Lecture I. GL_n crystals

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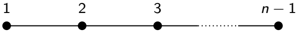
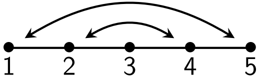
Set up: Cartan type A_{n-1}

- weight lattice $\Lambda = \mathbb{Z}^n$.
- $\mathcal{P}_n = \{\lambda \in \mathbb{Z}^n_{\geq 0} : \lambda = (\lambda_1 \geq \cdots \lambda_n \geq 0)\}$ the set of partitions with at most n parts.
- $I := [n-1]$, simple roots $\{\alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1},\ i \in I\}$, $\alpha_i^\vee = \alpha_i,\ i \in I$.
  $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ standard basis of $\mathbb{R}^n$
- fundamental weights $\varpi_i = \mathbf{e}_1 + \cdots + \mathbf{e}_i$, $i \in I$.
- Dynkin diagram I Cartan type A_{n-1},

A_{n-1} 

$$W = \mathfrak{S}_n = \langle s_1, \ldots, s_{n-1} : s_i^2 = 1, (s_i s_j)^2 = 1, |i-j| > 1, (s_i s_{i+1})^2 = 1 \rangle$$

- Dynkin diagram automorphism, $\theta : I \to I$, $\alpha_{\theta(i)} = -w_0 \alpha_i \Rightarrow \theta(i) = n - i$

$n = 6 \quad I = [5] \quad A_5$ 

GL_n- crystals

- The finite-dimensional irreducible polynomial representations of the $\mathfrak{gl}_n$ Lie algebra are parameterized by the partitions in $\mathcal{P}_n$. To any $\lambda \in \mathcal{P}_n$, we denote by $V(\lambda)$ the corresponding finite-dimensional representation (or $\mathfrak{gl}_n$-module).
- To each partition $\lambda \in \mathcal{P}_n$ corresponds a crystal graph $B(\lambda)$ which can be regarded as the combinatorial skeleton of the simple module $V(\lambda)$.

GL_n- crystals

A GL_n-crystal is a non-empty set B along with maps

$$\mathrm{wt} : B \to \mathbb{Z}^n, \quad e_i, f_i : B \to B \cup \{0\}, \varepsilon_i, \varphi_i : B \to \mathbb{Z},$$

such that for any $b, b' \in B$ and $i \in I = [n-1]$,

- $b' = e_i(b)$ if and only if $b = f_i(b')$;
- if $f_i(b) \neq 0$ then $\mathrm{wt}(f_i(b)) = \mathrm{wt}(b) - \alpha_i$;
  if $e_i(b) \neq 0$, then $\mathrm{wt}(e_i(b)) = \mathrm{wt}(b) + \alpha_i$;
- $\varepsilon_i(b) = \max\{a \in \mathbb{Z}_{\geq 0} : e_i^a(b) \neq 0\}$ and $\varphi_i(b) = \max\{a \in \mathbb{Z}_{\geq 0} : f_i^a(b) \neq 0\}$;
- $\varphi_i(b) = \varepsilon_i(b) + \langle \mathrm{wt}(b), \alpha_i \rangle$.

For any $i \in I$, the crystal B can be decomposed into its i-chains (or strings) which are obtained just by keeping the i-arrows,

Standard GL_n crystal

- Let $\mathbb{B}_n := B(\varpi_1) = \{1, \ldots, n\}$ be the standard Gl_n-crystal consisting of the words of a sole letter on the alphabet $[n]$ whose coloured crystal graph is

$$1 \xrightarrow{1} 2 \xrightarrow{2} \cdots \xrightarrow{n-2} n-1 \xrightarrow{n-1} n.$$

-
  - The Kashiwara operators f_i and e_i are defined for $i \in I = [n-1]$ as follows:

$$f_i(i) = i+1,$$
$$e_i(i+1) = i, \text{ otherwise, the letters are unchanged.}$$

  - The weight $\mathrm{wt(i)} = \mathbf{e}_\mathrm{i}$, for $i = 1, \ldots, n$.
  - The highest (lowest) weight element of $B(\varpi_1)$ is the word 1 (n), since

$$e_i(1) = 0 \ (f_i(n) = 0), \text{ for all } i \in I$$

and the highest (lowest) weight is $\mathbf{e}_1$ $(\mathbf{e}_n)$.

Tensor product of crystals

- B and C crystals.
- the crystal $B \otimes C$ has set of vertices the cartesian product of the sets of vertices of B and C, the elements denoted $u \otimes v$, $u \in B$ and $v \in C$, and crystal structure given by

$$e_i(u \otimes v) = \begin{cases} u \otimes e_i(v) \text{ if } \varepsilon_i(v) > \varphi_i(u) \\ e_i(u) \otimes v \text{ if } \varepsilon_i(v) \leq \varphi_i(u) \end{cases}$$

$$f_i(u \otimes v) = \begin{cases} f_i(u) \otimes v \text{ if } \varphi_i(u) > \varepsilon_i(v) \\ u \otimes f_i(v) \text{ if } \varphi_i(u) \leq \varepsilon_i(v) \end{cases}$$

- GL_3 standard tensor product:
$B(\varpi_1)^{\otimes 2} = B(2\varpi_1) \sqcup B(\varpi_2) = B(2\varpi_1) \bigoplus B(\varpi_2)$
$V(\varpi_1)^{\otimes 2} = V(2\varpi_1) \bigoplus V(\varpi_2)$

$$\begin{array}{ccccccccc} 1 \otimes 1 & \xrightarrow{1} & 2 \otimes 1 & \xrightarrow{2} & 3 \otimes 1 & & \\ & & {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle 1} & & \\ & & 2 \otimes 2 & \xrightarrow{2} & 3 \otimes 2 & \xrightarrow{2} & 3 \otimes 3 \end{array}$$

$$1 \otimes 2 \xrightarrow{2} 1 \otimes 3 \xrightarrow{1} 2 \otimes 3$$

Crystal of words

- The crystal $\mathcal{W}_n = \bigsqcup_{k>0} \mathbb{B}^{\otimes k} \sqcup \{\emptyset\}$ of all finite words on $[n]$ where $\emptyset$ is the empty word and the vertex $w_1 \otimes \cdots \otimes w_k \in \mathbb{B}^{\otimes k}$ is identified with the word $w = w_1 \cdots w_k$ of length k on $[n]$.
- *Signature rule:*

  1- substitute each letter w_j by $\begin{cases} + \text{ if } w_j = i \\ - \text{ if } w_j = i+1 \\ \text{erase it in any other case.} \end{cases}$

  2- successively erase any pair $+-$ until all the remaining letters form a word $-^a+^b$. Then $\varphi_i(w) = b$ and $\varepsilon_i(w) = a$.

  3- e_i (f_i) acts on the letter $i+1$ (i) associated to the rightmost (leftmost) $-$ ($+$) in $-^a+^b$: $e_i(i+1) = \begin{cases} i \text{ if } a > 0 \\ 0 \text{ if } a = 0 \end{cases}$ $\quad f_i(i) = \begin{cases} i+1 \text{ if } b > 0 \\ 0 \text{ if } b = 0 \end{cases}$
- $124211232113 \in \mathbb{B}_4^{\otimes 12} = B(\varpi_1)^{\otimes 12}$.

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- $124211232113 \in \mathbb{B}_4^{\otimes 12} = B(\varpi_1)^{\otimes 12}$.

  $122112211 \to (12)\mathbf{2}(1(12)2)\mathbf{11} \to \mathbf{211}$

  $2\mathbf{11} \underset{e_1}{\to} 1\mathbf{11} \Rightarrow e_1(124211232\mathbf{11}3) = 1241112321\ \mathbf{1}3$

  $\mathbf{2}11 \underset{f_1}{\to} 22\mathbf{1} \Rightarrow f_1(124\mathbf{2}1123221\mathbf{3}) = 12411123211\mathbf{13}$

Crystal of Young tableaux

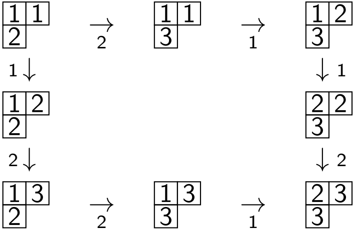
- $\lambda \in \mathcal{P}_n$
- $B(\lambda, n)$ the set of all Young tableaux of shape λ on the alphabet $[n]$
- the column reading word of a tableau

$$T = \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 4 & 5 \\ \cline{1-2}\end{array} \underset{\text{column reading}}{\longmapsto} w(T) = 22514 \in \mathcal{W}_5.$$

- a tableau in $B(\lambda, n)$ is uniquely recovered from its word.
- the map $T \mapsto w(T)$ gives an embedding of $B(\lambda, n)$ in $\mathcal{W}_n$ and we may think of $B(\lambda, n)$ as a subset of $\mathcal{W}_n$.
- $B(\lambda, n)$ is closed for the action of e_i and f_i, $i \in I$: $e_i(T) := e_i(w(T))$ and $f_i(T) := f_i(w(T)) \in B(\lambda, n)$.
- $B(\lambda, n)$ is a subcrystal of $\mathcal{W}_n$

GL_3 crystal $B((2,1,0),3) \subseteq \mathcal{W}_3$

$T_\lambda =$



- **Weyl group** $W = \mathfrak{S}_n$ **action on** $B(\lambda, n)$: the simple reflection s_i sends each vertex $b \in B(\lambda, n)$ to the unique vertex b' in the i-chain of b such that b' is the reflection of b with respect to the center of the i-chain containing b. Note $wt(s_i.b) = s_i wt(b)$.
- The **character** of $B(\lambda, n)$, $V(\lambda)$ finite-dimensional irreducible polynomial representation of Gl_n of highest weight λ, is the Schur polynomial

$$s_\lambda(x_1, \ldots, x_n) = \sum_{b \in B(\lambda,n)} x^{wt(b)} = \sum_{\mu \preceq \lambda} K(\lambda,\mu) x^\mu.$$

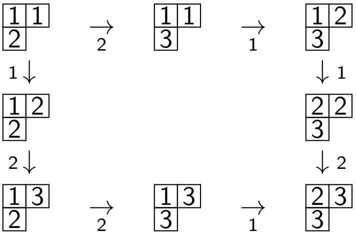
- **Weight space decomposition** of $V(\lambda)$

$$V(\lambda) = \bigoplus_{\mu \preceq \lambda} V(\lambda)_\mu, \quad \dim V(\lambda)_\mu = K(\lambda,\mu)$$

$$\dim V(\lambda) = \sum_{\mu \preceq \lambda} \dim V(\lambda)_\mu = \sum_{\mu \preceq \lambda} K(\lambda,\mu) = |B(\lambda,n)| = s_\lambda(1,\ldots,1).$$

GL_3 crystal $B((2,1,0),3) \subseteq \mathcal{W}_3$

$T_\lambda =$



- $B((2,1,0),3)$:

$$s_{(2,1,0)}(x_1,x_2,x_3) = x^{210} + x^{120} + x^{201} + x^{021} + x^{012} + x^{102} + 2x^{(111)}$$

$$\dim V(210) = s_{(2,1,0)}(1,1,1) = 8 = |B(210),3)|$$

$$\dim V(210)_\mu = \begin{cases} 1, & \mu \neq (1,1,1) \trianglelefteq (2,1,0) \\ 2, & \mu = (1,1,1). \end{cases}$$

GL_3 crystal $B((2,1,0),3) \subseteq \mathcal{W}_3$

$$
\begin{array}{ccccccc}
T_\lambda = & \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1} \end{array} & \xrightarrow[2]{} & \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1} \end{array} & \xrightarrow[1]{} & \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} \\
 & 1 \downarrow & & & & \downarrow 1 \\
 & \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1} \end{array} & & & & \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} \\
 & 2 \downarrow & & & & \downarrow 2 \\
 & \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array} & \xrightarrow[2]{} & \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1} \end{array} & \xrightarrow[1]{} & \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1} \end{array}
\end{array}
$$

- $$B((2,1,0),3) = \{f_{i_1}^{k_1} \cdots f_{i_l}^{k_\ell}(T_\lambda) \mid (k_1, \ldots, k_l) \in \mathbb{Z}_{\geq 0}^{\ell},\ i_1, \ldots, i_l \in I = [2]\} \backslash \{0\}.$$

Crystal equivalence and classical Young tableau combinatorics

- Knuth equivalence in $\mathcal{W}_n$/ Schützenberger jeu de taquin (sliding)

$$xyz \equiv \begin{cases} xzy, & y \le x < z \\ yxz, & y < z \le x \end{cases} \Leftrightarrow \begin{array}{|c|c|}\hline y & x \\ \hline z \\ \cline{1-1}\end{array} \equiv \begin{cases} \begin{array}{|c|c|}\hline & x \\ \hline y & z \\ \hline\end{array} & y \le x < z \\ \begin{array}{|c|c|}\hline & y \\ \hline z & x \\ \hline\end{array}, & y < z \le x \end{cases}$$

- Two words $w, w' \in \mathcal{W}_n$ are said Knuth equivalent if they can be transformed into each other by a sequence of Knuth transformations. $\mathcal{W}_n/\equiv$ is a monoid with $[u][v] = [uv]$ called plactic monoid.
- Robinson-Schensted correspondence $w = 31224 \in \mathcal{W}_4$
  -

$$(\emptyset \leftarrow 31224) = (3 \leftarrow 1224) = (\begin{array}{|c|c|}\hline 1 & 3 \\ \hline\end{array} \leftarrow 224) = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \leftarrow 24$$

$$= \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \leftarrow 4 = P(w) = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} \qquad Q(w) = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array}$$

  - For $w, w' \in \mathcal{W}_n$, $w \equiv w'$ iff $P(w) = P(w')$.

Crystal equivalence

- For $w, w' \in \mathcal{W}_n$, $P(w) = P(w')$ if and only if w and w' occur at the same place in two isomorphic connected components of the crystal graph of $\mathcal{W}_n$.
  An isomorphism of GL_n-crystals is an isomorphism of I-colored oriented graphs which preserves the weight and length functions for every $i \in I$.

- Each connected component in $\mathcal{W}_n$ is isomorphic to $B(\lambda, n)$ for some $\lambda \in \mathcal{P}_n$.

- For $w \in \mathcal{W}_n$, $e_i(w) = 0$, for all $i \in I = [n-1]$, $\Leftrightarrow P(w) = \begin{array}{|c|c|c|c|} \hline 1 & \cdots & 1 & 1 \\ \hline 2 & \cdots & 2 \\ \cline{1-3} & \cdots & \\ \cline{1-3} n & n \\ \cline{1-2} \end{array}$

- For $w \in \mathcal{W}_n$, $e_i(w) = 0$, for all $i \in I = [n-1]$, if and only if the number of occurrences of i in w is no less than that of $i+1$. These words are called Yamanouchi or lattice permutation or ballot words.

$$121132413 \equiv \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 \\ \cline{1-2} 3 & 3 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array}$$

Crystal equivalence

- $\mathbb{B}_3^{\otimes 3} \subseteq \mathcal{W}_3$
- The highest weight words in $\mathbb{B}_3^{\otimes 3}$ are:

$$111 = \begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array}\ ,\ 112 = \begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array}\ ,\ \text{and } \mathbf{121} \equiv \begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array}\ ,\ 123 = \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}$$

- $\mathbb{B}_3^{\otimes 3} = B(\varpi_1)^{\otimes 3} = B(3\varpi_1) \bigoplus B(\varpi_3) \bigoplus B(\varpi_1 + \varpi_2) \bigoplus B(\mathbf{121})$
- $B(\varpi_1 + \varpi_2) \simeq B(\mathbf{121})$

$$T_\lambda = \begin{array}{ccccc}
\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array} & \underset{2}{\rightarrow} & \begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array} & \underset{1}{\rightarrow} & \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array} \\
1\downarrow & & & & \downarrow 1 \\
\begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array} & & & & \begin{array}{|c|c|}\hline 2&2\\\hline 3\\\cline{1-1}\end{array} \\
2\downarrow & & & & \downarrow 2 \\
\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array} & \underset{2}{\rightarrow} & \begin{array}{|c|c|}\hline 1&3\\\hline 3\\\cline{1-1}\end{array} & \underset{1}{\rightarrow} & \begin{array}{|c|c|}\hline 2&3\\\hline 3\\\cline{1-1}\end{array}
\end{array}
\qquad
\begin{array}{ccccc}
\mathbf{121} & \underset{2}{\rightarrow} & 131 & \underset{1}{\rightarrow} & 231 \\
1\downarrow & & & & \downarrow 1 \\
122 & & & & 232 \\
2\downarrow & & & & \downarrow 2 \\
132 & \underset{2}{\rightarrow} & 133 & \underset{1}{\rightarrow} & 233
\end{array}$$

- $112 \equiv \mathbf{121},\ 113 \equiv 131,\ 213 \equiv 231,$
  $212 \equiv 122,\ 312 \equiv 132,\ 313 \equiv 133,\ 323 \equiv 233$

GL_n standard tensor product decomposition of $\mathbb{B}^{\otimes k}$

- $\mathbb{B}_3^{\otimes 3} = B(\varpi_1)^{\otimes 3} = B(3\varpi_1) \bigoplus B(\varpi_3) \bigoplus B(\varpi_1 + \varpi_2)^{\oplus 2}$, where $3\varpi_1,\ \varpi_3,\ \varpi_1 + \varpi_2$ are all the partitions of 3.
- How do I compute the number of isomorphic components?
  - The Robinson-Schensted correspondence is a bijection between the sets $\mathbb{B}^{\otimes k}$ and $\displaystyle\bigsqcup_{\lambda \in \mathcal{P}_n, |\lambda| = k} B(\lambda, n) \times SYT(\lambda, k)$,

$$w \mapsto (P(w), Q(w)).$$

  - For $w, w' \in \mathcal{W}_n$, $Q(w) = Q(w')$ if and only if w and w' occur in the same connected component of the graph of $\mathcal{W}_n$.

- $B(\varpi_1 + \varpi_2) \simeq B(\mathbf{121})$

$$T_\lambda = \begin{array}{ccccc}
\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} & \underset{2}{\rightarrow} & \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} & \underset{1}{\rightarrow} & \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \\
1\downarrow & & & & \downarrow 1 \\
\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} & & & & \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \\
2\downarrow & & & & \downarrow 2 \\
\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} & \underset{2}{\rightarrow} & \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} & \underset{1}{\rightarrow} & \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}
\end{array}
\qquad
\begin{array}{ccccc}
\mathbf{121} & \underset{2}{\rightarrow} & 131 & \underset{1}{\rightarrow} & 231 \\
1\downarrow & & & & \downarrow 1 \\
122 & & & & 232 \\
2\downarrow & & & & \downarrow 2 \\
132 & \underset{2}{\rightarrow} & 133 & \underset{1}{\rightarrow} & 233
\end{array}$$

-

$$\emptyset \leftarrow 112 = 1 \leftarrow 12 = 11 \leftarrow 2 = P(112) = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \qquad Q(112) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$$

$$\emptyset \leftarrow f_1(112) = 212 = 2 \leftarrow 12 = 12 \leftarrow 2 = P(212) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array},$$

$$Q(f_1(112)) = Q(112) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$$

- $$\emptyset \leftarrow \mathbf{121} = 1 \leftarrow 21 = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} \leftarrow 1 = P(112) = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1} \end{array}, \quad Q(\mathbf{121}) = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array}$$
  $$\emptyset \leftarrow f_1(\mathbf{121}) = 122 = 1 \leftarrow 22 = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} \leftarrow 2 = P(112) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1} \end{array},$$
  $$Q(f_1(\mathbf{121})) = Q(\mathbf{121}) = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array}$$
- $|SYT((2,1),3)| = 2$

- The RS correspondence gives the following GL_n crystal isomorphism

$$\mathbb{B}^{\otimes k} \simeq \bigoplus_{\substack{\lambda \in \mathcal{P}_n, |\lambda| = k \\ Q \in SYT(\lambda, k)}} B(Q) \simeq \bigoplus_{\lambda \in \mathcal{P}_n, |\lambda| = k} B(\lambda, n)^{\oplus |SYT(\lambda, k)|},$$

  where $B(Q) = B(\lambda, n) \times \{Q\}$.

- $\mathbb{B}^{\otimes k}$ decomposes into a disjoint union of crystals, each isomorphic to $B(\lambda, n)$, with multiplicity $|SYT(\lambda, k)|$, where λ is a partition of k of length $\leq n$.

An identity

- $$B(\varpi_1, n)^{\otimes k} = \bigoplus_{\lambda \in \mathcal{P}_n,\ |\lambda|=k} B(\lambda, n)^{\oplus |SYT(\lambda,k)|}$$

$$(x_1 + \cdots + x_n)^k = \sum_{\lambda \in \mathcal{P}_n,\ |\lambda|=k} |SYT(\lambda, k)| s_\lambda(x_1, \ldots, x_n)|$$

$$n^k = \sum_{\lambda \in \mathcal{P}_n,\ |\lambda|=k} |SYT(\lambda, k)||SSYT(\lambda, n)|$$

- $n = k = 3$

$$(x_1 + x_2 + x_3)^3 = \sum_{\lambda \vdash = 3} |SYT(\lambda, 3)| s_\lambda(x_1, x_2, x_3)|$$

$$3^3 = s_{(3)}(1,1,1) + 2s_{(2,1)}(1,1,1) + s_{(1,1,1)} = 10 + 2 \times 8 + 1$$

The Littlewood-Richardson rule

- For $\mu, \nu \in \mathcal{P}_n$, we have the following GL_n crystal isomorphism.
- Let $T \in B(\mu, n)$, $T' \in B(\nu, n)$, $T \otimes T' = w(T) \otimes w(T') = w(T)w(T')$ and

$$P(T \otimes T') = T \leftarrow w(T') \Rightarrow T \otimes T' \equiv P(T \otimes T')$$

- The recording tableau $Q(T \otimes T')$?

$$T = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\ , \quad T' = \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & 4 \\ \hline \end{array}$$

$$T \leftarrow 2423 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 3 \\ \hline \end{array} \leftarrow 423 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 3 \\ \hline 4 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \leftarrow 23 = \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 2 & 3 & \multicolumn{1}{c}{} \\ \cline{1-2} 4 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \leftarrow 3 = \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 2 & 3 & \multicolumn{1}{c}{} \\ \cline{1-2} 3 & 4 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}$$

- $Q(T \otimes T') = \begin{array}{|c|c|c|}\hline \ & \ & 1 \\ \hline \ & 1 & \multicolumn{1}{c}{} \\ \cline{1-2} 2 & 2 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}$ is a Littlewood-Richardson tableau of shape λ/μ, $\mu = (2, 1)$ and content $\nu = (2, 2)$ with word 1122 a Yamanouchi word.

- The map $T \otimes T' \longmapsto (P(T \otimes T'), Q(T \otimes T'))$ gives the following crystal isomorphism

$$B(\mu, n) \bigotimes B(\nu, n) \simeq \bigoplus_{\substack{\lambda \in \mathcal{P}_n \\ Q \in LR^{\lambda}_{\mu,\nu}}} B(Q) \simeq \bigoplus_{\substack{\lambda \in \mathcal{P}_n \\ |\lambda|=|\mu|+|\nu|}} B(\lambda, n)^{\oplus c^{\lambda}_{\mu,\nu}},$$

  where $B(Q) = B(\lambda, n) \times \{Q\}$ and $c^{\lambda}_{\mu,\nu} = |LR^{\lambda}_{\mu,\nu}|$

- $c^{(3,2,2)}_{(2,1),(2,2)} = 1$
- $s_\mu s_\nu = \sum_{\lambda \in \in \mathcal{P}_n} c^{\lambda}_{\mu,\nu} s_\lambda.$
- $$V(\mu) \otimes V(\nu) = \bigoplus_{\substack{\lambda \in \mathcal{P}_n \\ |\lambda|=|\mu|+|\nu|}} V(\lambda)^{\oplus c^{\lambda}_{\mu,\nu}}$$

Lusztig-Schützenberger involution

- The Schützenberger–Lusztig involution $\xi : B(\lambda) \to B(\lambda)$ is the unique map of sets such that, for all $b \in B(\lambda)$, and $i \in I$,
  - $e_i\xi(b) = \xi f_{\theta(i)}(b)$
  - $f_i\xi(b) = \xi e_{\theta(i)}(b)$
  - $\mathrm{wt}(\xi(b)) = w_0\mathrm{wt(b)}$

  where w_0 is the long element of the Weyl group W.
- Let $b \in B(\lambda)$ and $b = f_{j_r} \cdots f_{j_1}(u_\lambda)$, for $j_r, \ldots, j_1 \in I$. Then

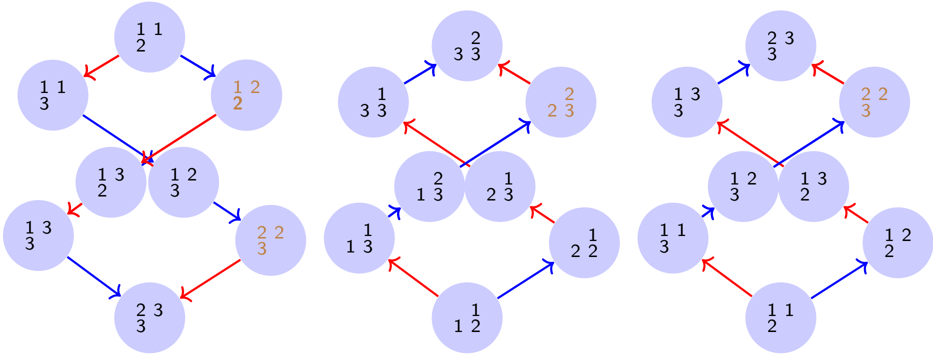
$$\xi(b) = e_{\theta(j_r)} \cdots e_{\theta(j_1)}(u_\lambda^{\mathrm{low}}), \quad \mathrm{wt}(\xi(b)) = w_0\mathrm{wt}(b)$$

  In particular,
  - in type A_{n-1}, $\xi(b) = e_{n-j_r} \cdots e_{n-j_1}(u_\lambda^{\mathrm{low}})$, and $\mathrm{wt}(\xi(b)) = rev\,\mathrm{wt}(b)$, where rev is the reverse permutation (long element) of $\mathfrak{S}_n$, ξ reverses all arrows and colors, and weight. In particular, it interchanges the highest and lowest weight elements.

Schützenberger–Lusztig involution in type A

Schützenberger, 70' reversal/evacuation : $\begin{smallmatrix} 1 & 1 \\ 2 & \end{smallmatrix} \overset{\text{rotate}}{\longrightarrow} \begin{smallmatrix} 2 & 3 \\ 3 & \end{smallmatrix} \overset{\text{rectification}}{\longrightarrow} \begin{smallmatrix} 2 & 3 \\ 3 & \end{smallmatrix}$

$1 =$ — $\quad 2 =$ —



$$B((2,1,0),3) \underset{\text{rotate}}{\rightarrow} B((2,2)/(1),3) \underset{\text{rectification}}{\rightarrow} B((2,1,0),3)$$

$$\xi\left(\begin{smallmatrix} 1 & 2 \\ \mathbf{2} & \end{smallmatrix}\right) = evac\left(\begin{smallmatrix} 1 & 2 \\ \mathbf{2} & \end{smallmatrix}\right) = \begin{smallmatrix} 2 & 2 \\ 3 & \end{smallmatrix}$$

Demazure crystals

- Let $G = GL_n(\mathbb{C})$ and B a Borel subgroup. Let the Lie algebras of G and B be $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ and $\mathfrak{b}$ a Borel subalgebra of $\mathfrak{g}$. Let $V(\lambda)$ be the irreducible G-module with highest weight λ.
- For $w \in W = \mathfrak{S}_n$, the Demazure module $V_w(\lambda) \subseteq V(\lambda)$ is the B-submodule defined

$$V_w(\lambda) = \mathcal{U}(\mathfrak{b}).V(\lambda)_{w\lambda},$$

  where $\mathcal{U}(\mathfrak{b})$ is the enveloping algebra of the Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$, and $V(\lambda)_{w\lambda}$ is the one dimensional weight space of $V(\lambda)$ with extremal weight $w\lambda$.
- The Kashiwara crystal $B(\lambda)$ is a combinatorial skeleton for the G-module $V(\lambda)$.
- Demazure characters are the characters of the B-submodules $V_w(\lambda)$.
- Kashiwara and Littelmann have shown that they can be obtained by summing the monomial weights over certain subsets $B_v = B_w(\lambda)$, $v \in W\lambda$, in the crystal $B(\lambda, n)$, called Demazure crystals.
- $B_v = B_w(\lambda)$ is the combinatorial skeleton of the Demazure module $V_w(\lambda)$, for $v \in W\lambda$.
- How to detect a Demazure crystal $B_v = B_w(\lambda)$ in $B(\lambda, n)$?
- How to detect the Demazure atom to which a vertex b of $B(\lambda, n)$ belongs?

Demazure keys: Dilatation of crystals

- Let m be a positive integer. There exists a unique embedding of crystals

$$\psi_m : B(\lambda) \hookrightarrow B(m\lambda)$$

such that for $b \in B(\lambda)$ and any path $b = f_{i_1} \cdots f_{i_l}(b_\lambda)$ in $B(\lambda)$, we have

$$\psi_m(b) = f_{i_1}^m \cdots f_{i_l}^m(b_{m\lambda}).$$

- $b_\lambda^{\otimes m}$ is of highest weight $m\lambda$ in $B(\lambda)^{\otimes m} \Rightarrow B(b_\lambda^{\otimes m})$ is a realization of $B(m\lambda)$ in $B(\lambda)^{\otimes m}$ with highest weight vertex $b_\lambda^{\otimes m}$.
- This gives a canonical embedding

$$\theta_m : \begin{cases} B(b_\lambda) \hookrightarrow B(b_\lambda^{\otimes m}) \subset B(b_\lambda)^{\otimes m} \\ \qquad b \longmapsto b_1 \otimes \cdots \otimes b_m \end{cases}$$

with important properties.

- For $\sigma \in W^\lambda$ the set of minimal coset representatives of W/W_λ, $\theta_m(b_{\sigma\lambda}) = b_{\sigma\lambda}^{\otimes m}$.
- When m has sufficiently many factors, there exist elements $\sigma_1, \dots, \sigma_m$ in W^λ such that $\theta_m(b) = b_{\sigma_1\lambda} \otimes \cdots \otimes b_{\sigma_m\lambda}$.
  - the elements $b_{\sigma_1\lambda}$ and $b_{\sigma_m\lambda}$ in $\theta_m(b)$ do not then depend on m,
  - up to repetition, the sequence $(\sigma_1\lambda, \dots, \sigma_m\lambda)$ in $\theta_m(b)$ does not depend on the realization of the crystal $B(\lambda)$ and we have $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_m$.

Demazure keys

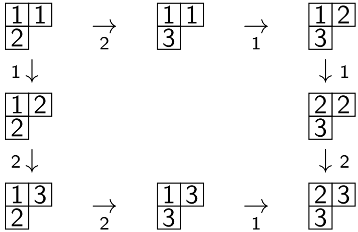
- the keys $K^{+}(b)$ and $K^{-}(b)$ of b are defined as follows:

$$K^{+}(b) = b_{\sigma_1 \lambda} \text{ and } K^{-}(b) = b_{\sigma_m \lambda}.$$

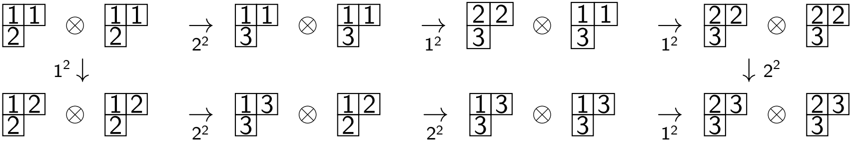
  In particular, $K^{+}(b_{\sigma\lambda}) = K^{-}(b_{\sigma\lambda}) = b_{\sigma\lambda}$ for any $\sigma \in W_n^{\lambda}$.
- $K^{-}(b) \leq K^{+}(b)$ for any $b \in B(\lambda)$, and
- $K^{-}(b) = K^{+}(b)$ if and only if b is in $O(\lambda) = \{b_{\sigma\lambda} : \sigma \in W_n^{\lambda}\}$.

$T_\lambda =$

$B((2,1),3)$

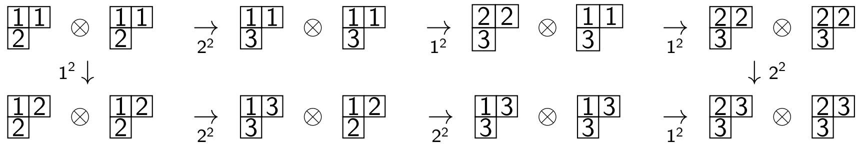


$2-\textit{dilatation},\ B((2,1),3) \hookrightarrow B(T_\lambda^{\otimes 2} \subseteq B((2,1))^{\otimes 2}$



$$K_+ \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array} = \begin{array}{|c|c|}\hline 1&3\\\hline 3\\\cline{1-1}\end{array}\,, \quad K^- \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array} = \begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}\,, \quad K_+ \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array} = \begin{array}{|c|c|}\hline 2&2\\\hline 3\\\cline{1-1}\end{array}\,, \quad K^- \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array} = \begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array}$$

Lakshmibai-Seshadri (LS) paths



$$\begin{array}{ccccccc}
(id;0,1) & \xrightarrow[2]{} & (s_2;0,1) & \xrightarrow[1]{} & (s_1s_2,s_2;0,1/2,1) & \xrightarrow[1]{} & (s_1s_2;0,1) \\
1\downarrow & & & & & & \downarrow 2 \\
(s_1;0,1) & \xrightarrow[2]{} & (s_2s_1,s_1;0,1/2,1) & \xrightarrow[2]{} & (s_2s_1;0,1) & \xrightarrow[1]{} & (s_1s_2s_1;0,1)
\end{array}$$

- Let $(\pi = (\tau, a)$ be an L–S path of shape λ, where $\tau = (\tau_0, \dots, \tau_r)$. Then $i(\pi) = \tau_0$ is the initial direction (right key) and $e(\pi) = \tau_r$, the final direction (left key) of the path.
- Lakshmibai-Seshadri (LS) path of shape λ is a pair $(\nu; a)$ of sequences $\nu : \nu_0 > \cdots > \nu_s$ of elements in W/W_λ in strictly decreasing order and $a : a_0 = 0 < a_1 < \cdots < a_s < a_{s+1} = 1$ of rational numbers in strictly increasing order, satisfying certain integrability conditions. We may regard π as a piecewise linear function such that

$$\pi(t) = \sum_{k=1}^{i-1}(a_k - a_{k-1})\nu_k\lambda + (t - a_{i-1})\nu_i\lambda, \quad a_{i-1} \le t \le a_i$$

- For $\pi = (id; 0, 1)$, one has $\pi_\lambda(t) = \lambda t$, $t \in [0, 1]$ and $\theta_m(K(\lambda))$ gives π_λ.

The crystal of LS paths

- We denote by $B^{LS}(\lambda)$ the set of all LS paths of shape λ.
- $\pi = (id; 0, 1)$ identified with $\pi_\lambda(t) = \lambda t,\ t \in [0,1]$ is in $B^{LS}(\lambda)$.
- $B^{LS}(\lambda)$ has crystal structure isomorphic to $B(\lambda)$ with highest weight element π_λ given by $\theta_m(K(\lambda))$. There is a unique isomorphism between $B(\lambda)$ and $B^{LS}(\lambda)$ that sends $\theta_m(K(\lambda))$ to π_λ.

Lascoux's keys: jeu de taquin