm{T}_\nu, \mathrm{T}_\lambda$ and with $\mathrm{T}_{\lambda^\vee/\mu}$ on the edge $\mu - \lambda^\vee$, by the symmetry of *jeu de taquin*, also contains a standard tableau $\mathrm{T}_{\lambda^\vee/\nu}$ on the edge $\nu - \lambda^\vee$. Denoting by ρ_1 the infusion corresponding to rectification and by ρ_2 the infusion corresponding to antirectification, each of the six faces has a pair of skew standard tableaux and a pair of standard tableaux of normal shape or of antinormal shape: $\rho_2(\mathrm{T}_{\lambda^\vee/\mu}, \mathrm{T}^{\mathrm{a}}_\lambda) = (\mathrm{T}_{\nu^\vee/\mu}, \mathrm{T}^{\mathrm{a}}_\nu)$, $\rho_1(\mathrm{T}_\mu, \mathrm{T}_{\nu^\vee/\mu}) = (\mathrm{T}_\lambda, \mathrm{T}_{\nu^\vee/\lambda})$, $\rho_2(T_{\nu^\vee/\lambda}, \mathrm{T}^{\mathrm{a}}_\nu) = (\mathrm{T}_{\mu^\vee/\lambda}, \mathrm{T}^{\mathrm{a}}_\mu)$ or $\rho_1(\mathrm{T}_\mu, \mathrm{T}_{\lambda^\vee/\mu}) = (\mathrm{T}_\nu, \mathrm{T}_{\lambda^\vee/\nu})$, $\rho_2(\mathrm{T}_{\lambda^\vee/\nu}, \mathrm{T}^{\mathrm{a}}_\lambda) = (\mathrm{T}_{\mu^\vee/\nu}, \mathrm{T}^{\mathrm{a}}_\mu)$, $\rho_1(\mathrm{T}_\nu, \mathrm{T}_{\mu^\vee/\nu}) = (\mathrm{T}_\lambda, \mathrm{T}_{\mu^\vee/\lambda})$. That is $\rho_1\rho_2\rho_1(\mathrm{T}_\mu, \mathrm{T}_{\lambda^\vee/\mu}, \mathrm{T}_\lambda = \rho_2\rho_1\rho_2(\mathrm{T}_\mu, \mathrm{T}_{\lambda^\vee/\mu}, \mathrm{T}_\lambda) = (\mathrm{T}_\lambda, \mathrm{T}_{\mu^\vee/\lambda}, \mathrm{T}^{\mathrm{a}}_\mu)$ $\mathtt{CARTONS}_{\mu,\nu,\lambda} \to \mathtt{CARTONS}_{\lambda,\nu,\mu}$

4.3. **LR companion tableaux and Lascoux's double crystal graph.** Let T be an LR tableau of shape λ/μ with companion G. We recall Lascoux's *double crystal graph structure on biwords* [L03] where a crystal operator consists of a left and a right operator, and a crystal string of a left and a right string. Let $\mathcal{B}(T,G)$ be the crystal graph whose vertices consist of the collection of biwords whose recording tableau is G in the RSK-correspondence, Subsection 2.5.3, with highest weight element the biword $W^{\lambda/\mu} = \binom{\mathbf{y}}{w(T)}$ on the LHS of (2.8) identified with the LR tableau T, and lowest weight element the biword $\binom{\mathbf{y}}{\sigma_0 w(T)}$ identified with $\sigma_0 T = T^e$. The vertices of $\mathcal{B}(T,G)$ describe simultaneously integer matrices and tableaux. The latter have the former as recording matrices. Instead of biwords we may consider integer matrices which are the recording matrices of the tableaux that they do describe.

Proposition 2.2, exhibits the Lascoux's double crystal graph structure on biwords [L03]. It shows that the strings of the crystal graph $\mathcal{B}(T,G)$, are transformed, by re-ordering the biwords (or transposing the corresponding matrices) as on the RHS of (2.8), into the strings of the *cocrystal* $\mathcal{CB}(T,G)$, the set of biwords whose insertion tableau is G [L03, p.103], with top biword $W^{\nu} = \binom{w(G)}{\mathbf{x}}$, and bottom biword $\binom{w(G^{\mathrm{a}})}{\mathbf{x}}$ with G^{a} the anti–normal form of G. Under this reordering of the biletters, the Kashiwara operators in $\mathcal{B}(T,G)$ are translated to the cocrystal operators or right operators, elementary *jeu de taquin* (*reverse jeu de taquin*) operations on two–row tableaux (see also [Az06]). Again instead of biwords we may consider as vertices of $\mathcal{CB}(T,G)$ the transpose of the matrices as the vertices of the crystal $\mathcal{B}(T,G)$. The matrices as vertices of $\mathcal{CB}(T,G)$ are the recording matrices of the tableaux that they do describe.

An i-string in $\mathcal{CB}(T,G)$ is an ordered string of two–row words in the plactic class of a two–row tableau [L03, p.100]. The two-row words being ordered according to the length of their bottom row, such that the two–row word on the top is a two-row tableau of partition shape (ν_i,ν_{i+1}), and on the bottom is the anti-normal form of the two-row tableau on the top, see [L03, Section 2]. Let θ_i be translation of the crystal reflection operators σ_i on the cocrystal under the reordering of the billeters in the biwords as explained above. Thus the symmetric group also acts on the cocrystal through the involutions θ_i which reflects each i-string about the middle, for $i \in 1,\dots,n-1$. In particular, θ_i sends the top of a i-string, a two–row tableau, to its anti-normal form in the bottom and vice versa. Indeed the entries of the j–th row of $\theta_i G$ are precisely the k's telling us in which rows of $\sigma_i T$ the j's are filled in. Indeed $\iota(\sigma_i T) = \theta_i G$ and put $\theta_0 := \theta_{i_N}\dots\theta_{i_1}$ where $\sigma_0 = \sigma_{i_N}\dots\sigma_{i_1}$. Thus $\iota(\sigma_0 T) = \iota(T^e) = \theta_0 G = G^{\mathrm{a}}$. This defines the commutative scheme

$$\begin{array}{ccccccccc} T & \longleftrightarrow & \sigma_{i_1}T & \longleftrightarrow & \sigma_{i_2}\sigma_{i_1}T & \longleftrightarrow & \cdots & \longleftrightarrow & \sigma_0 T = T^e \\ \iota\updownarrow & & \iota\updownarrow & & \iota\updownarrow & & & & \iota\updownarrow \\ G & \longleftrightarrow & \theta_{i_1}G & \longleftrightarrow & \theta_{i_2}\theta_{i_1}G & \longleftrightarrow & \cdots & \longleftrightarrow & \theta_0 G = G^{\mathrm{a}}. \end{array}$$

Example 4.2. Consider Example 2.4. The following exhibits the action of $\mathfrak{S}_3$ on the key tableaux (straight shape tableaux whose weight is a reordering of the partition shape) of $B(\nu,G)$ whose weight is a reordering of the partition shape)

Reordering the biletters so that the biword bottom row is a row word, equivalently, transposing the matrices defined by the biwords above, one obtains

Thus and taking into account Algorithm 3.6, we have the following result relating the tableaux T, T^e, $T^{e\bullet}$ and $T^{e\bullet\blacklozenge}$ respectively with their corresponding companion tableaux G, G^{a}, G^E and $G^{E\blacklozenge} = G^{\blacklozenge E}$. See also the construction of $G^{E\blacklozenge}$ in [LecLen17, Section 6.1].

$$\begin{array}{ccc}
 & T \equiv W^{\lambda/\mu} = \left(\begin{smallmatrix} 1&1&1&1&2&2&2&3&3&3\\ 1&1&1&1&2&2&1&3&3&2 \end{smallmatrix}\right) & \\
\swarrow \sigma_1 & & \sigma_2 \searrow \\
\sigma_1 T \equiv \left(\begin{smallmatrix} 1&1&1&1&2&2&2&3&3&3\\ 2&2&1&1&2&2&1&3&3&2 \end{smallmatrix}\right) & & \sigma_2 T \equiv \left(\begin{smallmatrix} 1&1&1&1&2&2&2&3&3&3\\ 1&1&1&1&2&2&1&3&3&3 \end{smallmatrix}\right) \\
\downarrow \sigma_2 & & \sigma_1 \downarrow \\
\sigma_2\sigma_1 T \equiv \left(\begin{smallmatrix} 1&1&1&1&2&2&2&3&3&3\\ 3&3&1&1&2&2&1&3&3&3 \end{smallmatrix}\right) & & \sigma_1\sigma_2 T \equiv \left(\begin{smallmatrix} 1&1&1&1&2&2&2&3&3&3\\ 2&2&1&1&2&2&2&3&3&3 \end{smallmatrix}\right) \\
\searrow \sigma_1 & & \sigma_2 \swarrow \\
 & \sigma_0 T \equiv \left(\begin{smallmatrix} 1&1&1&1&2&2&2&3&3&3\\ 3&3&1&1&2&2&2&3&3&3 \end{smallmatrix}\right) &
\end{array}$$

$$\begin{array}{ccc}
 & G \equiv W^{\nu} = \left(\begin{smallmatrix} 2&1&1&1&1&3&2&2&3&3\\ 1&1&1&1&1&2&2&2&3&3 \end{smallmatrix}\right) & \\
\swarrow \theta_1 & & \theta_2 \searrow \\
\theta_1 G \equiv \left(\begin{smallmatrix} 2&1&1&3&2&2&1&1&3&3\\ 1&1&1&2&2&2&2&2&3&3 \end{smallmatrix}\right) & & \theta_2 G \equiv \left(\begin{smallmatrix} 2&1&1&1&1&2&2&3&3&3\\ 1&1&1&1&1&2&2&3&3&3 \end{smallmatrix}\right) \\
\downarrow \theta_2 & & \theta_1 \downarrow \\
\theta_2\theta_1 G \equiv \left(\begin{smallmatrix} 2&1&1&2&2&3&3&3&1&1\\ 1&1&1&2&2&3&3&3&3&3 \end{smallmatrix}\right) & & \theta_1\theta_2 G \equiv \left(\begin{smallmatrix} 1&1&2&2&2&1&1&3&3&3\\ 1&1&2&2&2&2&2&3&3&3 \end{smallmatrix}\right) \\
\searrow \theta_1 & & \theta_2 \swarrow \\
 & \theta_0 G \equiv \left(\begin{smallmatrix} 1&1&2&2&2&1&1&3&3&3\\ 1&1&2&2&2&3&3&3&3&3 \end{smallmatrix}\right) &
\end{array}$$

Theorem 4.6. *Let* T *be a LR tableau with shape* λ/μ *and right LR companion tableau* G. *Then*

(*a*) *the following diagram is commutative:*

$$\begin{array}{ccccccc}
T & \overset{e}{\longleftrightarrow} & T^{e} & \overset{\bullet}{\longleftrightarrow} & \rho(T) = T^{e\bullet} & \overset{\blacklozenge}{\longleftrightarrow} & \varrho(T) = T^{e\bullet\blacklozenge} \\
\iota \updownarrow & & \iota \updownarrow & & \iota \updownarrow & & \iota \updownarrow \\
G & \underset{\mathrm{a}}{\longleftrightarrow} & G^{\mathrm{a}} & \underset{\bullet}{\longleftrightarrow} & G^{\mathrm{a}\bullet} = \operatorname{evac} \mathrm{G} & \underset{\blacklozenge}{\longleftrightarrow} & \blacklozenge \operatorname{evac} \mathrm{G}.
\end{array}$$

(*b*) *the involution symmetries* ρ *and* ϱ *translate to companion LR tableaux as follows*

$$\rho : \mathrm{LR}_{\nu,\lambda/\mu} \longrightarrow \mathrm{LR}_{\nu,(\lambda/\mu)^{\bullet}} : \mathrm{G} \mapsto \operatorname{evac} \mathrm{G} \ \textit{such that} \operatorname{evac} \iota(\mathrm{T}) = \iota(\rho(\mathrm{T}));$$

$$\varrho : \mathrm{LR}_{\nu,\lambda/\mu} \longrightarrow \mathrm{LR}_{\nu^{\mathrm{t}},(\lambda/\mu)^{\mathrm{t}}} : \mathrm{G} \mapsto \blacklozenge\mathrm{evacG} = \mathrm{evac}\blacklozenge\mathrm{G} \ \textit{such that} \operatorname{evac} \blacklozenge\iota(\mathrm{T}) = \iota(\varrho(\mathrm{T})).$$

4.4. **Illustration.** Consider the BSS'LR transposer [BSS96]

$$\begin{array}{rccc}
\varrho^{BSS} : & \mathrm{LR}(\mu,\nu,\lambda) & \rightarrow & \mathrm{LR}(\mu^{\mathrm{t}},\nu^{\mathrm{t}},\lambda^{\mathrm{t}}) \\
 & T & \mapsto & \varrho^{BSS}(T) = [Y(\nu^{t})]_{K} \cap [\widehat{T}^{t}]_{dK}
\end{array}.$$

The image of T by the BSS–bijection is the unique tableau of shape λ^t/μ^t whose rectification is $Y(\nu^t)$ and the Q–symbol of the column reading word is $Q(w(T))^{Et}$. The idea behind this bijection can be told as follows: $\widehat{T}$ constitutes a set of instructions telling where expanding slides can be applied to $Y(\mu)$. Then $\widehat{T}^t$ is a set of instructions telling where expanding slides can be applied to $Y(\mu)^t$. Tableau–switching provides an algorithm to give way to those instructions. In the following, s denotes switching:

$$\begin{array}{ccccccc}
Y(\mu)\cup T & \xrightarrow[\text{of T}]{\text{standardization}} & Y(\mu)\cup\widehat{T} & \xrightarrow[\text{of }\widehat{T}]{\text{transposition}} & Y(\mu^t)\cup\widehat{T}^t & & Y(\mu^t)\cup\varrho^{BSS}(T) \\
 & & & & \downarrow s & & \uparrow s \\
 & & & & (\widehat{T}^t)^{\mathrm{n}}\cup Z & \mapsto & Y(\nu^t)\cup Z
\end{array}.$$

Then $\varrho^{BSS}(T)\equiv Y(\nu^t)$ and $\varrho^{BSS}(T)\overset{d}{\equiv}\widehat{T}^t$.

Example 4.3. Let T in $\mathrm{LR}(\mu,\nu,\lambda)$ with $\mu=21$, $\nu=532$ and $\lambda=643$.

$$T=\begin{array}{|c|c|c|c|c|c|}\hline 2&3&3& & & \\\hline &1&2&2& & \\\hline & &1&1&1&1\\\hline\end{array}\ \rightarrow\ \widehat{T}=\begin{array}{|c|c|c|c|c|c|}\hline 6&9&10& & & \\\hline &1&7&8& & \\\hline & &2&3&4&5\\\hline\end{array}\ \rightarrow\ \widehat{T}^t=\begin{array}{|c|c|c|}\hline 5& & \\\hline 4& & \\\hline 3&8& \\\hline 2&7&10\\\hline &1&9\\\hline & &6\\\hline\end{array}\ \rightarrow$$

$$\begin{array}{ccc}
\rightarrow\ Y(\mu^t)\cup\widehat{T}^t=\begin{array}{|c|c|c|}\hline 5& & \\\hline 4& & \\\hline 3&8& \\\hline 2&7&10\\\hline \mathbf{2}&1&9\\\hline \mathbf{1}&\mathbf{1}&6\\\hline\end{array} & & \begin{array}{|c|c|c|}\hline 5& & \\\hline 4& & \\\hline 2&3& \\\hline 1&2&3\\\hline \mathbf{2}&1&2\\\hline \mathbf{1}&\mathbf{1}&1\\\hline\end{array}=Y(\nu^t)\cup\varrho^{BSS}(T) \\
s\downarrow & & \uparrow s \\
(\widehat{T}^t)^{\mathrm{n}}\cup Z=\begin{array}{|c|c|c|}\hline \mathbf{1}& & \\\hline 5& & \\\hline 4&\mathbf{2}& \\\hline 3&8&\mathbf{1}\\\hline 2&7&10\\\hline 1&6&9\\\hline\end{array} & \rightarrow & \begin{array}{|c|c|c|}\hline \mathbf{1}& & \\\hline 5& & \\\hline 4&\mathbf{2}& \\\hline 3&3&\mathbf{1}\\\hline 2&2&2\\\hline 1&1&1\\\hline\end{array}=Y(\nu^t)\cup Z
\end{array}.$$

Consider the involution $\varrho=e\bullet\blacklozenge$,

Example 4.4. Recall Example 4.1. Letting T as in the previous example, we get

$$\mathrm{T}=\begin{array}{|c|c|c|c|c|c|}\hline 2&3&3& & & \\\hline &1&2&2& & \\\hline & &1&1&1&1\\\hline\end{array}\ \overset{e}{\rightarrow}\ \mathrm{T}^e=\begin{array}{|c|c|c|c|c|c|}\hline 3&3&3& & & \\\hline &2&2&2& & \\\hline & &1&1&3&3\\\hline\end{array}\ \overset{\bullet}{\rightarrow}\ \mathrm{T}^{e\bullet}=\begin{array}{|c|c|c|c|c|c|}\hline 1&1&3&3& & \\\hline & &2&2&2& \\\hline & & &1&1&1\\\hline\end{array}\ \overset{\blacklozenge}{\rightarrow}$$

$$w=1111221332\ \rightarrow\ \sigma_0 w=3311222333\ \overset{*}{\rightarrow}\ (\sigma_0 w)^{\bullet}=1112223311\ \overset{\blacklozenge}{\rightarrow}$$

$$\overset{\blacklozenge}{\rightarrow}\ \varrho(T)=\mathrm{T}^{e\bullet\blacklozenge}=\begin{array}{|c|c|c|}\hline 5& & \\\hline 4& & \\\hline 2&3& \\\hline 1&2&3\\\hline &1&2\\\hline & &1\\\hline\end{array}.$$

$\overset{\blacklozenge}{\rightarrow}\ (\sigma_0 w)^{\bullet\blacklozenge}=1231231245$ column word of $\varrho(T)=\varrho^{BSS}(T)$.

On the left hand side one has the biwords associated to T and T^e and on the right hand side the reordered biwords giving the companions G, Example 2.5, and G^{a}

respectively

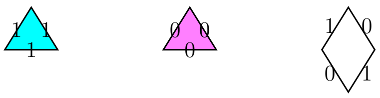
$$\begin{array}{ccccc} W^{\lambda/\mu} & = \left(\begin{smallmatrix} 1&1&1&1&2&2&2&3&3&3\\ 1&1&1&1&2&2&1&3&3&2 \end{smallmatrix}\right) & \rightarrow & W^{\nu} & = \left(\begin{smallmatrix} 2&1&1&1&1&3&2&2&3&3\\ 1&1&1&1&1&2&2&2&3&3 \end{smallmatrix}\right) \\ & \downarrow_{\sigma_0} & & & \downarrow_{\theta_0} \\ & \left(\begin{smallmatrix} 1&1&1&1&2&2&2&3&3&3\\ 3&3&1&1&2&2&2&3&3&3 \end{smallmatrix}\right) & \rightarrow & & \left(\begin{smallmatrix} 1&1&2&2&2&3&3&3&1&1\\ 1&1&2&2&2&3&3&3&3&3 \end{smallmatrix}\right). \end{array}$$

The companions of $T^e \in \mathrm{LR}(\mu, \nu^\bullet, \lambda)$, $T^{e\bullet} \in LR(\lambda, \nu, \mu)$ and $T^{e\bullet\blacklozenge} \in \mathrm{LR}(\mu^{\mathrm{t}}, \nu^{\mathrm{t}}, \lambda^{\mathrm{t}})$ are respectively

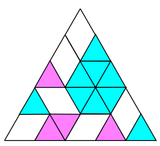
$$G^{\mathrm{a}} = \begin{array}{|c|c|c|c|c|c|} \hline & 1 & 1 & 3 & 3 & 3 \\ \hline & & & 2 & 2 & 2 \\ \hline & & & & 1 & 1 \\ \hline \end{array}, \quad G^{\mathrm{a}\bullet} = G^{\bullet\mathrm{n}} = G^{E} = \begin{array}{|c|c|c|c|c|c|} \hline 3 & 3 & & & & \\ \hline 2 & 2 & 2 & & & \\ \hline 1 & 1 & 1 & 3 & 3 & \\ \hline \end{array}, \quad G^{E\blacklozenge} = \begin{array}{|c|c|c|} \hline & & \\ \hline 6 & & \\ \hline 5 & & \\ \hline 3 & 4 & \\ \hline 2 & 3 & 4 \\ \hline 1 & 2 & 3 \\ \hline \end{array}, \ \mu^\bullet = 012$$

5. The $\mathbb{Z}_2 \times \mathfrak{S}_3$–symmetries and the subgroup $\mathcal{H}$ of KT puzzle dualities and rotations

5.1. **Knutson–Tao puzzles and Tao's bijection.** A KT puzzle of size n [KTW04] is a tiling of an equilateral triangle of side length n with three kind of puzzle pieces: (a) unit equilateral triangles with all edges labeled 1 (here also represented in blue colour); (b) unit equilateral triangles with all edges labeled 0 (here also represented in pink colour); and (c) unit rhombi (two equilateral triangles joined together) with the two edges, clockwise, of acute vertices labeled 0, and the other two labeled 1,



such that whenever two pieces share an edge, the labels on the edge must agree. Puzzle pieces may be rotated in any orientation but rhombi can not be reflected. The boundary data of the KT puzzle is the partition triple (μ, ν, λ) where the partitions μ, ν and λ appear clockwise, starting in the lower-left corner, as 01–words. The partitions μ, ν and λ as 01–words have exactly d 1's and $n-d$ 0's. This means that the blue unitary triangles constitute a triangle of size d, the d-*triangle*, and the pink unitary triangles a triangle of size $n-d$, the $(n-d)$-*triangle*. For instance, the following is a puzzle with $n = 5$, $d = 3$ and boundary $\mu = 01011 = (100)$, $\nu = 01101 = (110)$ and $\lambda = 10101 = (210)$, read clockwise starting in the lower-left corner.

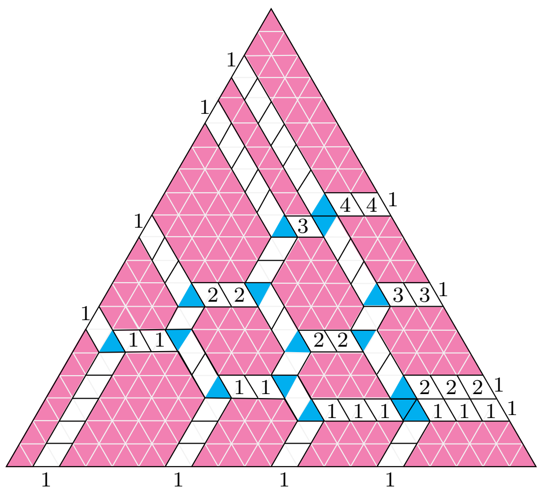


(5.1)

The number of puzzles with μ, ν and λ appearing clockwise as 01–strings along the boundary is equal to $c_{\mu\,\nu\,\lambda}$ [KTW04]. KT puzzles are in bijection with LR tableaux [KTW04]. We use *Tao's bijection "without words"* in [Va06] also used in [Pu08, Figure 9]. Tao's bijection defines a one-to-one correspondence between puzzles of size n and boundary (μ, ν, λ) as 01 strings with d 1's and $n-d$'s 0's, and LR tableaux with boundary (λ, μ, ν), inside a rectangle D of size $d \times (n-d)$. From now on we just write puzzle to mean KT puzzle, as no other puzzles will be considered. To get the LR tableau filling from a puzzle, as illustrated in Example 5.1, we follow Pechenik's wording [Pe16]. We construct disjoint trails of puzzle pieces, one for each 1 along the bottom side, the λ-edge.

Then these trails will be read to produce the row fillings of the LR tableau. We think of the puzzle pieces as rooms and the 1-labeled edges as doorways. We enter through one of the doors on the λ-edge. Whenever we enter a room, we leave it by a different door edge. We traverse right leaned rhombi from bottom to to top and left leaned rhombi from left to right. When we enter the base of a triangle, we exit through the door on our right, and when we enter the lower left door of a upsidedown triangle, we exit through the door on our left. Thus we will be always moving northeast and eventually we exit through a door on the ν-side. The recording of the rooms together with the filling along this walk gives the track of the initial door on the λ-side. Reading the filling of the track of each basement door gives the row filling of each row in the LR tableau.

Example 5.1. Tao's bijection on the puzzle below, with $n = 20$, $d = 4$ and boundary $(\mu = (10, 7, 3, 2), \nu = 8522, \lambda = (11, 8, 5, 1))$, gives, on the right, the LR tableau T with boundary $(\lambda = (11, 8, 5, 1); \mu = (10, 7, 3, 2); \nu = 8522)$, inside the rectangle D of size 4×16. From bottom to top, there are exactly $\ell(\mu) = 4$ left leaned rhombi corridors, SE to NW, of lengths μ_1, μ_2, μ_3 and μ_4 respectively. Those corridors are filled in with μ_i, i's, respectively. In our example, the filling of the track in each base door, left to right, gives the words 1122344, 112233, 111222, and 111.



(μ, ν, λ)

$T =$

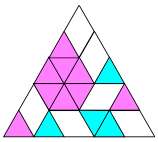
	1	1	2	2	3	4	4								
					1	1	2	2	3	3					
								1	1	1	2	2	2		
											1	1	1		

(λ, μ, ν).

5.2. **The KT puzzle H–subgroup of symmetries.** The reflection of a (upright or left or right leaned) rhombus in a puzzle swaps 0 and 1 labels which propagates in a unique way through the puzzle and eventually swaps 0 and 1 labels in the boundary. To get a puzzle again we have to make one of three diagonal reflections of the puzzle (a diagonal is a line joining a vertex to the midpoint of the opposite side). Recalling Section 2.1, with respect to operations on 01 words and partitions, this procedure defines, according to the chosen diagonal, an involution on puzzles. The vertical reflection while swapping all 01 labels of a puzzle of boundary (μ, ν, λ), defines the *dual puzzle* of boundary $(\nu^t, \mu^t, \lambda^t)$. This procedure is an involution, denoted ♠, on puzzles. It exhibits the symmetry $c_{\mu\nu\lambda} = c_{\nu^t\mu^t\lambda^t}$ on puzzles. Performing the two remaining diagonal left and right reflections, respectively, with 01 swapping, these procedures define the involutions ♦ (left) and ♣ (right) respectively. They exhibit the symmetries (puzzle dualities) $c_{\mu\nu\lambda} = c_{\lambda^t\nu^t\mu^t}$ and $c_{\mu,\nu,\lambda} = c_{\mu^t\lambda^t\nu^t}$ on puzzles, respectively. The duals of the puzzles counted by $c_{\mu\nu\lambda}$ are exactly those counted by $c_{\nu^t\mu^t\lambda^t}$ or $c_{\lambda^t\nu^t\mu^t}$ or $c_{\mu^t\lambda^t\nu^t}$. The involutions ♦, ♠ and ♣ are the unique involutions which swap pink (label 0) and blue (label 1) colours in a puzzle such that the resulting tiled triangle is still a puzzle, equivalently, which swap the blue d-triangle with the pink $(n-d)$-triangle. As an apart, one observes that the symmetry with respect to complement operation on partitions, equivalently, reversing the 01 word, Section 2.1, is indeed not easy to exhibit due to the restriction of the reflection operation on rhombi. The same difficulty has been already observed in [BerZel92, Remarks] in the case of BZ triangles. The same difficulty applies to the transposition of partitions, reversing and 01 swapping a 01 string.

The action of $H = \langle \tau\varsigma_1, \tau\varsigma_2\rangle = \langle \tau\varsigma_1, \tau\varsigma\rangle = \langle \tau\varsigma_2, \tau\varsigma\rangle$ on puzzles is defined via any two of the involutions $\blacklozenge, \spadesuit, \clubsuit$. The compositions $\clubsuit\blacklozenge = \blacklozenge\spadesuit = \spadesuit\clubsuit$ and $\blacklozenge\clubsuit = \spadesuit\blacklozenge = \clubsuit\spadesuit$ rotate the puzzle clockwise $2\pi/3$ and $4\pi/3$ radians about the center respectively, and realize the corresponding rotational (cycle) symmetries $c_{\mu,\nu,\lambda} = c_{\lambda,\mu,\nu}$ and $c_{\mu,\nu,\lambda} = c_{\nu,\lambda,\mu}$. The group of symmetries of a puzzle, generated by the action of H, is realized by the three diagonal reflections with 01 swapping and clockwise 0, $2\pi/3$ and $4\pi/3$ radians rotations about the center.

Example 5.2. Illustration of the action of $\tau\varsigma_1 \in H$ on puzzles through the involution $\spadesuit$. The involution $\spadesuit$ on the puzzle (5.1), realizes the action of $\tau\varsigma_1$ on that puzzle, giving the puzzle



whose boundary data is ($\nu^t = 01001$; $\mu^t = 00101; \lambda^t = 01010$).

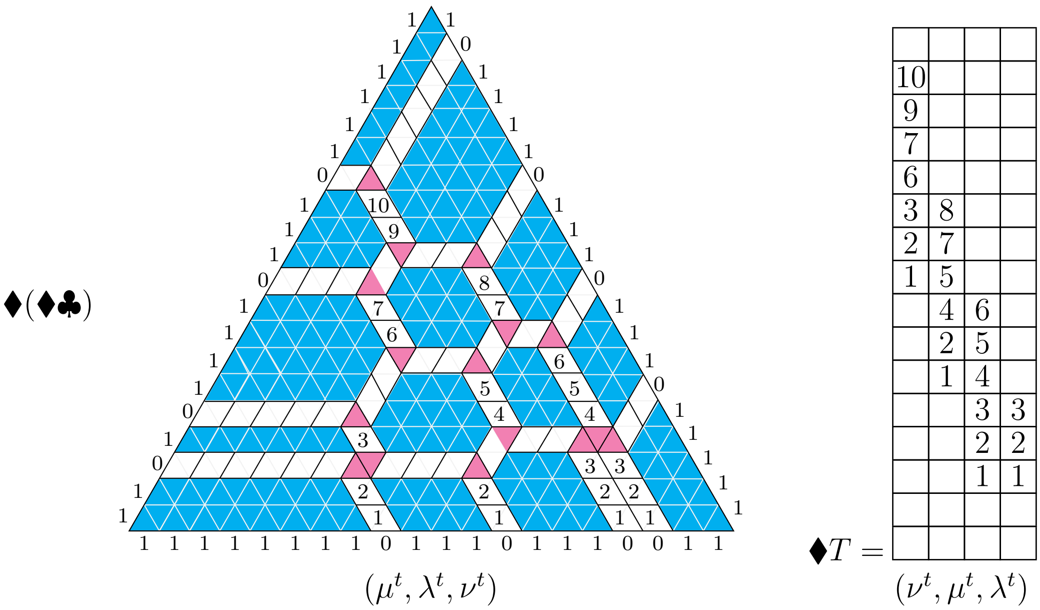
The group action of H on puzzles is faithful and we may identify H with its representation in $Sym(\mathcal{LR})$, $H \simeq \langle \spadesuit, \blacklozenge\rangle = \langle \clubsuit, \blacklozenge\rangle = \langle \spadesuit, \clubsuit\rangle$, and a possible presentation is

$$H = \{\spadesuit, \blacklozenge : \spadesuit^2 = \blacklozenge^2 = \mathbf{1} = (\blacklozenge\spadesuit)^{\mathbf{3}}\} \simeq \mathcal{D}_{\mathbf{3}} = \mathcal{S}_{\mathbf{3}}, \tag{5.2}$$

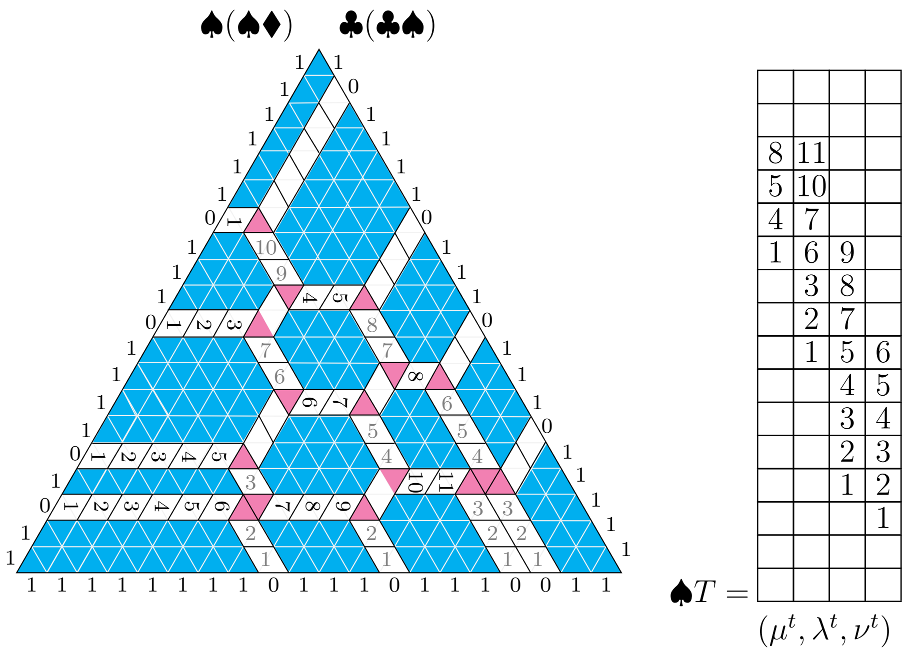
the dihedral group of an equilateral triangle. Since through Tao's bijection, LR tableau and puzzle boundaries differ by a cyclic permutation, first rotate counterclockwise the puzzle $2\pi/3$ radians about the center, follow with the involution $\blacklozenge$, $\spadesuit$ or $\clubsuit$, and then apply Tao's bijection to obtain the involutions $\blacklozenge$, $\spadesuit$ or $\clubsuit$ on LR tableaux respectively. That is, on puzzles, $\clubsuit = \blacklozenge(\blacklozenge\clubsuit)$, $\blacklozenge = \spadesuit(\spadesuit\blacklozenge)$ and $\spadesuit = \clubsuit(\clubsuit\spadesuit)$ are translated to LR tableaux, by Tao's bijection, to $\blacklozenge$, $\spadesuit$ or $\clubsuit$, respectively. In particular, we show that the bijection $\blacklozenge$: $\mathrm{LR}(\mu,\nu,\lambda) \to \mathrm{LR}(\lambda^{\mathrm{t}},\nu^{\mathrm{t}},\mu^{\mathrm{t}})$, defined in Section 3.3, exhibiting the identity $c_{\mu\nu\lambda} = c_{\lambda^t\nu^t\mu^t}$ on LR tableaux is translated to puzzles in this fashion by exhibiting the identity $c_{\mu,\nu,\lambda} = c_{\mu^t,\lambda^t,\nu^t}$. We also define bijections $\spadesuit : \mathrm{LR}(\mu,\nu,\lambda) \to \mathrm{LR}(\nu,\mu,\lambda)$, $\clubsuit : \mathrm{LR}(\mu,\nu,\lambda) \to \mathrm{LR}(\mu,\lambda,\nu)$, exhibiting the identities $c_{\mu,\nu,\lambda} = c_{\mu^t,\nu^t,\lambda^t}$, $c_{\mu,\nu,\lambda} = c_{\mu^t\lambda^t\nu^t}$ respectively, which are also translated to puzzles in the same fashion by exhibiting the identities $c_{\mu,\nu,\lambda} = c_{\lambda^t,\mu^t,\nu^t}$, $c_{\mu,\nu,\lambda} = c_{\nu^t,\mu^t,\lambda^t}$.

Example 5.3. If we first rotate clockwise the puzzle $4\pi/3$ radians about the center and then apply $\blacklozenge$, that is, we perform $\blacklozenge(\blacklozenge\clubsuit)$, it is not difficult to see that Tao's bijection translates the involution $\blacklozenge$ (the left diagonal reflection with 01 swapping) on the $4\pi/3$ radians clockwise rotated puzzle to the involution $\blacklozenge$ on LR tableaux as we have defined in Subsection 3.3. Consider the puzzle in Example 5.1 together with the left leaned rhombi filling. After rotating the puzzle clockwise $\frac{4}{3}\pi$ radians about the center, to obtain a puzzle of boundary (ν,λ,μ), apply the involution $\blacklozenge$ (left diagonal reflection while 01 label swapping) to give a puzzle of boundary (μ^t,λ^t,ν^t). Then replace East-West: (i) the ten 1's word, inside the left leaned rhombi, with the word 123456789 (10); (ii) the seven

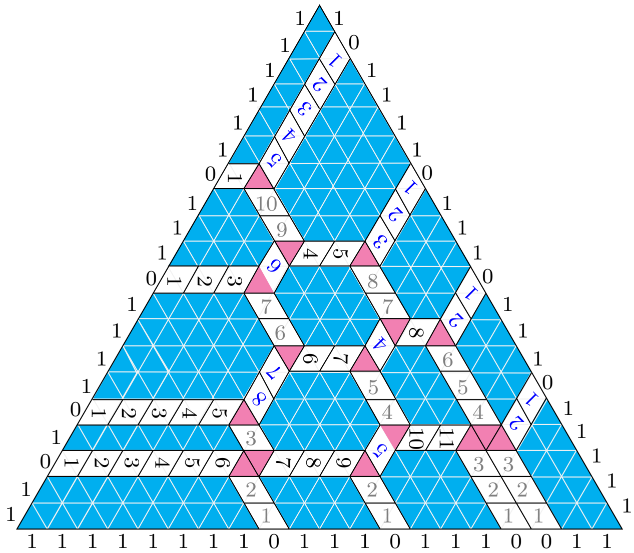
2's word with the word 1234567; (iii) the three 3's word by 123; and (iv) the two 4's word with 12. The puzzle-LR tableau pair with boundaries (μ,ν,λ) and (λ,μ,ν) respectively, in Example 5.1, is transformed into a puzzle-LR tableau pair of boundaries (μ^t,λ^t,ν^t) and (ν^t,μ^t,λ^t), where $\mu^t=4^2 3 2^4 1^3$ and $\ell(\mu^t)=10$, respectively. See figure below



Performing ♠(♠♦) = ♠(♦♣) on the puzzle in Example 5.1, or, equivalently, rotate counter clockwise the puzzle just above by $2\pi/3$ radians about the center ♠(♠♦) = ♠♦(♦(♦♣)) to get a puzzle of boundary (ν^t,λ^t,μ^t). Tao's bijection translates the action of ♠ on puzzles to LR tableaux. The puzzle-LR tableau pair with boundaries (μ,ν,λ) and (λ,μ,ν) respectively, in Example 5.1, is transformed into a puzzle-LR tableau pair of boundaries (λ^t,ν^t,μ^t) and (μ^t,λ^t,ν^t) respectively. See figure below, to obtain the LR filling on the right, one has to rotate the puzzle counterclockwise $2\pi/3$ radians about the center, where $\lambda^t=43^4 2^3 1^3$, and $\ell(\lambda^t)=11$, and ignore the old filling in gray



Performing ♣(♦♣) = ♣(♣♠) on the puzzle in Example 5.1 (or ♦♠(♦(♦♣)), that is, rotating the previous puzzle clockwise $2\pi/3$ radians about the center), we get a puzzle of boundary $(\nu^t, \mu^t, \lambda^t)$ where $\nu^t = 4^2 2^3 1^3$. It is not difficult to see that the corridors consisting of upright rhombi, in the previous puzzle, now filled in blue below, give rise to the filling of ♣T of boundary $(\lambda^t, \nu^t, \mu^t)$.



5			
2	8		
1	7		
	4		
	2		
	1	6	
		3	
		2	
		1	5
			4
			3
			2
			1

♣T = $(\lambda^t, \nu^t, \mu^t)$

Tao's bijection translates the involutions ♠, ♣ and ♦ on puzzles to LR tableaux which in turn can also be explained using Puhrboo's mosaics [Pu08] naturally in bijection with puzzles. In [Pu08, Section 5.1] it is discussed how the operation *migration* of a single rhombus in a mosaic is related with *jeu de taquin* slides on tableaux. *Migration* is an invertible operation on tableau-like structures on the rhombi of a mosaic, called flocks, that allows to identify a mosaic (equivalently, a puzzle) with an LR tableau. It gives a bijection between mosaics (equivalently puzzles) and LR tableaux, and, with appropriate orientation of the flocks in the mosaic, it coincides with Tao's bijection. More importantly, *migration* allows to relate operations on puzzles with *jeu de taquin* operations, like tableau-switching [BSS96], on LR tableaux. This explains the correspondence between the action of H on puzzles and on LR tableaux. Technical details and illustrations on mosaics are deferred to Appendix A. Our concern next is to show that although involutions ♠, ♣ and ♦ may be executed using *jeu de taquin slides* on LR tableaux, those slides do not need to be performed upon a scan of the neighbours. The slides are independent of the relative size of the neighbours and are reduced to simple procedures defining linear cost involutions.

5.3. **The LR tableau H–symmetries.** Migration on Puhrboo's mosaics [Pu08] is related with *jeu de taquin* and translates H-symmetries on puzzles to LR tableaux through tableau switching and *standardization*. This translation coincides with Tao's bijection as illustrated in Example 5.3. We want to avoid the computational complexity of standardization and *jeu de taquin* procedure which we succeed through hybrid tableau pairs.

Given a decomposition of the rectangle D into shapes μ, $\lambda^\vee/\mu$, and $D/\lambda^\vee$, a triple of tableaux (U_1, U_2, U_3) is said to be a three–fold multitableau of shape $(\mu, \lambda^\vee/\mu, D/\lambda^\vee)$ if U_1 is a filling of the shape μ, U_2 is a filling of $\lambda^\vee/\mu$, and U_3 is a filling of $D/\lambda^\vee$. A three–fold LR multitableau of boundary data (μ, ν, λ) is a three–fold multitableau where the inner tableau is the Yamanouchi tableau $Y(\mu)$, the middle one is the LR tableau of shape $\lambda^\vee/\mu$ and content ν, and the outer tableau is $Y(\lambda)^{\mathrm{a}}$ the *anti-normal form* of $Y(\lambda)$, defined to be the filling of the anti-normal shape of λ such that each column, right to left, is filled with consecutive integers bottom to top starting with 1. For instance, see (2.5). Tableau switching can be adjusted to move a pair of tableaux through each other where one is a column strict tableau and the other is a row strict tableau see [BSS96, Section 2, pp. 22]. Our next definitions are just a particular case where the left or right tableau is the transpose of a (or antinormal) Yamanouchi tableau row strict) and therefore switch moves can a priori be prescribed.

We define the bijection $\spadesuit : \mathrm{LR}(\mu,\nu,\lambda) \longrightarrow \mathrm{LR}(\nu^{\mathrm{t}},\mu^{\mathrm{t}},\lambda^{\mathrm{t}})$ as a five step procedure.

Definition 5.1 (Map $\spadesuit$)**.** Let $T \in \mathrm{LR}(\mu,\nu,\lambda)$ inside the rectangle D of size $d\times(n-d)$.

(1) Fill the inner shape μ, using a completely ordered alphabet different from the "numerical" filling of T, so that its transpose is the Yamanouchi tableau $Y(\mu^t)$.
(2) For $i = 1,\dots,d$, slide down vertically the i's in the filling of T to the ith row.
(3) For $i = 1,\dots,d$, slide horizontally all the numbers i's to the left so that we get the Yamanouchi tableau $Y(\nu)$. Erase $Y(\nu)$.
(4) Transpose the resulting filling to obtain $T^\spadesuit \in \mathrm{LR}(\nu^{\mathrm{t}},\mu^{\mathrm{t}},\lambda^{\mathrm{t}})$.

Clearly, the last step can also be the first step with obvious adaptations in the next steps. An illustration of this procedure follows.

Example 5.4. Let $T \in LR(\mu,\nu,\lambda)$ with $d = 4$, $n = 11$, $\mu = 4210$, $\nu = 5420$ and $\lambda = 5320$. Then considering the twofold hybrid tableau

$$T = \begin{array}{|c|c|c|c|c|c|c|}\hline 1&3&&&&&\\\hline &2&2&3&&&\\\hline &&1&2&2&&\\\hline &&&&1&1&1\\\hline\end{array} \to ([Y(\mu)]^t,T) = \begin{array}{|c|c|c|c|c|c|c|}\hline 1&3&&&&&\\\hline \mathrm{a}&2&2&3&&&\\\hline \mathrm{a}&\mathrm{b}&1&2&2&&\\\hline \mathrm{a}&\mathrm{b}&\mathrm{c}&\mathrm{d}&1&1&1\\\hline\end{array} \to \begin{array}{|c|c|c|c|}\hline 1&&&\\\hline 1&&&\\\hline 1&2&&\\\hline \mathrm{d}&2&3&\\\hline \mathrm{c}&1&2&\\\hline \mathrm{b}&\mathrm{b}&2&3\\\hline \mathrm{a}&\mathrm{a}&\mathrm{a}&1\\\hline\end{array} = (Y(\mu^t),T^t)$$

$$\to \begin{array}{|c|c|c|c|}\hline 1&&&\\\hline 1&&&\\\hline 1&2&&\\\hline \mathrm{d}&2&3&\\\hline 1&2&\mathrm{c}&\\\hline \mathrm{b}&2&3&\mathrm{b}\\\hline 1&\mathrm{a}&\mathrm{a}&\mathrm{a}\\\hline\end{array} \to \begin{array}{|c|c|c|c|}\hline \mathrm{d}&&&\\\hline \mathrm{b}&&&\\\hline 1&\mathrm{a}&&\\\hline 1&2&\mathrm{c}&\\\hline 1&2&\mathrm{a}&\\\hline 1&2&3&\mathrm{b}\\\hline 1&2&3&\mathrm{a}\\\hline\end{array} = ([Y(\nu)]^t,T^\spadesuit) \to \begin{array}{|c|c|c|c|}\hline \mathrm{d}&&&\\\hline \mathrm{b}&&&\\\hline &\mathrm{a}&&\\\hline &&\mathrm{c}&\\\hline &&\mathrm{a}&\\\hline &&&\mathrm{b}\\\hline &&&\mathrm{a}\\\hline\end{array} = \mathrm{T}^\spadesuit.$$

The procedure is clearly reversible and an involution on $\mathcal{LR}$. Next we check that it yields the desired tableau. Let $\mathbf{s}_i$, $i = 1,2$, denote the tableau–switching operation on the LR–multitableau of boundary data (μ,ν,λ) (recall Subsection 2.9) which switches the first two LR tableaux and the last two respectively, see [BSS96]. Compare the procedure with the explanation given by the migration for the operation $\spadesuit$ on mosaics in Appendix.

Proposition 5.1. *The map* $\spadesuit$ *is such that*

$$T \longrightarrow (Y(\mu^t),T^t,Y(\lambda^t)^{\mathrm{a}}) \longrightarrow \mathbf{s}_1(Y(\mu^t),T^t,Y(\lambda^t)^{\mathrm{a}}) = ([Y(\nu)]^t,T^\spadesuit,Y(\lambda^t)^{\mathrm{a}}).$$

Proof. Since T is column strict then T^t is row strict The second and third steps of the definition of the map $\spadesuit$ coincides with the action of the switching operation $\mathbf{s}_1$ on the hybrid twofold tableau $(Y(\mu^t),T^t)$, that is, $\mathbf{s}_1(Y(\mu^t),T^t) = ([Y(\nu)]^t,T^\spadesuit)$, see [BSS96, Section 2]. □

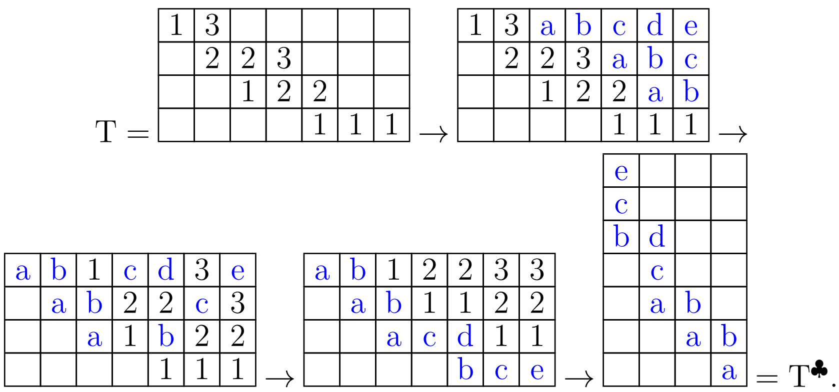
The bijection $\mathrm{LR}(\mu,\nu,\lambda) \xrightarrow{\clubsuit} \mathrm{LR}(\mu^{\mathrm{t}},\lambda^{\mathrm{t}},\nu^{\mathrm{t}})$ is defined similarly as a five step procedure.

Definition 5.2 (Map $\clubsuit$)**.** Let $T \in LR(\mu,\nu,\lambda)$ inside the rectangle D of size $d\times(n-d)$ and $\nu^t = (\nu^t_1,\nu^t_2,\dots,\nu^t_{\nu_1})$.

(1) Fill the outer shape λ, using a completely ordered alphabet different from the "numerical" filling of T, so that its transpose is $Y(\lambda^t)^{\mathrm{a}}$.
(2) For $i=1,\dots,\nu^t_1$, slide horizontally the rightmost i of T to the $(n-d)$th column of D; for $i=1,\dots,\nu^t_2$, slide horizontally the rightmost i in the first $(n-d-1)$ columns of T to the $(n-d-1)$th column of D; ..., lastly slide horizontally the remaining ν_1th 1 to the $(n-d-\nu_1+1)$th column of D.
(3) Slide up vertically the numbers along each column so that we get $Y(\nu)^{\mathrm{a}}$. Erase $Y(\nu)^{\mathrm{a}}$.
(4) Transpose the resulting filling to obtain $T^{\clubsuit} \in LR(\mu^t,\lambda^t,\nu^t)$.

The example illustrates the procedure.

Example 5.5. Consider again the LR tableau T of Example 5.4 with $d=4$ and $n-d=7$. One has $\nu=542$, $\nu^t=33221$, $\lambda=532$, $\lambda^t=33211$, and



The procedure is clearly an involution on $\mathcal{LR}$ and as before we check that it yields the desired tableau. Compare the procedure with the explanation given by the migration on mosaics for the operation $\clubsuit$ in Appendix. Again we have avoided the standardization of T.

Proposition 5.2. *The map $\clubsuit$ is such that*

$$T \longrightarrow (Y(\mu^t),T^t,Y(\lambda^t)^{\mathrm{a}}) \longrightarrow \mathbf{s}_2(Y(\mu^t),T^t,Y(\lambda^t)^{\mathrm{a}}) = (Y(\mu^t),T^{\clubsuit},[Y(\nu)^{\mathrm{a}}]^t).$$

Proof. The second and third steps of the definition of the map $\clubsuit$ correspond exactly to the action of the switching operation $\mathbf{s}_2$ on the hybrid two-fold tableau $(T^t,Y(\lambda^t)^{\mathrm{a}})$. ☐

Theorem 5.3. *The involutions $\spadesuit$ and $\clubsuit$ on $\mathcal{LR}$ have linear cost.*

Proof. The use of hybrid tableaux show clearly that the maps are performed using *jeu de taquin* without need to scan the neighbours. ☐

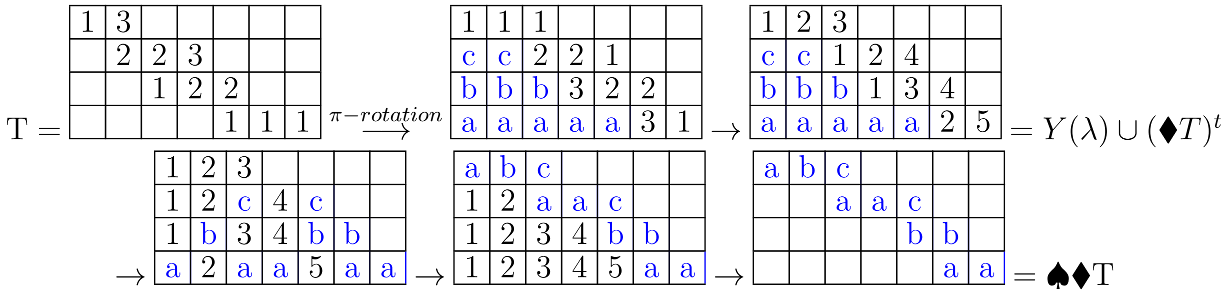
The rotation symmetries on puzzles are explicit and exhibit the cyclic symmetries of LR coefficients $c_{\mu\nu\lambda} = c_{\lambda\mu\nu} = c_{\nu\lambda\mu}$. On LR tableaux although less explicitly they are easily performed, noting that $\clubsuit\blacklozenge = \blacklozenge\spadesuit = \spadesuit\clubsuit$ and $\blacklozenge\clubsuit = \spadesuit\blacklozenge = \clubsuit\spadesuit$. Migration on mosaics explains with *jeu de taquin* the rotation symmetries of LR tableaux using standardisation. For instance, the rotation below $\blacklozenge\clubsuit = \spadesuit\blacklozenge = \clubsuit\spadesuit$ on puzzles is translated to LR tableaux in [Pu08, Corollary 5.3]. Again we may avoid standardisation operation with hybrid tableau pairs as shown next.

Definition 5.3 ($2\pi/3$ radians counterclockwise rotation symmetry map $\blacklozenge\clubsuit = \spadesuit\blacklozenge = \clubsuit\spadesuit$). Let $T \in \mathrm{LR}(\mu, \nu, \lambda)$ inside the rectangle D of size $d \times (n-d)$.

(1) Rotate T by π radians.
(2) Fill the inner shape λ using a completely ordered alphabet different from the "numerical" filling of T so that we get the Yamanouchi tableau $Y(\lambda)$.
(3) For each i, replace the ν_i i's with $1, 2, \dots, \nu_i$ according to the standard order on the boxes.
(4) Slide horizontally each 1 to the first column, each 2 to the second column, each 3 to the third column and so on.
(5) Slide down vertically along each column the numbers.
(6) Erase the numerical filling to obtain $\spadesuit\blacklozenge\mathrm{T} \in \mathrm{LR}(\nu, \lambda, \mu)$.

The example illustrates the procedure from which we see that it is reversible.

Example 5.6.



The $4\pi/3$ radians counterclockwise rotation symmetry $\clubsuit\blacklozenge = \blacklozenge\spadesuit$ can be performed in a similar way, considering $Y(\mu)^{\mathrm{a}}$ as the filling of the outer shape μ after rotating T by π radians. We may now observe that all these symmetries can be exhibited using the *hybrid* tableau–switching involution where the slides are executed without scanning the neighbours.

The H action on puzzles (equivalently mosaics) or LR–tableaux is defined by the group with presentation (5.2) or $\langle \clubsuit, \spadesuit : \clubsuit^2 = \spadesuit^2 = 1 = (\spadesuit\clubsuit)^3 \rangle = \{\mathbf{1}, \spadesuit, \clubsuit, \spadesuit\clubsuit, \clubsuit\spadesuit, \clubsuit\spadesuit\clubsuit = \spadesuit\clubsuit\spadesuit\} \simeq \mathcal{D}_3$, where $\blacklozenge$, $\spadesuit$ and $\clubsuit$ are given for LR tableaux in (3.1) and definitions 5.1 and 5.2 respectively. From Theorem 5.3 (which agrees with the computational cost of Algorithm 3.4), we may say that H is a linear time subgroup of index 2 of $\mathbb{Z}_2 \times \mathfrak{S}_3$. In particular, the $\mathfrak{S}_3$ index two subgroup of symmetries given by the cyclic group $R \simeq \{\spadesuit\clubsuit : (\spadesuit\clubsuit)^3 = 1\} = \{\mathbf{1}, \spadesuit\clubsuit, (\spadesuit\clubsuit)^2 = \clubsuit\spadesuit\}$ is of linear cost as already shown in [PV05]. Since $\clubsuit\spadesuit\clubsuit = \spadesuit\clubsuit\spadesuit$ is equivalent to $(\spadesuit\clubsuit)^3 = 1$, from propositions 5.1 and 5.2, we may conclude the following identity in the hybrid threefold

$$s_1 s_2 s_1 (Y(\mu^t), T^t, Y(\lambda^t)^{\mathrm{a}}) = s_2 s_1 s_2 (Y(\mu^t), T^t, Y(\lambda^t)^{\mathrm{a}}) = (Y(\lambda^t), T^{\clubsuit\spadesuit\clubsuit} = T^{\spadesuit\clubsuit\spadesuit}, Y(\mu^t)^{\mathrm{a}}).$$

Therefore, switching in a three fold multi-tableau consisting of Yamanouchi tableaux on the left and right and a standard tableau in the middle satisfy braid relations, $s_1 s_2 s_1 (Y(\alpha), U, Y(\gamma)^{\mathrm{a}}) = s_2 s_1 s_2 (Y(\alpha), U, Y(\gamma)^{\mathrm{a}})$ where U is a standard tableau. (In general, braid relations are not satisfied and the result depends on the factorization of the permutation, see [BSS96, Lemma 3.2, Section 3].)

5.4. **The LR companion tableau H–symmetries.** As in the case of the map $\blacklozenge$, Subsection 3.3, the linear cost of the maps $\spadesuit$ and $\clubsuit$ allows to give bijections between the companion tableaux of T and $T^{\spadesuit}$ or $T^{\clubsuit}$ whenever $T \in \mathrm{LR}(\mu, \nu, \lambda^{\vee})$. We describe the procedure for the map $\spadesuit$ on companion tableaux. We construct a bijection between the sets of companion LR tableaux of shape ν and weight λ/μ and those of shape μ^t and content λ^t/ν^t,

$$\spadesuit : \mathrm{LR}_{\nu,\lambda/\mu} \to \mathrm{LR}_{\mu^t,\lambda^t/\nu^t}, \mathrm{G}_\nu \mapsto \mathrm{G}^{\spadesuit}_{\mu^t} \tag{5.3}$$

where G is the right LR companion of some $T \in \mathrm{LR}(\mu,\nu,\lambda^\vee)$ and $\iota(T^\spadesuit) = (\iota(T))^\spadesuit$, that is, the recording matrix of $\spadesuit G$ is the transpose of the recording matrix of $\spadesuit T$, equivalently, G is the LR companion of T if and only if $\spadesuit G$ is the LR companion of $\spadesuit T$. The construction has the following three steps.

Algorithm 5.4. [Construction of $G^\spadesuit$.] Let $G \in \mathrm{LR}_{\nu,\lambda/\mu}$ and $T \in \mathrm{LR}(\mu,\nu,\lambda^\vee)$ with right LR companion G. The construction of $G^\spadesuit$ has the following three steps on the track of Definition 5.1 to construct $T^\spadesuit$.

Step 1: For $i = 1,\dots,\ell(\lambda)$, consider the i-horizontal strip (the horizontal strip consisting of all boxes filled with i) of G of size $\lambda_i - \mu_i$ and replace the entries with $\mu_i+1, \mu_i+2,\dots,\lambda_i$, scanned from SE to NW. In the resulting filling of shape ν and content λ^t/μ^t, sort by decreasing order the entries of the rows to obtain the companion plane partition C of T^t.

The entries in row r of C tell which columns of T contain r as an entry.

Step 2: For $k = 1,\dots,\ell(\lambda^t)$, let R_k consists of the row indices of C containing the entry k,

$$R_k := \{r \in \{1,\dots,\ell(\nu)\} : k \text{ is an entry in row } r \text{ of } C\},$$

with $\#R_k = \lambda^t_k - \mu^t_k$. R_k consists of the entries in the column k of T. Put

$$F_k := [\lambda^t_k] \setminus R_k,$$

with $\#F_k = \mu^t_k$. Note that $F_k = \emptyset$, for $k > \ell(\mu^t)$.

Step 3: Define the filling of shape $\mu^t = (|F_1|,\dots,|F_{\mu^t_k}|,0,\dots,0)$ whose i-th row consists of the elements in F_i by decreasing order, for $i = 1,\dots,\ell(\mu^t)$. Its content is $(\epsilon_j)_{j=1}^{\lambda^t_1}$ the multiplicity vector of the multiset union $\bigcup_{i=1}^{\ell(\mu^t)} F_i$. For each $j = 1,\dots,\lambda^t_1$, replace the entries of the j-vertical strip of size ϵ_j with $\nu_j+1,\dots,\nu_j+\epsilon_j$ scanned from bottom to top. The resulting tableau is $G^\spadesuit$ of shape μ^t and content λ^t/ν^t.

It is easy to see that if G is the companion of $T \in LR(\mu,\nu,\lambda)$, then our construction of $G^\spadesuit$ gives the companion of $T^\spadesuit \in LR(\nu^t,\mu^t,\lambda^t)$.

Example 5.7. Consider the LR tableau T of shape $\lambda^\vee/\mu$ and content ν in Example 5.4, where $d = 4$, $\mu = 4210$, $\nu = 5420$, $\lambda = 5320$, $\lambda^\vee = 7542$; $\mu^t = 3211000$, $\nu^t = 3322100$, $\lambda^{\vee t} = 4433211$, $\ell(\lambda^{\vee t}) = 7$, and its right companion tableau $G \in LR_{\nu,\lambda^\vee/\mu}$ of shape ν and content $\lambda^\vee/\mu$,

$T =$

1	3					
	2	2	3			
		1	2	2		
				1	1	1

$G =$

3	4					
2	2	3	3			
1	1	1	2	4		

.

We explain Algorithm 5.4 as a track of Example 5.4.

Step 1 produces the companion plane partition C of T^t of shape ν and content $\alpha := \lambda^{\vee t}/\mu^t = 1222211$,

$$T^t = \begin{array}{|c|c|c|c|}\hline 1 & & & \\ \hline 1 & & & \\ \hline 1 & 2 & & \\ \hline & 2 & 3 & \\ \hline & 1 & 2 & \\ \hline & & 2 & 3 \\ \hline & & & 1 \\ \hline \end{array} \qquad C = \begin{array}{|c|c|c|c|c|c|c|}\hline \ & & & & & & \\ \hline 4 & 2 & & & & & \\ \hline 5 & 4 & 3 & 2 & & & \\ \hline 7 & 6 & 5 & 3 & 1 & & \\ \hline \end{array}$$

The entries in row r of C tell which rows of T^t contain r as an entry.

Step 2 produces the sets $R_1 = \{1\}, R_2 = \{2,3\}$, $R_3 = \{1,2\}$, $R_4 = \{2,3\}$, $R_5 = \{1,2\}$, $R_6 = \{1\}$, $R_7 = \{1\}$ where R_k consists of the row indices of C containing the entry k, for $k = 1,\dots,7 = \ell(\lambda^{\vee t})$. R_k also consists of the entries in column k of T (that is, in row k of T^t).

Let $F_1 = [4] \setminus R_1 = \{2,3,4\}, F_2 = [4] \setminus R_2 = \{1,4\}$, $F_3 = [3] \setminus \{1,2\} = \{3\}$, $F_4 = [3] \setminus \{2,3\} = \{1\}$, $\ell(\mu^t) = 4$. That is, $\mu^t = (|F_1|,|F_2|,|F_3|,|F_4|,0,0,0)$.

Step 3 constructs the filling of shape μ^t, below on the right, whose row i entries, bottom to top, consist of the elements in F_i, $i = 1,\dots,\ell(\mu^t) = 4$, with content $(1^2,2,3^2,4^2,0^3)$,

$$\begin{array}{|c|c|c|c|}\hline 1 & & & \\ \hline 1 & & & \\ \hline 1 & 2 & & \\ \hline \mathrm{d} & 2 & 3 & \\ \hline 1 & 2 & \mathrm{c} & \\ \hline \mathrm{b} & 2 & 3 & \mathrm{b} \\ \hline 1 & \mathrm{a} & \mathrm{a} & \mathrm{a} \\ \hline \end{array} \qquad \begin{array}{|c|c|c|c|}\hline \ & & & \\ \hline \ & & & \\ \hline \ & & & \\ \hline 1 & & & \\ \hline 3 & & & \\ \hline 4 & 1 & & \\ \hline 4 & 3 & 2 & \\ \hline \end{array}$$

The number of j's in the jth column of the tableau on the left is ν_j, for $j = 1,\dots,\ell(\nu)$. The second part of Step 3, equivalent to push down the ν_j's j's in column j of the tableau above on the left

$$\begin{array}{|c|c|c|c|}\hline \mathrm{d} & & & \\ \hline \mathrm{b} & & & \\ \hline 1 & \mathrm{a} & & \\ \hline 1 & 2 & \mathrm{c} & \\ \hline 1 & 2 & \mathrm{a} & \\ \hline 1 & 2 & 3 & \mathrm{b} \\ \hline 1 & 2 & 3 & \mathrm{a} \\ \hline \end{array}, \qquad \text{constructs} \qquad G^{\spadesuit} = \begin{array}{|c|c|c|c|}\hline \ & & & \\ \hline \ & & & \\ \hline \ & & & \\ \hline 7 & & & \\ \hline 4 & & & \\ \hline 2 & 6 & & \\ \hline 1 & 3 & 5 & \\ \hline \end{array}$$

of shape $\mu^t = 3211$ and content $\lambda^{\vee t}/\nu^t = 1^7$, the companion tableau of $T^{\spadesuit}$. Note that $A = \begin{pmatrix} 3&0&0&0\\ 1&2&0&0\\ 0&2&1&0\\ 1&0&1&0 \end{pmatrix}$ is the recording matrix of T, A^t is the recording matrix of G, $B = \begin{pmatrix} 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 1&0&0&0&0&0&0\\ 0&0&0&1&0&0&0 \end{pmatrix}$ is the recording matrix of $\spadesuit T$ and B^t is the recording matrix of $\spadesuit G$, that is $\spadesuit\iota(T) = \iota(\spadesuit T)$.

5.5. **The symmetries outside of H.** We have discussed the H-symmetries of KT puzzles, LR tableaux, and LR companion tableaux and how do they do translate to each other. We now discuss the symmetries under the action of the other coset in $\mathbb{Z}_2 \times \mathfrak{S}_3/H$, that is, the coset $\mathbb{Z}_2 \times \mathfrak{S}_3 - H = \zeta H = H\zeta \neq H$, for $\zeta = \varsigma_1,\ \varsigma_2, \tau,\ \varsigma = \varsigma_1\varsigma_2\varsigma_1,\ \tau\varsigma_2\varsigma_1,$

$\tau\varsigma_1\varsigma_2$. On KT puzzles the symmetries outside of H are also explained by the *migration* on Purbhoo mosaics which correspond to *jeu de taquin* slides and tableau switching on LR tableaux. Recall theorems 4.1, 4.3, and the LR commuter involution $\rho = \bullet e$, $\rho : \mathrm{LR}(\mu,\nu,\lambda) \longrightarrow \mathrm{LR}(\lambda,\nu,\mu)$, $\rho(T) = \bullet e\, T$, and the LR transposer (involution) $\varrho = \blacklozenge\rho$, $\varrho : \mathrm{LR}(\mu,\nu,\lambda) \longrightarrow \mathrm{LR}(\mu^{\mathrm{t}},\nu^{\mathrm{t}},\lambda^{\mathrm{t}})$, $\varrho(T) = \blacklozenge \bullet eT$. Let us now denote by ρ_1 and ρ_2 the LR commutativity bijections $\rho_1 : \mathrm{LR}(\mu,\nu,\lambda) \to \mathrm{LR}(\nu,\mu,\lambda)$ and $\rho_2 : \mathrm{LR}(\mu,\nu,\lambda) \to \mathrm{LR}(\mu,\lambda,\nu)$ defined by the tableau switching involutions s_1 and s_2 respectively. In [Az17] it has been shown that all known LR commutors, known in the literature, exhibiting the identity $c_{\mu\,\nu\,\lambda} = c_{\nu\,\mu\,\lambda}$ coincide with the tableau switching involution, that is, with ρ_1. In [Pu08, Corollary 3.4], the proof of the commutativity for LR tableaux, using *migration* on mosaics, is equivalent to *tableau switching.* We have seen in theorems 4.3 and 4.5 that the LR commutor ρ and the LR transposer ϱ on LR tableaux are related through the linear cost involution $\blacklozenge$, $\varrho = \blacklozenge\rho = \rho\blacklozenge$. We next show that the same holds for the LR commutors ρ_1, ρ_2 and ρ and the LR transposer ϱ via the linear cost involutions $\spadesuit$, $\clubsuit$ and $\blacklozenge$ in H,

$$\spadesuit\,\rho_1 = \rho_1\,\spadesuit = \blacklozenge\,\rho = \rho\,\blacklozenge = \varrho = \clubsuit\,\rho_2 = \rho_2\clubsuit. \tag{5.4}$$

Theorem 5.5. *Consider the set up as above. Then*

(*a*) $\rho_1 = \spadesuit\blacklozenge\,\rho = \clubsuit\spadesuit\,\rho = \blacklozenge\clubsuit\rho = \spadesuit\varrho.$

(*b*) $\rho_2 = \blacklozenge\spadesuit\,\rho = \spadesuit\clubsuit\,\rho = \clubsuit\blacklozenge\,\rho = \clubsuit\varrho = \blacklozenge\spadesuit\blacklozenge\,\varrho.$

All known LR commutors and LR transposers are linear time reducible to each other and to the tableau switching involution, in particular, to the reversal involution, equivalently, Schützenberger involution.

Proof. We use the LR tableau model.

(*a*) We prove $\spadesuit\,\rho_1 = \blacklozenge\,\rho = \varrho$. The key ingredient is to show that, in Algorithm 2.6, the second stage of the calculation of the reversal of an LR tableau, performed by the switching involution s_1 on the right hand side of (5.5) below, can be performed in a special way, without scanning the neighbours, and thus is a linear cost involution. That is, when $W = Y(\mu)$ is a Yamanouchi tableau, $Q \equiv Y(\nu)$ is an LR tableau, and $V = Q^{\mathrm{nE}} = Y(\nu^\bullet)$, Algorithm 2.6 calculates $Q^e = [Q^{\mathrm{nE}}]_K \cap [Q]_{dK} = [Y(\nu^\bullet)]_K \cap [Q]_{dK}$,

$$\begin{array}{ccc} Y(\mu)\cup Q & & Y(\mu)\cup Q^e \\ \mathbf{s_1}\!\downarrow & & \uparrow\mathbf{s_1} \\ Y(\nu)\cup\rho_1(Q) & \to & Y(\nu^\bullet)\cup\rho_1(Q). \end{array} \tag{5.5}$$

First, we observe that the last step of (5.5), $Y(\nu^\bullet)\cup\rho_1(Q) \xleftrightarrow{s_1} Y(\mu)\cup Q^e$ can be performed by a linear cost map as follows: first, for $i = 1,\dots,\ell(\mu)$, the i-horizontal strip of length μ_i of $\rho_1(Q)$, an LR tableau of weight μ, slides down to the ith row (see Example 5.8); then, for $i = 1,\dots,\ell(\mu)$, sliding horizontally, justify to the left the μ_i, i's, to get $Y(\mu)$. Simultaneously it produces Q^e. This is possible thanks to the filling of inner tableau, the reverse Yamanouchi $Y(\nu^\bullet)$.

Let $\nu^t = (n_1, n_2, \dots, n_{\nu_1})$. The column i of $Y(\nu^\bullet)$ is $C_i := n_1 > n_1 - 1 > \cdots > n_1 - n_i + 1$ for $i = 1,\dots,\nu_1$. When the 1 in column i of $\rho_1(Q)$ slides down to the first row of $Y(\nu^\bullet)$, column i of $Y(\nu^\bullet)$ is shifted one box up. If there is an entry in column $i-1$ of $\rho_1(Q)$, an LR tableau, next to the left of 1, then this entry is also 1. In this case, $C_{i-1} = C_i$ and C_{i-1} is also shifted one box up. If there is no entry in column $i-1$ of $\rho_1(Q)$, next to the left of 1, then $\ell(C_{i-1} > \ell(C_i)$ and the entry of C_{i-1} next to the left of 1 is $\le n_1$. Thus when C_i is shifted one box up the semistandardness along rows is preserved. Since $\ell(C_i) \ge \ell(C_{i+1})$, when column C_i is shifted one box up the semistandardness along rows is preserved. Therefore when the 1's of $\rho_1(Q)$ slide down to the first row of $Y(\nu^\bullet)$, we get a perforated tableau pair. This perforated tableau pair has the following property: ignoring the first row the result is a tableau pair consisting of an opposite Yamanouchi tableau and the LR tableau obtained from $\rho_1(Q)$ removing the border strip consisting of 1's. By induction on $\ell(\mu)$ we conclude the validity of the procedure above. This is illustrated in (5.9) and (5.10).

Second, thanks to remarks 3.1 and 4.2, for $i = 1, \dots, \ell(\nu)$, replacing, from NW to SE, the entries of the i-horizontal strip of length $\ell(\nu) - i + 1$ in Q^e, with $1, \dots, \ell(\nu) - i + 1$, gives $(\blacklozenge\rho(Q))^t$. We then get the sequence

$$Y(\nu^\bullet) \cup \rho_1(Q) \xleftrightarrow{s_1} Y(\mu) \cup Q^e \longleftrightarrow Y(\mu) \cup (\blacklozenge\rho(Q))^t. \tag{5.6}$$

In fact, to obtain $(\blacklozenge\rho(Q))^t$, we do not need the last step $Q^e \longleftrightarrow (\blacklozenge\rho(Q))^t$ in the sequence (5.6). Replace on the left hand side of (5.6) $Y(\nu^\bullet)$ with $[Y(\nu^t)]^t$ (they have the same shape ν) and apply the map involution $\spadesuit$ without transposing (see Definition 5.1) to $[Y(\nu^t)]^t \cup \rho_1(Q)$. The sequence of instructions dictated by $\spadesuit$, without transposing, on $[Y(\nu^t)]^t \cup \rho_1(Q)$, is exactly the same as the ones described above to realize $Y(\nu^\bullet) \cup \rho_1(Q) \xleftrightarrow{s_1} Y(\mu) \cup Q^e$ in (5.6). At the end $\spadesuit$ produces $Y(\mu) \cup (\blacklozenge\rho(Q))^t$ instead of $Y(\mu) \cup Q^e$, that is,

$$[Y(\nu^t)]^t \cup \rho_1(Q) \xrightarrow{\spadesuit} Y(\mu) \cup (\spadesuit\rho_1(Q))^t = Y(\mu) \cup (\blacklozenge\rho(Q))^t.$$

Henceforth, $\spadesuit\rho_1 = \blacklozenge\rho = \varrho$.

(*b*) We prove $\blacklozenge\rho = \clubsuit\rho_2$. Thanks to Remark 2.1, Algorithm 2.6 also calculates $Q^e = [Y(\nu^\bullet)^{\mathrm{a}}]_K \cap [Q]_{dK}$,

$$\begin{array}{ccc} Q \cup Y(\lambda)^{\mathrm{a}} & & Q^e \cup Y(\lambda)^{\mathrm{a}} \\ \mathbf{s_2}\downarrow & & \uparrow\mathbf{s_2} \\ \rho_2(Q) \cup Y(\nu)^{\mathrm{a}} & \to & \rho_2(Q) \cup Y(\nu^\bullet)^{\mathrm{a}}. \end{array} \tag{5.7}$$

First, we observe that the last step of (5.7), $\rho_2(Q) \cup Y(\nu^\bullet)^{\mathrm{a}} \xleftrightarrow{s_2} Q^e \cup Y(\lambda)^{\mathrm{a}}$ can be performed by a linear cost map as follows. Let $\lambda^t = (m_1, m_2, \dots, m_{\lambda_1})$. For $i = 1, \dots, m_1$, slide horizontally the rightmost i of $\rho_2(Q)$ to the $(n-d)$th column of D, and put $C_1 := 12\cdots m_1$; for $i = 1, \dots, m_2$, slide horizontally the rightmost i of $\rho_2(Q)$, in the first $(n-d-1)$ columns, to the $(n-d-1)$th column of D, and put $C_2 := 12\cdots m_2$; ...; lastly, slide horizontally the remaining $1, 2, \dots, m_{\lambda_1}$ of $\rho_2(Q)$ to the $(n-d-\lambda_1+1)$th column of D, and put $C_{\lambda_1} := 12\cdots m_{\lambda_1}$. Then, for each for $j = 1, 2, \dots, \lambda_1$, slide up vertically the column word $C_j := 12\cdots m_j$ in the $n-d-j+1$ of D, to justify on the Northeast corner of D the λ_i, i's of $\rho_2(Q)$ so that we get $Y(\lambda)^{\mathrm{a}}$. Simultaneously one produces Q^e. The procedure is illustrated in Example 5.9.

Second, for $i = 1, \dots, \ell(\nu)$, replacing, from NW to SE, the entries of the i-horizontal strip of length $\nu_{\ell(\nu)-i+1}$ in Q^e, with $1, \dots, \nu_{\ell(\nu)-i+1}$, gives $(\blacklozenge\rho(Q))^t$. We then get the sequence

$$\rho_2(Q) \cup Y(\nu^\bullet)^{\mathrm{a}} \xleftrightarrow{s_2} Q^e \cup Y(\lambda)^{\mathrm{a}} \longleftrightarrow (\blacklozenge\rho(Q))^t \cup Y(\lambda)^{\mathrm{a}}. \tag{5.8}$$

In fact, to obtain $(\blacklozenge\rho(Q))^t$, we do not need the last step $Q^e \longleftrightarrow (\blacklozenge\rho(Q))^t$ in the sequence (5.8). Replace on the left hand side of (5.8) $Y(\nu^\bullet)^{\mathrm{a}}$ with $[Y(\nu^t)^{\mathrm{a}}]^t$ (they have the same anti normal shape $rev\,\nu$) and apply the map involution $\clubsuit$ without transposing (see Definition 5.2) to $\rho_2(Q)$. The sequence of instructions dictated by $\clubsuit$, without transposing, on $\rho_2(Q) \cup [Y(\nu^t)^{\mathrm{a}}]^t$, is exactly the same as the ones described above to produce $\rho_2(Q) \cup Y(\nu^\bullet)^{\mathrm{a}} \xleftrightarrow{s_2} Q^e \cup Y(\lambda)^{\mathrm{a}}$ in (5.8). At the end $\clubsuit$ returns $(\blacklozenge\rho(Q))^t \cup Y(\lambda)^{\mathrm{a}}$ instead of $Q^e \cup Y(\lambda)^{\mathrm{a}}$, that is,

$$\rho_2(Q) \cup [Y(\nu^t)^{\mathrm{a}}]^t \xrightarrow{\clubsuit} (\clubsuit\rho_2(Q))^t \cup Y(\lambda)^{\mathrm{a}} = (\blacklozenge\rho_2(Q))^t \cup Y(\lambda)^{\mathrm{a}}.$$

□

Example 5.8. Let $\nu = 552$ and $\mu = 5321$, $Q \equiv Y(\nu)$ and $Y(\mu) \cup Q \overset{s_1}{\longleftrightarrow} Y(\nu) \cup \rho_1(Q)$, where $\rho_1(Q) \equiv Y(\mu)$. Then

$$Y(\nu^\bullet) \cup \rho_1(Q) = \begin{array}{ccccccc} 1&2&3&4&&&\\ 3&3&1&1&2&3&\\ 2&2&3&3&3&2&\\ 1&1&2&2&2&1&1 \end{array} \leftrightarrow \begin{array}{ccccccc} 3&2&3&4&&&\\ 2&3&3&3&2&3&\\ 1&2&2&2&3&2&\\ 1&1&1&1&2&1&1 \end{array} \leftrightarrow \begin{array}{ccccccc} 3&3&3&4&&&\\ 2&2&3&3&3&3&\\ 1&2&2&2&2&2&\\ 1&1&1&1&2&1&1 \end{array} \tag{5.9}$$

$$\leftrightarrow \begin{array}{ccccccc} 3&3&3&4&&&\\ 2&2&3&3&3&3&\\ 1&2&2&2&2&2&\\ 1&1&1&1&2&1&1 \end{array} \leftrightarrow Y(\mu) \cup Q^e = \begin{array}{ccccccc} 4&3&3&3&&&\\ 3&3&2&2&3&3&\\ 2&2&2&1&2&2&\\ 1&1&1&1&1&1&2 \end{array} \tag{5.10}$$

On the righthand of (5.10), replacing, from NW to SE, the entries of the $(\ell(\nu) - i + 1)$-horizontal strip of Q^e, with $1, \ldots, \nu_i$, for $i = 1, \ldots, \ell(\nu)$, gives

$$\leftrightarrow \begin{array}{ccccccc} 4&1&2&3&&&\\ 3&3&1&2&4&5&\\ 2&2&2&1&3&4&\\ 1&1&1&1&1&2&5 \end{array} = Y(\mu) \cup (\blacklozenge\rho(Q))^t = Y(\mu) \cup (\varrho(Q))^t.$$

This is equivalent to replace, on the left hand side of (5.9), $Y(\nu^\bullet)$ with $[Y(\nu^t)]^t$ (they have the same shape ν) and apply the map involution $\spadesuit$, Definition 5.1, to $[Y(\nu^t)]^t \cup \rho_1(Q)$. It produces the same sequence of steps as in (5.9) and (5.10) with the appropriate replacement of $Y(\nu^\bullet)$ by $[Y(\nu^t)]^t$,

$$[Y(\nu^t)]^t \cup \rho_1(Q) = \begin{array}{ccccccc} 1&2&3&4&&&\\ 1&2&1&1&2&3&\\ 1&2&3&4&5&2&\\ 1&2&3&4&5&1&1 \end{array} \leftrightarrow \begin{array}{ccccccc} 1&2&3&4&&&\\ 1&2&3&4&2&3&\\ 1&2&3&4&5&2&\\ 1&2&1&1&5&1&1 \end{array} \leftrightarrow \begin{array}{ccccccc} 1&2&3&4&&&\\ 1&2&3&3&5&3&\\ 1&2&3&4&2&2&\\ 1&2&1&1&5&1&1 \end{array}$$

$$\leftrightarrow \begin{array}{ccccccc} 1&2&3&4&&&\\ 1&2&3&4&5&3&\\ 1&2&3&4&2&2&\\ 1&2&1&1&5&1&1 \end{array} \leftrightarrow \begin{array}{ccccccc} 4&1&2&3&&&\\ 3&3&1&2&4&5&\\ 2&2&2&1&3&4&\\ 1&1&1&1&1&2&5 \end{array} = Y(\mu) \cup (\spadesuit\rho_1(Q))^t = Y(\mu) \cup (\blacklozenge\rho(Q))^t.$$

Henceforth, $\spadesuit\rho_1(Q) = \blacklozenge\rho(Q) = \varrho(Q)$.

Example 5.9. Let $\lambda = 542$, $\lambda^t = 33221$ and $\nu^\bullet = 235$, $Q \equiv Y(\nu)$ and $\rho_2(Q) \cup Y(\nu^\bullet)^{\mathrm{a}} \overset{s_2}{\longleftrightarrow} Q^e \cup Y(\lambda)^{\mathrm{a}}$, where $\rho_2(Q) \equiv Y(\lambda)$. Then

$$\rho_2(Q) \cup Y(\nu^\bullet)^{\mathrm{a}} = \begin{array}{ccccccc} 1&3&3&3&3&3&3\\ &2&2&3&2&2&2\\ &&1&2&2&1&1\\ &&&&1&1&1 \end{array} \leftrightarrow \begin{array}{ccccccc} 1&3&3&3&3&3&3\\ &2&2&2&2&2&3\\ &&1&2&1&1&2\\ &&&&1&1&1 \end{array} \leftrightarrow \begin{array}{ccccccc} 1&3&3&3&3&3&3\\ &2&2&2&2&2&3\\ &&1&1&1&2&2\\ &&&&1&1&1 \end{array} \tag{5.11}$$

$$\leftrightarrow \begin{array}{ccccccc} 1&3&3&3&3&3&3\\ &2&2&2&2&2&3\\ &&1&1&1&2&2\\ &&&&1&1&1 \end{array} \leftrightarrow \begin{array}{ccccccc} 1&3&3&3&3&3&3\\ &2&2&2&2&2&3\\ &&1&1&1&2&2\\ &&&&1&1&1 \end{array} \leftrightarrow \begin{array}{ccccccc} 3&3&1&3&3&3&3\\ &2&2&2&2&2&3\\ &&1&1&1&2&2\\ &&&&1&1&1 \end{array}$$

$$\leftrightarrow Q^e \cup Y(\lambda)^{\mathrm{a}} = \begin{array}{ccccccc} 3&3&1&2&2&3&3\\ &2&2&1&1&2&2\\ &&1&3&3&1&1\\ &&&&1&2&3 \end{array} \tag{5.12}$$

On the righthand of (5.12), replacing, from NW to SE, the entries of the i-horizontal strip of Q^e, with $1,\dots,\nu_{\ell(\nu)-i+1}$, for $i=1,\dots,\ell(\nu)$, gives

$$\leftrightarrow \begin{array}{|c|c|c|c|c|c|c|}\hline 1&2&1&2&2&3&3\\\hline &1&2&1&1&2&2\\\hline &&1&3&4&1&1\\\hline &&&&2&3&5\\\hline\end{array} = (\blacklozenge\rho(Q))^t\cup Y(\lambda)^{\mathrm{a}} = (\varrho(Q))^t\cup Y(\lambda)^{\mathrm{a}}.$$

This is equivalent to replace, on the left hand side of (5.11), $Y(\nu^\bullet)^{\mathrm{a}}$ with $[Y(\nu^t)^{\mathrm{a}}]^t$ (they have the same antinormal shape $rev\,\nu$) and apply the map involution $\clubsuit$, Definition 5.2, to $\rho_2(Q)\cup[Y(\nu^t)]^t$. It produces the same sequence of steps as in (5.11) and (5.12) with the appropriate replacement of $Y(\nu^\bullet)^{\mathrm{a}}$ by $[Y(\nu^t)^{\mathrm{a}}]^t$,

$$\rho_2(Q)\cup[Y(\nu^t)^{\mathrm{a}}]^t = \begin{array}{|c|c|c|c|c|c|c|}\hline 1&3&1&2&3&4&5\\\hline &2&2&3&1&2&3\\\hline &&1&2&2&1&2\\\hline &&&&1&1&1\\\hline\end{array} \leftrightarrow \begin{array}{|c|c|c|c|c|c|c|}\hline 1&3&1&2&3&4&5\\\hline &2&2&1&2&3&3\\\hline &&1&2&1&2&2\\\hline &&&&1&1&1\\\hline\end{array} \leftrightarrow \begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&2&3&4&3&5\\\hline &2&2&1&2&3&3\\\hline &&1&1&2&2&2\\\hline &&&&1&1&1\\\hline\end{array}$$

$$\leftrightarrow \begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&2&3&4&3&5\\\hline &2&1&2&2&3&3\\\hline &&1&1&2&2&2\\\hline &&&&1&1&1\\\hline\end{array} \leftrightarrow \begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&2&3&4&3&5\\\hline &1&2&2&2&3&3\\\hline &&1&1&2&2&2\\\hline &&&&1&1&1\\\hline\end{array} \leftrightarrow \begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&2&3&4&3&5\\\hline &1&2&2&2&3&3\\\hline &&1&1&2&2&2\\\hline &&&&1&1&1\\\hline\end{array} \leftrightarrow \begin{array}{|c|c|c|c|c|c|c|}\hline 1&2&1&3&4&3&5\\\hline &1&2&2&2&3&3\\\hline &&1&1&2&2&2\\\hline &&&&1&1&1\\\hline\end{array}$$

$$\leftrightarrow \begin{array}{|c|c|c|c|c|c|c|}\hline 1&2&1&2&2&3&3\\\hline &1&2&1&1&2&2\\\hline &&1&3&4&1&1\\\hline &&&&2&3&5\\\hline\end{array} = (\clubsuit\,\rho_2(Q))^t\cup Y(\lambda)^{\mathrm{a}} = (\blacklozenge\rho_2(Q))^t\cup Y(\lambda)^{\mathrm{a}}.$$

Henceforth, $\clubsuit\rho_2(Q)=\blacklozenge\rho_2(Q)=\varrho(Q)$.

Corollary 5.6. *Consider the symmetries outside of H. The following holds in $\mathcal{LR}$:*

- (*a*) $\varrho=\blacklozenge\rho=\rho\blacklozenge$, *and* $\rho=\varrho\blacklozenge=\blacklozenge\varrho$.
- (*b*) $\rho_1=\spadesuit\blacklozenge\rho=\rho\blacklozenge\spadesuit=\clubsuit\spadesuit\rho=\rho\spadesuit\clubsuit=\blacklozenge\clubsuit\rho.=\rho\clubsuit\blacklozenge=\spadesuit\varrho=\varrho\spadesuit$.
- (*c*) $\rho_2=\rho\spadesuit\blacklozenge=\blacklozenge\spadesuit\rho=\rho\clubsuit\spadesuit=\spadesuit\clubsuit\rho=\rho\blacklozenge\clubsuit=\clubsuit\blacklozenge\rho=\varrho\clubsuit=\clubsuit\varrho$.
- (*d*) $\rho\spadesuit=\spadesuit\blacklozenge\spadesuit\rho=\clubsuit\rho=\blacklozenge\spadesuit\varrho=\varrho\blacklozenge\spadesuit$.
- (*e*) $\spadesuit\rho=\rho\clubsuit=\spadesuit\blacklozenge\varrho=\varrho\spadesuit\blacklozenge$.
- (*f*) $\rho_1\rho_2\rho_1=\rho_2\rho_1\rho_2=(\spadesuit\blacklozenge)^3\rho=\rho=\bullet\,e$.
- (*g*) $(\rho_1\rho_2)^3=1$.
- (*h*) ρ_1 *and* ρ_2 *generate a representation of* $\mathfrak{S}_3$ *in Sym*$(\mathcal{LR})$.

Proof. (*a*) It follows from Theorem 4.3. (*b*) and (*c*) follow from Theorem 5.5, (*a*) and (*b*), respectively: $\rho_1^2=1=\spadesuit\blacklozenge\rho\spadesuit\blacklozenge\rho=\clubsuit\spadesuit\rho\clubsuit\spadesuit\rho=\blacklozenge\clubsuit\rho\blacklozenge\clubsuit\rho=\varrho\spadesuit\varrho\spadesuit\Leftrightarrow\rho\blacklozenge\spadesuit=\spadesuit\blacklozenge\rho=\clubsuit\spadesuit\rho=\spadesuit\clubsuit\rho=\blacklozenge\clubsuit\rho=\rho\clubsuit\blacklozenge=\varrho\spadesuit=\spadesuit\varrho$; $\rho_2^2=1=\blacklozenge\spadesuit\rho\blacklozenge\spadesuit\rho\Leftrightarrow\blacklozenge\spadesuit\rho=\rho\spadesuit\blacklozenge$ and $\varrho\clubsuit=\rho\spadesuit\blacklozenge=\blacklozenge\spadesuit\rho=\clubsuit\varrho$. (*d*)From (*c*), $\spadesuit\rho\spadesuit=\spadesuit\blacklozenge\spadesuit\rho\blacklozenge=\clubsuit\rho\blacklozenge=\clubsuit\blacklozenge\rho=\blacklozenge\spadesuit\rho$. (*e*) From (*d*) and (*c*), $\spadesuit\rho=\spadesuit\clubsuit\rho\spadesuit=\rho\blacklozenge\clubsuit\spadesuit=\rho\clubsuit\spadesuit\spadesuit=\rho\clubsuit$. $\square$

Remark 5.1. From the previous corollary, we conclude that the tableau-switching composition $s_1s_2s_1=s_2s_1s_2$ on (three fold tableau model) $\mathcal{LR}$ satisfy braid relations and coincides with reversal composed with rotation. That is, if $T\in LR(\mu,\nu,\lambda)$ then

$$\begin{aligned} & s_1s_2s_1(Y(\mu)\cup T\cup Y(\lambda)^{\mathrm{a}}) = s_2s_1s_2(Y(\mu)\cup T\cup Y(\lambda)^{\mathrm{a}})\\ = \;& Y(\lambda)\cup\rho_1\rho_2\rho_1(T)\cup Y(\mu)^{\mathrm{a}} = Y(\lambda)\cup\rho_2\rho_1\rho_2(T)\cup Y(\mu)^{\mathrm{a}}\\ = \;& Y(\lambda)\cup\rho(T)\cup Y(\mu)^{\mathrm{a}} = Y(\lambda)\cup\bullet\,e(T)\cup Y(\mu)^{\mathrm{a}}. \end{aligned} \tag{5.13}$$

The same happens for the tableau switching on pairs of Yamanouchi and antinormal Yamanouchi tableaux inside D. From (5.13), $s_1s_2s_1(\emptyset\cup Y(\mu)\cup Y(\mu^\vee)^{\mathrm{a}})=Y(\mu^\vee)\cup$

$\bullet \text{evac} Y(\mu) \cup \emptyset = s_2 s_1 s_2(\ Y(\mu) \cup Y(\mu^\vee)^{\text{a}} \cup \emptyset) = \emptyset \cup \text{evac} \bullet Y(\mu^\vee)^{\text{a}} \cup Y(\mu)^{\text{a}}$. Therefore, $Y(\mu^\vee) \cup \bullet\text{evac}Y(\mu) = \text{evac} \bullet Y(\mu^\vee)^{\text{a}} \cup Y(\mu)^{\text{a}}$, and $Y(\mu)^{\text{a}} = \bullet\text{evac}Y(\mu) \Leftrightarrow Y(\mu) = \text{evac} \bullet Y(\mu)^{\text{a}}$. On the other hand, $s(Y(\mu) \cup Y(\mu^\vee)^{\text{a}}) = Y(\mu^\vee) \cup Y(\mu)^{\text{a}} = Y(\mu^\vee) \cup \bullet\text{evac}Y(\mu)$.

Taking into account Theorem 4.6 and algorithms 3.6 and 5.4, replacing ρ with evac in the previous corollary, in particular, one has

Corollary 5.7. *The following holds on LR companions*

(a) $\blacklozenge\text{evac} = \text{evac}\blacklozenge$.
(b) $\spadesuit\blacklozenge\text{evac} = \text{evac}\blacklozenge\spadesuit = \clubsuit\spadesuit\text{evac} = \text{evac}\spadesuit\clubsuit = \blacklozenge\clubsuit\text{evac} = \text{evac}\clubsuit\blacklozenge$.
(c) $\text{evac}\spadesuit\blacklozenge = \blacklozenge\spadesuit\text{evac} = \text{evac}\clubsuit\spadesuit = \spadesuit\clubsuit\text{evac} = \text{evac}\blacklozenge\clubsuit = \clubsuit\blacklozenge\text{evac}$.
(d) $\text{evac}\spadesuit = \spadesuit\blacklozenge\spadesuit\text{evac} = \clubsuit\text{evac}$.
(e) $\spadesuit\text{evac} = \text{evac}\clubsuit$.

Remark 5.2. From Theorem 4.6 and algorithms 3.6 and 5.4, if G is the companion of T, the companion of $\rho_1(T) = \spadesuit\blacklozenge\rho(T) = \rho\blacklozenge\spadesuit(T)$ is $\text{evac}\blacklozenge\spadesuit G = \spadesuit\blacklozenge\text{evac}G$. Similarly, for ρ_2 and ϱ. This identity does not depend on the straight shape tableau G because every tableau of straight shape is the right companion of some LR tableau with appropriate inner shape and outer shape. Henceforth, the identities above are valid for any tableaux of straight shape.

The cyclic group $R \simeq \{\spadesuit\clubsuit : (\spadesuit\clubsuit)^3 = 1\} = \{\mathbf{1}, \spadesuit\clubsuit, (\spadesuit\clubsuit)^2 = \clubsuit\spadesuit\}$ of rotations involutions, where $\clubsuit\blacklozenge = \blacklozenge\spadesuit = \spadesuit\clubsuit$ is the $4\pi/3$ radians counterclockwise rotation symmetry involution, and $\blacklozenge\clubsuit = \spadesuit\blacklozenge = \clubsuit\spadesuit$ is the $2\pi/3$ radians counterclockwise rotation symmetry involution, is an index two subgroup of $\mathfrak{S}_3 \simeq <\rho_1, \rho_2 : \rho_1^2 = \rho_2^2 = (\rho_1\rho_2)^3 = 1 >= \{1, \rho_1, \rho_2, \rho_1\rho_2, \rho_2\rho_1, \rho_1\rho_2\rho_1 = \rho_2\rho_1\rho_2 = \rho\} = R \cup \rho_1 R$ with $\rho_1 R = R\rho_1 = \rho_2 R = R\rho_2 = \rho R = R\rho = \{\rho_1, \rho_2, \rho\} = \{\spadesuit\blacklozenge\rho, \blacklozenge\spadesuit\rho, \rho\} = \{\spadesuit\blacklozenge \bullet e, \blacklozenge\spadesuit \bullet e, \bullet e\}$. That is,

$$\mathfrak{S}_3 \simeq \{1, \spadesuit\blacklozenge, \blacklozenge\spadesuit, \spadesuit\blacklozenge \bullet e, \blacklozenge\spadesuit \bullet e, \bullet e\}, \tag{5.14}$$

and, from Corollary 5.6, $\blacklozenge\mathfrak{S}_3 = \mathfrak{S}_3\blacklozenge = \blacklozenge\mathfrak{S}_3 = \mathfrak{S}_3\spadesuit = \spadesuit\mathfrak{S}_3 = \mathfrak{S}_3\clubsuit = \clubsuit\mathfrak{S}_3$, where

$$\mathbb{Z}_2 \times \mathfrak{S}_3 - \mathfrak{S}_3 = \blacklozenge\mathfrak{S}_3 = \mathfrak{S}_3\blacklozenge = \{\blacklozenge, \clubsuit, \spadesuit, \blacklozenge\spadesuit\varrho, \spadesuit\blacklozenge\varrho, \varrho\}. \tag{5.15}$$

The faithful permutation representation of $\mathbb{Z}_2 \times \mathfrak{S}_3$ affords the following presentation

$$\mathbb{Z}_2 \times \mathfrak{S}_3 =< \blacklozenge, \rho_1, \rho_2 : \blacklozenge^2 = \rho_1^2 = \rho_2^2 = (\rho_1\rho_2)^3 = (\blacklozenge\rho_2)^2 = (\blacklozenge\rho_1)^2 = 1 > .$$

therefore, $\varsigma_1 R = R\varsigma_1 = \varsigma_2 R = R\varsigma_2 = \varsigma R = R\varsigma = \mathfrak{S}_3 - R$.

The symmetries outside of H, commutativity and conjugation symmetries, (1.16), are given by the involutions ρ, ρ_1, ρ_2 and ϱ, and the two others combining rotation and conjugation, (1.17), $c_{\mu\,\nu\,\lambda} = c_{\lambda^t\,\mu^t\,\nu^t}$, $c_{\mu\,\nu\,\lambda} = c_{\nu^t\,\lambda^t\,\mu^t}$, and are given by the bijections $\clubsuit\spadesuit\varrho = \blacklozenge\clubsuit\varrho = \spadesuit\blacklozenge\varrho = \spadesuit\rho$ and $\spadesuit\clubsuit\varrho = \blacklozenge\clubsuit\varrho = \blacklozenge\spadesuit\varrho = \clubsuit\rho$ respectively. That is,

$$\begin{aligned}\mathbb{Z}_2 \times \mathfrak{S}_3 - H = \rho_1 H = H\rho_1 = \rho_2 H = H\rho_2 = \rho H = H\rho = \varrho H \\ = H\varrho = \spadesuit\rho H = H\spadesuit\rho = \clubsuit\rho H = H\clubsuit\rho \\ = \{\rho_1, \rho_2, \rho, \varrho, \clubsuit\rho, \spadesuit\rho\} = \{\rho_1, \rho_2, \rho, \blacklozenge\rho, \clubsuit\rho, \spadesuit\rho\}\end{aligned} \tag{5.16}$$

Equivalently,

$$\mathbb{Z}_2 \times \mathfrak{S}_3 - H = \{\spadesuit\blacklozenge \bullet e, \blacklozenge\spadesuit \bullet e, \bullet e, \blacklozenge \bullet e, \clubsuit \bullet e, \spadesuit \bullet e\}. \tag{5.17}$$

The symmetries outside of H are therefore linearly reducible to each other, and, in particular, to the reversal involution e. On its turn, as shown in Section 3.2, the reversal map e is linearly reducible to the Schützenberger involution E. The linear time maps ι and $\blacklozenge\spadesuit\iota$ define bijections between tableaux of normal (straight) shape and LR tableaux, see [**?**, Lee01, Ou05, PV10]. We can thus state the following result.

Theorem 5.8. *LR transposers and commutors are linearly reducible to each other, in particular, to the reversal involution e, equivalently, Schützenberger involution* evac*.*

Hence, we may use two involutions of H and an LR commutor or an LR transposer to realize the action of $\mathbb{Z}_2 \times \mathfrak{S}_3$ on $\mathcal{LR}$. For instance, $\mathbb{Z}_2 \times \mathfrak{S}_3$ has the following presentations

$$\begin{aligned}\mathbb{Z}_2 \times \mathfrak{S}_3 \simeq \langle \spadesuit, \blacklozenge, \varrho\rangle &= \langle \spadesuit, \blacklozenge, \varrho : \varrho^2 = \spadesuit^2 = \blacklozenge^2 = (\spadesuit\blacklozenge)^3 = (\spadesuit\varrho)^2 = (\blacklozenge\varrho)^2 = 1\rangle \\ &= \langle \blacklozenge, \spadesuit, \rho : \rho^2 = \spadesuit^2 = \blacklozenge^2 = (\spadesuit\blacklozenge)^3 = (\spadesuit\blacklozenge\rho)^2 = (\blacklozenge\rho)^2 = 1\rangle, \\ &= \langle \blacklozenge, \spadesuit, \rho_i : \rho_i^2 = \spadesuit^2 = \blacklozenge^2 = (\spadesuit\blacklozenge)^3 = (\spadesuit\blacklozenge\rho_i)^2 = (\blacklozenge\rho_i)^2 = 1\rangle,\ i = 1, 2 \\ &= \langle \blacklozenge, \spadesuit, \varrho\spadesuit\blacklozenge : \spadesuit^2 = \blacklozenge^2 = (\spadesuit\blacklozenge)^3 = (\spadesuit\varrho\spadesuit\blacklozenge)^2 = (\blacklozenge\varrho\spadesuit\blacklozenge)^2 = 1\rangle,\end{aligned}$$

where $\spadesuit\varrho, \blacklozenge\,\varrho$ determine an action of $\mathfrak{S}_3$ on puzzles and LR tableaux.

6. The action of $\mathbb{Z}_2 \times \mathfrak{S}_3$ on LR companion pairs and hives

Theorems 4.1 and 5.5 show that LR commutors and LR transposers are linear time reducible to each other, in particular, to the Schützenberger involution. As for the LR companion tableaux, the Henriques-Kamnitzer LR commutor version of ρ_1 shows that if (L, G) is the companion pair (left and right) of $T \in LR(\mu, \nu, \lambda)$ then $(\operatorname{evac} G, \operatorname{evac} L)$ is the companion pair (left and right) of $\rho_1(T) \in LR(\nu, \mu, \lambda)$. On the other hand, Theorem 4.6 shows that $\operatorname{evac} G$ is also the right companion of $\rho(T)$. We now translate the action of $\mathbb{Z}_2 \times \mathfrak{S}_3$ on LR tableaux to their companion pairs.

Proposition 6.1. *Let (L_μ, G_ν) be the companion pair of T in $LR(\mu, \nu, \lambda)$. The following holds*

(*a*) *the companion pair of $\rho_1(T) \in LR(\nu, \mu, \lambda)$ is*

$$(\blacklozenge\spadesuit\operatorname{evac} L_\mu, \operatorname{evac} L_\mu) = (\operatorname{evac} G_\nu, \spadesuit\blacklozenge\operatorname{evac} G_\nu).$$

(*b*) $G_\nu = \spadesuit\blacklozenge L_\mu$, *equivalently,* $L_\mu = \blacklozenge\spadesuit G_\nu$.

Proof. From Henriques-Kamnitzer LR commutor version of ρ_1 and Theorem 4.6, if G is the right companion of T in $LR(\mu, \nu, \lambda)$ then evac$G$ is simultaneously the left companion of $\rho_1(T) \in LR(\nu, \mu, \lambda)$ and the right companion of $\rho(T) \in LR(\lambda, \nu, \mu)$. Being evac$G$ the right companion of $\rho(T) \in LR(\lambda, \nu, \mu)$, Algorithm 3.6 says that $\blacklozenge$evac$G$ is the right companion of $\blacklozenge\rho(T) \in LR(\mu^t, \nu^t, \lambda^t)$. Then, from Algorithm 5.4 and Corollary 5.6, $\spadesuit\blacklozenge$evac$G$ = evac$\blacklozenge\spadesuit G$ is the right companion of $\rho_1(T) = \spadesuit\blacklozenge\rho(T) = \rho\blacklozenge\spadesuit(T) \in LR(\nu, \mu, \lambda)$. Finally from Henriques-Kamnitzer LR commutor version of ρ_1, (evacG_ν, evacL_μ) is the companion pair of $\rho_1(T)$. Henceforth, evacL_μ = $\spadesuit\blacklozenge$evacG_ν = evac$\blacklozenge\spadesuit G_\nu$ and thereby $L_\mu = \blacklozenge\spadesuit G_\nu$ and (evacG_ν, $\spadesuit\blacklozenge$evacG_ν) is the companion pair of $\rho_1(T)$ in $LR(\nu, \mu, \lambda)$. Similarly, since $G_\nu = \spadesuit\blacklozenge L_\mu$, from Corollary 5.6 one has evacG_ν = $\blacklozenge\spadesuit$evacL_μ, and then ($\blacklozenge\spadesuit$evacL_μ, evacL_μ) = (evacG_ν, $\spadesuit\blacklozenge$evacG_ν) is the companion pair of $\rho_1(T)$. □

Corollary 6.2. *Let $T \in LR(\mu, \nu, \lambda)$ and (L, G) be a pair of straight semistandard tableaux of shape μ, weight* $\operatorname{rev}(\lambda/\nu)$, *and shape ν, weight $\lambda^\vee/\mu$ respectively. The pair (L, G) is a companion pair of T if and only if $L = \blacklozenge\spadesuit G$ and L or G is the left or right companion of T respectively.*

Proof. The "only if " part is the content of previous proposition. We now prove the "if" part. We assume that $L = \blacklozenge\spadesuit G$ and G is the right companion of T. Then from previous proposition (evac$G$, $\spadesuit\blacklozenge$evac$G$) is the companion pair of $\rho_1(T)$. In particular, $\spadesuit\blacklozenge$evac$G$ = evac$\blacklozenge\spadesuit G$ = evac$L$ is the right companion of $\rho_1(T)$. From the Henriques-Kamnitzer commutor, L is the left companion of T. □

Example 6.1. Considering Example 5.4, one has $\blacklozenge\spadesuit G = \begin{array}{|c|c|c|c|}\hline 3&&&\\\hline 2&4&&\\\hline 1&1&2&4\\\hline\end{array} = L_\mu$ of shape μ and content $(\lambda^\vee/\nu)^\bullet = (7542 - 5420)_{rev} = 2212$. The left companion tableau of T in Example 5.4.

$$\rho_1 : LR_{\nu,\lambda/\mu} \to LR_{\mu,\lambda/\nu} : G \mapsto \spadesuit\blacklozenge\, \mathrm{evac\, G} \tag{6.1}$$

such that $\iota(\rho_1(T)) = \rho_1(\iota(T)) = \spadesuit\blacklozenge\mathrm{evac\, G} = \mathrm{evac\, L}$.

$$\rho_2 : LR_{\nu,\lambda/\mu} \to LR_{\lambda^\vee,\nu^\vee/\mu} : G \mapsto \blacklozenge\spadesuit\, \mathrm{evac\, G} \tag{6.2}$$

such that $\iota(\rho_2(T)) = \rho_2(\iota(T)) = \blacklozenge\spadesuit\mathrm{evac\, G}$.

6.1. **The action of $\mathbb{Z}_2 \times \mathfrak{S}_3$ on hives.** *Hives* were first introduced by Knutson and Tao [KT99] with properties described in more detail by Buch [Buc00]. Hives have also a *edge representation* as introduced by [KiTolTou06] and used in [KiTolTou09]. In this representation, a hive is specified by superposing three Gelfand-Tsetlin patterns where two of them constitute the companion pair of an LR tableau and the third is a consequence of the triangle condition on the edge labels of the hive. On the other hand, superposing a companion pair of an LR tableau always specifies a unique LR hive (for details see [AzKiTe16]). Let $\mathcal{H}(\mu, \nu, \lambda)$ be the set of LR hives whose left, right and lower boundary edge labels are specified by the parts of the partitions μ, ν and λ fitting a $d \times n - d$ rectangle. Hives are thereby naturally in bijection with LR tableaux: $T \mapsto (\blacklozenge\spadesuit G, G)$ where G is the right LR companion of $T \in LR(\mu, \nu, \lambda)$. We define the action of $\mathbb{Z}_2 \times \mathfrak{S}_3$ on $\mathcal{H}(\mu, \nu, \lambda)$ via its action on LR companion pairs. Henceforth, the following theorem exhibits the twelve symmetries of LR coefficients in the hive model via LR companion pairs.

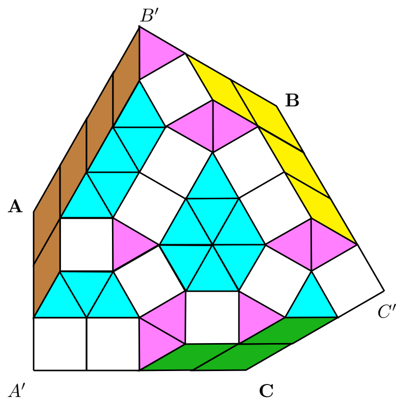
Theorem 6.3. *Let $(L = \blacklozenge\spadesuit G, G)$ be the LR companion pair of $T \in LR(\mu, \nu, \lambda)$. Then we have the following LR companion pairs under the action of $\mathbb{Z}_2 \times \mathfrak{S}_3$:*

(1)	$\spadesuit T$	$\mapsto$	$\spadesuit G \mapsto (\blacklozenge G, \spadesuit G)$
(2)	$\blacklozenge T$	$\mapsto$	$\blacklozenge G \mapsto (\spadesuit L = \blacklozenge\spadesuit\blacklozenge G, \blacklozenge G)$
(3)	$\blacklozenge\spadesuit\blacklozenge T$	$\mapsto$	$\blacklozenge\spadesuit\blacklozenge G \mapsto (\spadesuit G, \blacklozenge\spadesuit\blacklozenge G = \spadesuit L)$
(4)	$\blacklozenge\spadesuit T$	$\mapsto$	$\blacklozenge\spadesuit G \mapsto (\blacklozenge\spadesuit L = \spadesuit\blacklozenge G, \blacklozenge\spadesuit G = L)$
(5)	$\spadesuit\blacklozenge T$	$\mapsto$	$\spadesuit\blacklozenge G \mapsto (G, \spadesuit\blacklozenge G)$
(6)	$\rho(T)$	$\mapsto$	$\rho G \mapsto (\blacklozenge\spadesuit \mathrm{evac}\, G, \mathrm{evac}\, G)$
(7)	$\rho_1(T)$	$\mapsto$	$\rho_1 G \mapsto (\mathrm{evac}\, G, \spadesuit\blacklozenge \mathrm{evac}\, G = \mathrm{evac} L)$
(8)	$\rho_2(T)$	$\mapsto$	$\rho_2 G \mapsto (\spadesuit\blacklozenge \mathrm{evac}\, G = \mathrm{evac} L, \blacklozenge\spadesuit \mathrm{evac}\, G)$
(9)	$\varrho(T)$	$\mapsto$	$\varrho G \mapsto (\blacklozenge \mathrm{evac}\, L, \blacklozenge \mathrm{evac}\, G)$
(10)	$\spadesuit\rho(T)$	$\mapsto$	$\spadesuit\rho G \mapsto (\blacklozenge \mathrm{evac}\, G, \spadesuit \mathrm{evac}\, G)$
(11)	$\blacklozenge\spadesuit\blacklozenge\rho(T)$	$\mapsto$	$\blacklozenge\spadesuit\blacklozenge\rho G \mapsto (\spadesuit \mathrm{evac}\, G, \blacklozenge\spadesuit\blacklozenge \mathrm{evac}\, G) = (\spadesuit \mathrm{evac}\, G, \blacklozenge \mathrm{evac}\, L)$.

Proof. The proof is a direct consequence of Theorem 4.6, Algorithm 5.4, Theorem 5.5 and Theorem 6.1. We leave the calculations for the interested reader. □

Appendix A. Purbhoo mosaics, a rhombus–square–triangle model, and migration

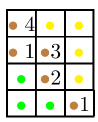
In this section, we follow closely [Pu08] (to which we refer the reader for more details) except that, an LR tableau of boundary (λ, μ, ν) here it is written there $(\mu, \lambda, \nu^\vee)$. Consider a puzzle of side length n and boundary (μ, ν, λ), and replace the rhombi by unitary squares. The puzzle will be distorted and a convex diagram can be recovered by adding thin rhombi with angles of $\frac{5}{6}\pi$ and $\frac{\pi}{6}$ radians to the three distorted edges of the puzzle. If one ignores the labels on the puzzle pieces, the resulting diagram, called *mosaic*, is a tiled hexagon by three shapes: equilateral triangles with side length 1; squares with side length 1; and rhombi with side length 1 and angles $\frac{\pi}{6}$ and $\frac{5}{6}\pi$ radians, in a way that all rhombi are packed into three nests A, B and C of the hexagon. See the picture below, where the mosaic was built on a puzzle with boundary $\mu = 2111$, $\nu = 2110$ and $\lambda = 2100$, the colours should be looked at this point as decoration. The mosaic has side lengths $A'A = B'B = C'C = n-d$ and $AB' = BC' = CA' = d$. The collections of rhombi in the nests A, B and C, denoted respectively by α, β, and γ, define the boundary data (α, β, γ) of the mosaic. The collections of extra rhombi α, β and γ, with the standard orientation, given by the edges of the nests clockwise, can be regarded as the three Young diagrams μ, ν, and λ, respectively, clockwise encoded by the 01–words on the boundary of the puzzle.



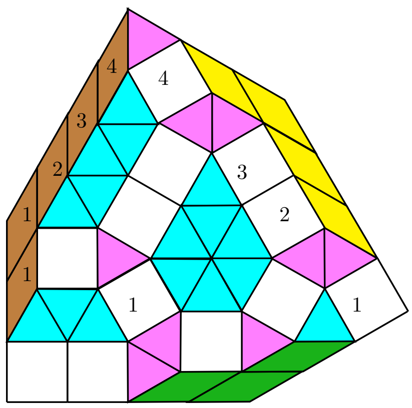
This construction exhibits a natural bijection between mosaics and puzzles. Removing the extra rhombi and straightening the resulting shape, we can go from a mosaic to a puzzle. Walking from A' to B', the shape that is left, by removing α, turns into the string of 0's and 1's, 0 for each unit step west, and a 1 for each unit step north; and straightening the squares they will become $\frac{\pi}{6}$ / $\frac{\pi}{3}$ radians rhombi. This will determine the remain labels of the puzzle pieces. (Similarly, walking anticlockwise from C' to B', we get the dual puzzle, that is, the one obtained by mirror reflection along the vertical axis and label swapping.) In the standard orientation, that is, read clockwise, a mosaic of boundary (α, β, γ) can be identified with the corresponding puzzle of boundary data (μ, ν, λ), where α is identified with the Young diagram of μ, β with ν, and γ with λ. The number of mosaics with boundary data (α, β, γ) is equal to the number of puzzles of boundary (μ, ν, λ).

One of the advantages of mosaics over puzzles is that we can give different orientations to the nests A, B, and C. This allows us to relate the symmetry bijections on puzzles and LR tableaux. Define unit vectors E_A, N_A, E_B, N_B, E_C, N_C in the directions of AA', AB', BB', BC', CC', CA' respectively, and fix orientations (E_A, N_A), (E_B, N_B), (E_C, N_C) on the nests at A, B, and C respectively. The letters E, N, $-E$, $-N$ are thought as compass directions east, north, west and south, respectively. Thus the orientations (E, N) and (N, E) in a nest means the standard or clockwise orientation, and the counterclockwise orientation respectively. Flocks are (skew) tableau–like structures, defined on the thin rhombi in a mosaic, packed into one of the nests A, B or C. Four orientations can be given to a nest. Each orientation uniquely determines the flock as a LR tableau. Fix a nest, say A, the rhombi in α under the orientation (E, N) define the Yamanouchi tableau $Y(\mu)$; under (N, E), $Y(\mu^t)$; under $(-E, -N)$, $Y(\mu)^{\mathrm{a}}$; and under $(-N, -E)$, $Y(\mu^t)^{\mathrm{a}}$. (The second compass direction indicates in which direction the entries of the LR tableau strictly increase. In the standard orientation (E, N), the entries of an LR tableau weakly increase eastward and strictly increase northward. This is consistent with the representation of partitions and their linear transformations as 01 words in (2.1) and (2.2), Section 2.1.) *Migration* is an invertible operation on mosaics that takes a flock from a nest to a new nest whose shape can be interpreted as an LR tableau. In this case the initial mosaic is identified with that LR tableau. This operation gives a bijection between mosaics (equivalently puzzles) and LR tableaux. More precisely, with appropriate orientation of the flocks in the mosaic, *migration* coincides with Tao's bijection and allows to relate operations on puzzles with *jeu de taquin* operations (or switching) on LR tableaux. The rhombi must move in the standard order of a tableau (recall the definition in Section 2.2). Choose the target nest. Rhombi move in the chosen direction of migration, inside a smallest hexagon in which the thin rhombus ♢ is contained:

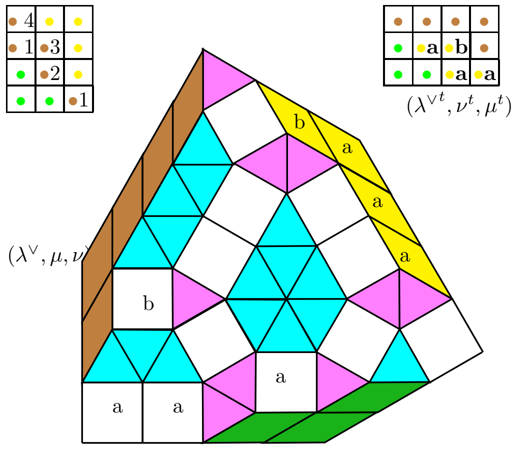
Migration from the nest A to the nest B of the mosaic, of the flock α with standard orientation (E, N), gives a bijection between mosaics of boundary (α, β, γ) and LR tableaux of boundary (λ, μ, ν) where $\nu = \beta$, $\lambda = \gamma$ and $\mu = \alpha$. This bijection coincides with the Tao's bijection, "without words" in [Va06] (see also [Pu08, Fig. 9]), between puzzles of boundary (μ, ν, λ), or the corresponding mosaics, and LR tableaux of boundary data (λ, μ, ν). On the other hand, migration from the left nest A to the bottom nest C of the mosaic, of the flock α with the orientation $(-E, -N)$, gives the same bijection [Pu08, Proposition 5.1]. This coincidence with respect to the two orientations given to the flock α is consistent with the definition of LR tableau. An LR tableau of boundary (λ, μ, ν) rectifies to the Yamanouchi tableau $Y(\mu)$, and by reverse *jeu de taquin*, to its antinormal form $Y(\mu)^{\mathrm{a}}$. An illustration of Tao's bijection which coincides with the migration of the flock $Y(\mu)$ to the nest B of the mosaic, and with the migration of the flock $Y(\mu)^{\mathrm{a}}$ to the nest C:



$(\lambda^\vee, \mu, \nu)$

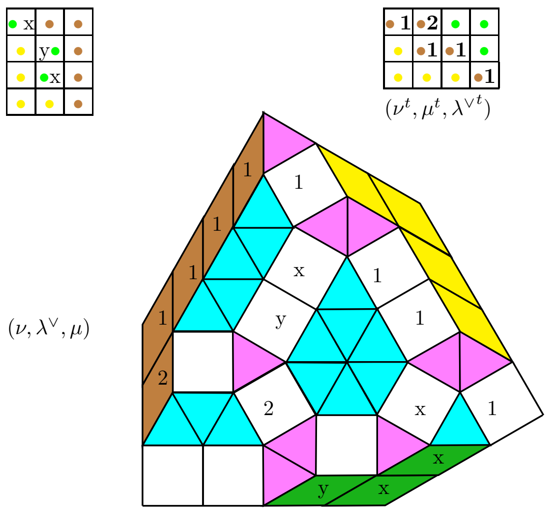


Following [Pu08, Table 1], we next see that the involutions ♣, ♠, ♦ defined by the action of the group H on puzzles and on LR tableaux are exactly what we get with migration on mosaics. In this discussion it is better to consider the presentation $H = <$ ♣, ♣♦ $>$. Migration from the nest B to the nest A of the mosaic, of the flock β with orientation (N, E) (read counterclockwise), thus identified with $Y(\nu^t)$ coincides with Tao's bijection on the back side of the mosaic, that is, on the mosaic of boundary (β, α, γ) defined by the reflection of our current mosaic along the vertical axis (also corresponding to the puzzle obtained by reflecting along the vertical axis and swapping the colours). This defines the ♠ involution on puzzles which translates to the ♣ involution on LR tableaux, as illustrated below.



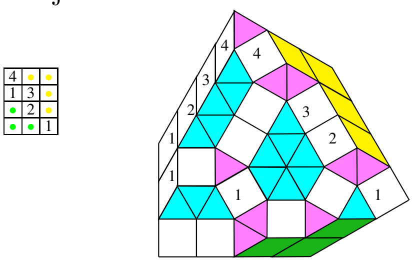
Migration of the flock α with orientation (N, E) (read counterclockwise), from the nest A to the nest C of the mosaic, coincides with what Tao's bijection gives when applied on the back side of the mosaic after rotating it $\frac{2}{3}\pi$ radians clockwise (reflection of the corresponding puzzle, after rotating it $\frac{2}{3}\pi$ radians clockwise, along the vertical axis and swapping the colours). This defines the involution ♣ on puzzles which translates to the ♦ involution on LR tableaux as illustrated below with the LR tableau on the right.

Migration from the nest C to the nest A of the mosaic, of the flock γ with standard orientation (E, N), coincides with what Tao's bijection gives when applied to the mosaic after a rotation of $\frac{2}{3}\pi$ radians clockwise. This defines the $\frac{2}{3}\pi$ clockwise rotation ♣♦ on puzzles which is translated to ♦♠ on LR tableaux. Illustrated below with the LR tableau on the left. The bijections are illustrated on the mosaic at the same time.



Similarly, migration of the flock γ with orientation (N, E) (read counterclockwise), from the nest C to the nest B of the mosaic, coincides with Tao's bijection on the back side of the mosaic after rotating it $\frac{2}{3}\pi$ radians counterclockwise (or $\frac{4}{3}\pi$ radians clockwise). It defines the involution ♦ on puzzles which translates to the involution ♠ on LR tableaux.

In [Pu08] it is discussed how the *migration* of a single rhombus in a mosaic is related with *jeu de taquin* slides on tableaux. This explains the correspondence between the action of $\mathbb{Z}_2 \times \mathfrak{S}_3$ on puzzles and on LR tableaux. We have seen that ♣ on LR tableaux can be described as the migration of the flock β with orientation (N, E), thus identified with $Y(\nu^t)$, from the nest B to the nest A of the mosaic. However ♣ can also be described on mosaics as the migration of the flock β with orientation $(-N, -E)$, thus identified with $Y(\nu^t)^{\mathrm{a}}$, from the nest B to the nest C of the mosaic. On the back mosaic, it is the migration of the flock ν^t with orientation $(-E, -N)$, thus identified with $Y(\nu^t)^{\mathrm{a}}$, to the nest C. Combining the Tao's bijection on the the mosaic of boundary (α, β, γ), in standard orientation, giving a LR tableau of boundary (λ, μ, ν), with the migration of the flock β with orientation $(-N, -E)$, identified with $Y(\nu^t)^{\mathrm{a}}$, to the nest C, we get Proposition 5.2. This is illustrated below. Consider our current mosaic with the LR tableau of boundary (λ, μ, ν) produced by the migration of the flock $Y(\mu)$ to the nest B or Tao's bijection.



Reflecting our mosaic vertically we get the "back" mosaic with boundary (β, α, γ) naturally in bijection with the dual puzzle of boundary $(\nu^t, \mu^t, \lambda^t)$ of the current mosaic. Tao's bijection on the back mosaic gives the LR tableau

	a	b	
		a	a

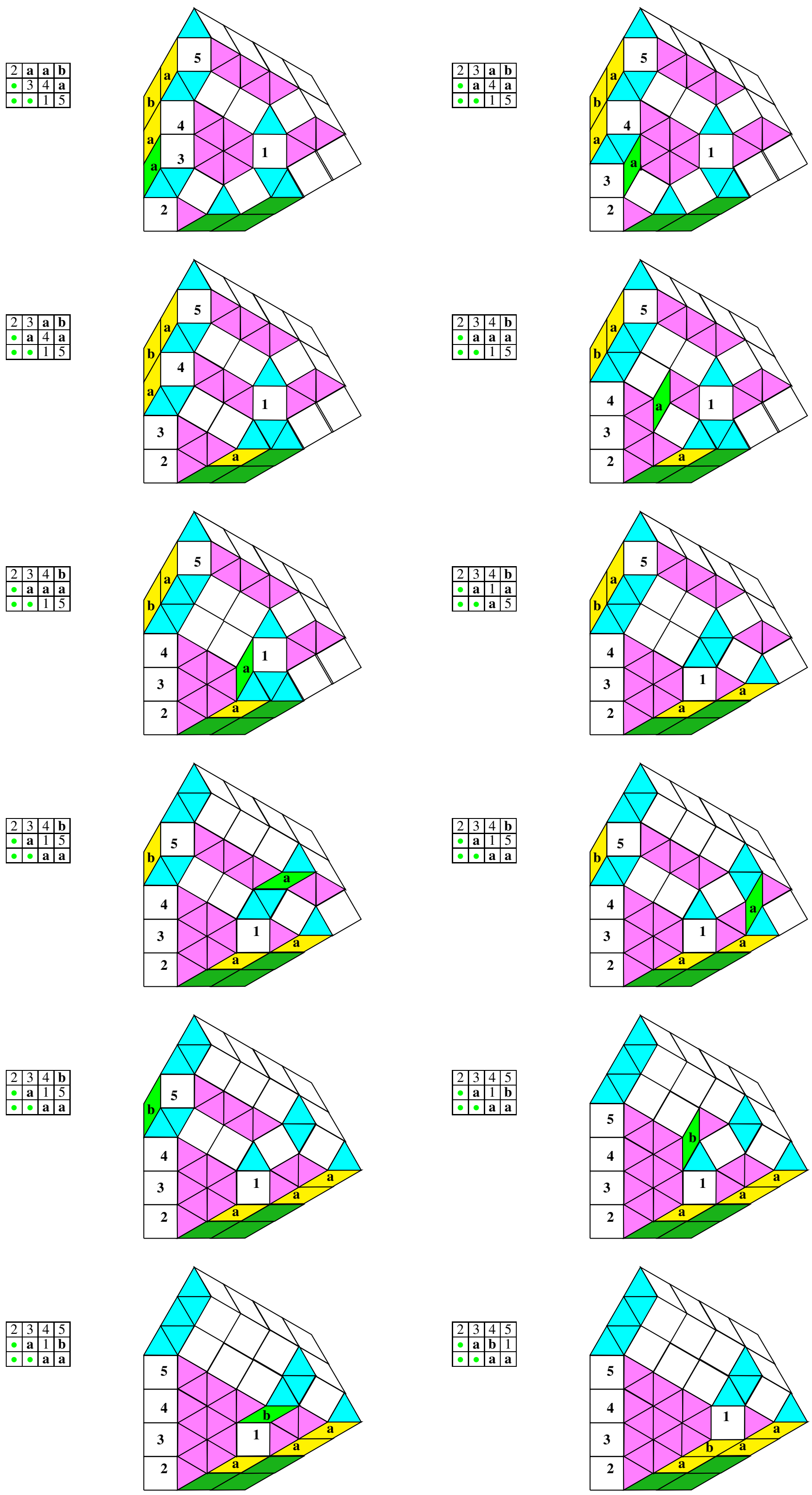
of boundary $(\lambda^t, \nu^t, \mu^t)$ which coincides with the migration on the back mosaic of $Y\nu^t)^{\mathrm{a}}$ to C.

Transposing the standardized LR tableau produced by Tao's bijection on the mosaic of boundary (α, β, γ) just above and filling the outer partition ν^t in the antinormal form to obtain $Y(\nu^t)^{\mathrm{a}}$,

2	a	a	b
	3	4	a
		1	5

, (A.1)

we verify that the migration of the flock $Y(\nu^t)^{\mathrm{a}}$ (flock β with orientation $(-E, -N)$) to the nest C, on the back mosaic, is also explained by the tableau switching on (A.1). Begin with the minimal rhombus in the standard order of $Y(\nu^t)^{\mathrm{a}}$ and proceed in standard order with the remain rhombi in $Y(\nu^t)^{\mathrm{a}}$. (Migration preserves the standard order of the flock.) The migrated flock packed in C is identified with an LR tableau of boundary $(\lambda^t, \nu^t, \mu^t)$ which is the same LR tableau of boundary $(\lambda^t, \nu^t, \mu^t)$ as Tao's bijection on the dual puzzle. (Note that some of the moves in the migration are omitted. For a complete animation, see [AzCoMa09a].)



The last mosaic, on the right, has boundary $(\emptyset, \alpha, \gamma')$ where, in standard orientation, α is identified with μ^t, and γ' identified with $(\mu^t)^\vee$ gives the LR tableau of boundary $(\lambda^t, \nu^t, \mu^t)$ already obtained with Tao's bijection above.

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CMUC, Centro de Matemática, Universidade Coimbra, 3001–454 Coimbra, Portugal

Email address: oazenhas@mat.uc.pt

Email address: alessandro.conflitti@gmail.com

Email address: mamede@mat.uc.pt