Tableau combinatorics in types A and C with interactions in representation theory and beyond

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$\mathfrak{gl}_n$− crystals

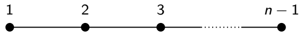
- The finite-dimensional irreducible polynomial representations of the $\mathfrak{gl}_n$ Lie algebra are parameterized by the partitions in $\mathcal{P}_n$ (partitions with at most n parts). For each $\lambda \in \mathcal{P}_n$, $V(\lambda)$ denotes the corresponding finite-dimensional representation (or $\mathfrak{gl}_n$-module).

- To each partition $\lambda \in \mathcal{P}_n$ corresponds a crystal graph $B(\lambda)$ which can be regarded as the combinatorial skeleton of the simple module $V(\lambda)$.

Set up: Cartan type A_{n-1}

- Weight lattice $\Lambda = \mathbb{Z}^n$.
- $\mathcal{P}_n = \{\lambda \in \mathbb{Z}^n_{\geq 0} : \lambda = (\lambda_1 \geq \cdots \lambda_n \geq 0)\}$ the set of partitions with at most n parts.
- $I := [n-1]$, simple roots $\{\alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1},\ i \in I\}$, $\alpha_i^\vee = \alpha_i,\ i \in I$.
  $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ standard basis of $\mathbb{R}^n$
- Fundamental weights $\varpi_i = \mathbf{e}_1 + \cdots + \mathbf{e}_i$, $i \in I$.
- Dynkin diagram I Cartan type A_{n-1},

A_{n-1}

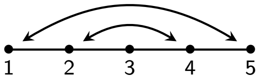


Weyl group:

$$W = \mathfrak{S}_n = \langle s_1, \ldots, s_{n-1} : s_i^2 = 1, (s_i s_j)^2 = 1, |i-j| > 1, (s_i s_{i+1})^2 = 1\rangle$$

- Dynkin diagram automorphism, $\theta : I \to I$, $\alpha_{\theta(i)} = -w_0\alpha_i \Rightarrow \theta(i) = n - i$

$n = 6 \quad I = [5] \quad A_5$



$\mathfrak{gl}_n-$ crystals

A $\mathfrak{gl}_n$-crystal is a non-empty set B along with maps

$$\mathrm{wt}: B \to \mathbb{Z}^n, \quad e_i, f_i : B \to B \cup \{0\}, \, \varepsilon_i, \varphi_i : B \to \mathbb{Z},$$

such that for any $b, b' \in B$ and $i \in I = [n-1]$,

- $b' = e_i(b)$ if and only if $b = f_i(b')$;
- if $f_i(b) \neq 0$ then $\mathrm{wt}(f_i(b)) = \mathrm{wt}(b) - \alpha_i$;
  if $e_i(b) \neq 0$, then $\mathrm{wt}(e_i(b)) = \mathrm{wt}(b) + \alpha_i$;
- $\varepsilon_i(b) = \max\{a \in \mathbb{Z}_{\geq 0} : e_i^a(b) \neq 0\}$ and $\varphi_i(b) = \max\{a \in \mathbb{Z}_{\geq 0} : f_i^a(b) \neq 0\}$;
- $\varphi_i(b) = \varepsilon_i(b) + \langle \mathrm{wt}(b), \alpha_i \rangle$.

We associate to the crystal B a directed graph with vertices in B and edges labelled by $i \in I$.

$$f_i(x) = y, \; x, y \in B \; \Leftrightarrow x \underset{i}{\to} y$$

For any $i \in I$, the crystal B can be decomposed into its i-chains (or strings) which are obtained just by keeping the i-arrows,

Standard $\mathfrak{gl}_n$ crystal

- Let $\mathbb{B}_n := B(\varpi_1) = \{1, \dots, n\}$ be the standard Gl_n-crystal consisting of the words of a sole letter on the alphabet $[n]$ whose coloured crystal graph is

$$1 \xrightarrow{1} 2 \xrightarrow{2} \cdots \xrightarrow{n-2} n-1 \xrightarrow{n-1} n.$$

-
  - The Kashiwara operators f_i and e_i are defined for $i \in I = [n-1]$ as follows:

$$f_i(i) = i + 1,$$
$$e_i(i+1) = i,$$

otherwise, the letters are unchanged.

  - The weight $\mathrm{wt(i)} = \mathbf{e}_\mathrm{i}$, for $i = 1, \dots, n$.
  - The highest (lowest) weight element of $B(\varpi_1)$ is the word 1 (n), since

$$e_i(1) = 0 \; (f_i(n) = 0), \text{ for all } i \in I$$

    and the highest (lowest) weight is $\mathbf{e}_1$ ($\mathbf{e}_n$).
  - The length functions are what is expected from the crystal graph.

Tensor product of crystals

- B and C crystals.
- The crystal $B \otimes C$ has set of vertices the cartesian product of the sets of vertices of B and C, the elements denoted $u \otimes v$, $u \in B$, $v \in C$, and crystal structure given by $\mathrm{wt}(u \otimes v) = \mathrm{wt}(u) + \mathrm{wt}(v)$ and the following rules where we follow the Kashiwara convention

$$e_i(u \otimes v) = \begin{cases} u \otimes e_i(v) \text{ if } \varepsilon_i(v) > \varphi_i(u) \\ e_i(u) \otimes v \text{ if } \varepsilon_i(v) \leq \varphi_i(u) \end{cases}$$

$$f_i(u \otimes v) = \begin{cases} f_i(u) \otimes v \text{ if } \varphi_i(u) > \varepsilon_i(v) \\ u \otimes f_i(v) \text{ if } \varphi_i(u) \leq \varepsilon_i(v) \end{cases}$$

- GL_3 standard tensor product: $B(\varpi_1)^{\otimes 2} = B(2\varpi_1) \sqcup B(\varpi_2) = B(2\varpi_1) \bigoplus B(\varpi_2)$

$$V(\varpi_1)^{\otimes 2} = V(2\varpi_1) \bigoplus V(\varpi_2)$$

$$1 \xrightarrow{1} 2 \xrightarrow{2} 3$$

$$\begin{array}{ccccccccc}
1 \otimes 1 & \overset{1}{\to} & 2 \otimes 1 & \overset{2}{\to} & 3 \otimes 1 & & \\
 & & 1\downarrow & & \downarrow 1 & & \\
 & & 2 \otimes 2 & \overset{2}{\to} & 3 \otimes 2 & \overset{2}{\to} & 3 \otimes 3
\end{array}$$

$$1 \otimes 2 \overset{2}{\to} 1 \otimes 3 \overset{1}{\to} 2 \otimes 3$$

Crystal of words

- The crystal $\mathcal{W}_n = \bigsqcup_{k>0} \mathbb{B}^{\otimes k} \sqcup \{\emptyset\}$ of all finite words on $[n]$ where $\emptyset$ is the empty word and the vertex $w_1 \otimes \cdots \otimes w_k \in \mathbb{B}^{\otimes k}$ is identified with the word $w = w_1 \cdots w_k$ of length k on $[n]$.
- *Signature rule:*

  1- substitute each letter w_j by $\begin{cases} + \text{ if } w_j = i \\ - \text{ if } w_j = i+1 \\ \text{erase it in any other case.} \end{cases}$

  2- successively erase any pair $+-$ until all the remaining letters form a word $-^a+^b$. Then $\varphi_i(w) = b$ and $\varepsilon_i(w) = a$.

  3- e_i (f_i) acts on the letter $i+1$ (i) associated to the rightmost (leftmost) $-$ $(+)$ in $-^a+^b$: $e_i(i+1) = \begin{cases} i \text{ if } a > 0 \\ 0 \text{ if } a = 0 \end{cases}$ $\quad f_i(i) = \begin{cases} i+1 \text{ if } b > 0 \\ 0 \text{ if } b = 0 \end{cases}$

- $124211232113 \in \mathbb{B}_4^{\otimes 12} = B(\varpi_1)^{\otimes 12}$.

  $122112211 \to (12)\mathbf{2}(1(12)2)\mathbf{11} \to \mathbf{211}$

  $2\mathbf{11} \underset{e_1}{\to} 1\mathbf{11} \Rightarrow e_1(124211232\mathbf{11}3) = 1241112321\ \mathbf{1}3$

  $\mathbf{2}1\mathbf{1} \underset{f_1}{\to} 22\mathbf{1} \Rightarrow f_1(124\mathbf{2}11232 1\mathbf{1}3) = 124211232\mathbf{21}3$

Crystal of Young tableaux

- $\lambda \in \mathcal{P}_n$
- $B(\lambda, n)$ the set of all Young tableaux of shape λ on the alphabet $[n]$
- the column reading word of a tableau

$$T = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 2 & 4 \\ \hline 4 & 5 \\ \cline{1-2}\end{array} \quad \underset{\text{column reading}}{\longmapsto} \quad w(T) = 422514 \in \mathcal{W}_5.$$

$wt(w(T)) = (1, 2, 0, 2, 1)$

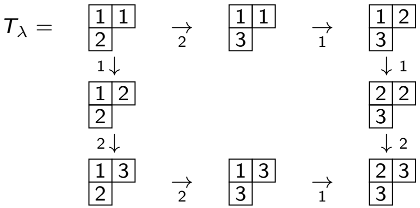
- a tableau in $B(\lambda, n)$ is uniquely recovered from its word.
- the map $T \mapsto w(T)$ gives an embedding of $B(\lambda, n)$ in $\mathcal{W}_n$ and we may think of $B(\lambda, n)$ as a subset of $\mathcal{W}_n$.
- $B(\lambda, n)$ is closed for the action of e_i and f_i, $i \in I$:
  $e_i(T) := e_i(w(T))$ and $f_i(T) := f_i(w(T)) \in B(\lambda, n)$.
- $B(\lambda, n)$ is a subcrystal of $\mathcal{W}_n$

$\mathfrak{gl}_3$ crystal $B((2,1,0),3) \subseteq \mathcal{W}_3$

$$
T_\lambda = \quad
\begin{array}{ccccc}
\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} & \underset{2}{\rightarrow} & \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} & \underset{1}{\rightarrow} & \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \\
{\scriptstyle 1}\downarrow & & & & \downarrow{\scriptstyle 1} \\
\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} & & & & \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \\
{\scriptstyle 2}\downarrow & & & & \downarrow{\scriptstyle 2} \\
\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} & \underset{2}{\rightarrow} & \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} & \underset{1}{\rightarrow} & \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}
\end{array}
$$

- $$B((2,1,0),3) = \{f_{i_1}^{k_1} \cdots f_{i_l}^{k_\ell}(T_\lambda) \mid (k_1, \dots, k_l) \in \mathbb{Z}_{\geq 0}^{\ell},\ i_1, \dots, i_l \in I = [2]\} \setminus \{0\}.$$

$\mathfrak{gl}_3$ crystal $B((2,1,0),3) \subseteq \mathcal{W}_3$



$W = \mathfrak{S}_3$

- **Weyl group** $W = \mathfrak{S}_n$ **action on** $B(\lambda, n)$: the simple reflection s_i sends each vertex $b \in B(\lambda, n)$ to the unique vertex b' in the i-chain of b such that b' is the reflection of b with respect to the center of the i-chain containing b. Note $wt(s_i.b) = s_i wt(b)$.
- The **character** of $B(\lambda, n)$, $V(\lambda)$ finite-dimensional irreducible polynomial representation of $\mathfrak{gl}_n$ of highest weight λ, is the Schur polynomial

$$s_\lambda(x_1, \dots, x_n) = \sum_{b \in B(\lambda,n)} x^{wt(b)} = \sum_{\mu \preceq \lambda} K(\lambda, \mu) x^\mu.$$

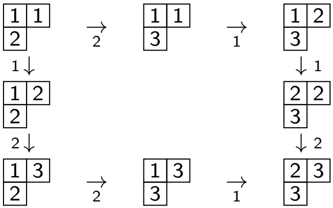
- **Weight space decomposition** of $V(\lambda)$

$$V(\lambda) = \bigoplus_{\mu \preceq \lambda} V(\lambda)_\mu, \quad \dim V(\lambda)_\mu = K(\lambda, \mu), \text{ Kostka number}$$

$$\dim V(\lambda) = \sum_{\mu \preceq \lambda} \dim V(\lambda)_\mu = \sum_{\mu \preceq \lambda} K(\lambda, \mu) = |B(\lambda, n)| = s_\lambda(1, \dots, 1).$$

$\mathfrak{gl}_3$ crystal $B((2,1,0),3) \subseteq \mathcal{W}_3$

$T_\lambda =$



- $B((2,1,0),3)$:

$$s_{(2,1,0)}(x_1,x_2,x_3) = x^{210} + x^{120} + x^{201} + x^{021} + x^{012} + x^{102} + 2x^{(111)}$$

$$\dim V(210) = s_{(2,1,0)}(1,1,1) = 8 = |B(210),3)|$$

$$\dim V(210)_\mu = \begin{cases} 1, & \mu \neq (1,1,1) \preceq (2,1,0) \\ 2, & \mu = (1,1,1). \end{cases}$$

Crystal equivalence and classical Young tableau combinatorics

- Knuth equivalence in $\mathcal{W}_n/$ Schützenberger jeu de taquin (sliding)

$$xyz \equiv \begin{cases} xzy, & y \le x < z \\ yxz, & y < z \le x \end{cases} \Leftrightarrow \begin{array}{|c|c|}\hline y & x \\ \hline z & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \equiv \left\{ \begin{array}{ll} \begin{array}{|c|c|}\hline & x \\ \hline y & z \\ \hline \end{array} & y \le x < z \\ \begin{array}{|c|c|}\hline & y \\ \hline z & x \\ \hline \end{array}, & y < z \le x \end{array} \right.$$

- Two words $w, w' \in \mathcal{W}_n$ are said Knuth equivalent if they can be transformed into each other by a sequence of Knuth transformations. $\mathcal{W}_n/\equiv$ is a monoid with $[u][v] = [uv]$ called plactic monoid.
- Robinson-Schensted correspondence $w = 31224 \in \mathcal{W}_4$
  - $$(\emptyset \leftarrow 31224) = (3 \leftarrow 1224) = (\begin{array}{|c|c|}\hline 1 & 3 \\ \hline \end{array} \leftarrow 224) = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \leftarrow 24$$
    $$= \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 2 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \leftarrow 4 = P(w) = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 2 & \multicolumn{2}{c}{} \\ \cline{1-1} 4 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \qquad Q(w) = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & \multicolumn{2}{c}{} \\ \cline{1-1} 5 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}$$
  - For $w, w' \in \mathcal{W}_n$, $w \equiv w'$ iff $P(w) = P(w')$.

Crystal equivalence

- For $w, w' \in \mathcal{W}_n$, $P(w) = P(w')$ if and only if w and w' occur at the same place in two isomorphic connected components of the crystal graph of $\mathcal{W}_n$.
  An isomorphism of GL_n-crystals is an isomorphism of I-colored oriented graphs which preserves the weight and length functions for every $i \in I$.

- Each connected component in $\mathcal{W}_n$ is isomorphic to $B(\lambda, n)$ for some $\lambda \in \mathcal{P}_n$.

- For $w \in \mathcal{W}_n$, $e_i(w) = 0$, for all $i \in I = [n-1]$ (w is highest weight element)$\Leftrightarrow$ $P(w) = \begin{array}{|c|c|c|c|} \hline 1 & \cdots & 1 & 1 \\ \hline 2 & \cdots & 2 \\ \cline{1-3} & \cdots & \\ \cline{1-3} n & n \\ \cline{1-2} \end{array}$

- For $w \in \mathcal{W}_n$, $e_i(w) = 0$, for all $i \in I = [n-1]$ if and only if the number of occurrences of i in w is no less than that of $i+1$, for all $i \in I = [n-1]$. These words are called Yamanouchi or lattice permutation or ballot words.

$$12113241333 \equiv \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 3 \\ \cline{1-3} 3 & 3 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array} .$$

Crystal equivalence

- $\mathbb{B}_3^{\otimes 3} \subseteq \mathcal{W}_3$, all words of length 3 on the alphabet $[3]$.
- The highest weight words in $\mathbb{B}_3^{\otimes 3}$ are the ballot words in $\mathcal{W}_3$ of length 3:

$$111 = \begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array}\ ,\ 112 = \begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array}\ ,\ \text{and } \mathbf{121} \equiv \begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array}\ ,\ 123 = \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}$$

- $\mathbb{B}_3^{\otimes 3} = B(\varpi_1)^{\otimes 3} = B(3\varpi_1) \bigoplus B(\varpi_3) \bigoplus B(\varpi_1+\varpi_2) \bigoplus B(\mathbf{121})$

- $B(\varpi_1+\varpi_2) \simeq B(\mathbf{121})$

$$T_\lambda = \begin{array}{ccccc}
\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array} & \underset{2}{\rightarrow} & \begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array} & \underset{1}{\rightarrow} & \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\\
1\downarrow & & & & \downarrow 1\\
\begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array} & & & & \begin{array}{|c|c|}\hline 2&2\\\hline 3\\\cline{1-1}\end{array}\\
2\downarrow & & & & \downarrow 2\\
\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array} & \underset{2}{\rightarrow} & \begin{array}{|c|c|}\hline 1&3\\\hline 3\\\cline{1-1}\end{array} & \underset{1}{\rightarrow} & \begin{array}{|c|c|}\hline 2&3\\\hline 3\\\cline{1-1}\end{array}
\end{array}
\qquad
\begin{array}{ccccc}
\mathbf{121} & \underset{2}{\rightarrow} & 131 & \underset{1}{\rightarrow} & 231\\
1\downarrow & & & & \downarrow 1\\
122 & & & & 232\\
2\downarrow & & & & \downarrow 2\\
132 & \underset{2}{\rightarrow} & 133 & \underset{1}{\rightarrow} & 233
\end{array}$$

- $\simeq$ connected componentes/ Knuth equivalence of vertices in a same position

  $112 \equiv \mathbf{121},\ 113 \equiv 131,\ 213 \equiv 231,$

  $212 \equiv 122,\ 312 \equiv 132,\ 313 \equiv 133,\ 323 \equiv 233$

$\mathfrak{gl}_n$ standard tensor product decomposition of $\mathbb{B}^{\otimes k}$

- $\mathbb{B}_3^{\otimes 3} = B(\varpi_1)^{\otimes 3} = B(3\varpi_1) \bigoplus B(\varpi_3) \bigoplus B(\varpi_1 + \varpi_2)^{\oplus 2}$, where $3\varpi_1, \ \varpi_3, \ \varpi_1 + \varpi_2$ are all the partitions of 3.

- For $\lambda \in \mathcal{P}_n$, how do I compute in $\mathcal{W}_n$ the number of isomorphic components to $B(\lambda, n)$?

  - The Robinson-Schensted correspondence is a bijection between the sets $\mathbb{B}^{\otimes k}$ and $\bigsqcup_{\lambda \in \mathcal{P}_n, |\lambda| = k} B(\lambda, n) \times SYT(\lambda, k)$,

$$w \mapsto (P(w), Q(w)).$$

  - For $w, w' \in \mathcal{W}_n$, $Q(w) = Q(w')$ if and only if w and w' occur in a same connected component of the graph of $\mathcal{W}_n$.

  - For $\lambda \in \mathcal{P}_n$ of size k, there exist $|SYT(\lambda, k)|$ crystal isomorphic components to $B(\lambda, n)$ in $\mathcal{W}_n$.

$\mathfrak{gl}_3$ standard tensor product decomposition of $\mathbb{B}^{\otimes 3}$

- $B(\varpi_1 + \varpi_2) \simeq B(\mathbf{121})$

$$T_\lambda = \begin{array}{ccccc}
\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array} & \xrightarrow[2]{} & \begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array} & \xrightarrow[1]{} & \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array} \\
{\scriptstyle 1}\downarrow & & & & \downarrow{\scriptstyle 1} \\
\begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array} & & & & \begin{array}{|c|c|}\hline 2&2\\\hline 3\\\cline{1-1}\end{array} \\
{\scriptstyle 2}\downarrow & & & & \downarrow{\scriptstyle 2} \\
\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array} & \xrightarrow[2]{} & \begin{array}{|c|c|}\hline 1&3\\\hline 3\\\cline{1-1}\end{array} & \xrightarrow[1]{} & \begin{array}{|c|c|}\hline 2&3\\\hline 3\\\cline{1-1}\end{array}
\end{array}
\qquad
\begin{array}{ccccc}
\mathbf{121} & \xrightarrow[2]{} & 131 & \xrightarrow[1]{} & 231 \\
{\scriptstyle 1}\downarrow & & & & \downarrow{\scriptstyle 1} \\
122 & & & & 232 \\
{\scriptstyle 2}\downarrow & & & & \downarrow{\scriptstyle 2} \\
132 & \xrightarrow[2]{} & 133 & \xrightarrow[1]{} & 233
\end{array}$$

-

$$\emptyset \leftarrow 112 = 1 \leftarrow 12 = 11 \leftarrow 2 = P(112) = \begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array} \qquad Q(112) = \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}$$

$$\emptyset \leftarrow f_1(112) = 212 = 2 \leftarrow 12 = 12 \leftarrow 2 = P(212) = \begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array},$$

$$Q(f_1(112)) = Q(112) = \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}$$

- $$\emptyset \leftarrow \mathbf{121} = 1 \leftarrow 21 = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} \leftarrow 1 = P(112) = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1} \end{array}, \quad Q(\mathbf{121}) = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array}$$
  $$\emptyset \leftarrow f_1(\mathbf{121}) = 122 = 1 \leftarrow 22 = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} \leftarrow 2 = P(112) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1} \end{array},$$
  $$Q(f_1(\mathbf{121})) = Q(\mathbf{121}) = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array}$$
- $|SYT((2,1),3)| = 2$

RS correspondence and $\mathfrak{gl}_3$ standard tensor product decomposition

- The RS correspondence gives the following $\mathfrak{gl}_n$ crystal isomorphism

$$\mathbb{B}^{\otimes k} \simeq \bigoplus_{\substack{\lambda \in \mathcal{P}_n, |\lambda|=k \\ Q \in SYT(\lambda,k)}} B(Q) \simeq \bigoplus_{\lambda \in \mathcal{P}_n, |\lambda|=k} B(\lambda, n)^{\oplus |SYT(\lambda,k)|},$$

  where $B(Q) = B(\lambda, n) \times \{Q\}$.
- For $k \geq 1$, $\mathbb{B}^{\otimes k}$ decomposes into a disjoint union of crystals, each isomorphic to $B(\lambda, n)$, with multiplicity $|SYT(\lambda, k)|$, where λ is a partition of k of length $\leq n$.
- $\mathfrak{gl}_n$

$$V(\varpi_1)^{\otimes k} = \bigoplus_{\lambda \in \mathcal{P}_n, |\lambda|=k} V(\lambda)^{\oplus |SYT(\lambda,k)|}$$

- $\mathfrak{gl}_3$

$$n = 3 \quad k = 2, \;\; V(\varpi_1)^{\otimes 2} = V(2\varpi_1) \bigoplus V(\varpi_2)$$

$$n = 3 \quad k = 3, \;\; V(\varpi_1)^{\otimes 3} = V(3\varpi_1) \bigoplus V(\varpi_3) \bigoplus V(\varpi_1 + \varpi_2)^{\oplus 2}$$

An identity

- $$B(\varpi_1, n)^{\otimes k} = \bigoplus_{\lambda \in \mathcal{P}_n,\ |\lambda| = k} B(\lambda, n)^{\oplus |SYT(\lambda, k)|}$$

$$(x_1 + \cdots + x_n)^k = \sum_{\lambda \in \mathcal{P}_n,\ |\lambda| = k} |SYT(\lambda, k)| s_\lambda(x_1, \ldots, x_n)$$

$$n^k = \sum_{\lambda \in \mathcal{P}_n,\ |\lambda| = k} |SYT(\lambda, k)||SSYT(\lambda, n)|$$

- $n = k = 3$, $\mathbb{B}_3^{\otimes 3} = B(\varpi_1)^{\otimes 3} = B(3\varpi_1) \bigoplus B(\varpi_3) \bigoplus B(\varpi_1 + \varpi_2)^{\oplus 2}$

$$(x_1 + x_2 + x_3)^3 = \sum_{\lambda \vdash 3} |SYT(\lambda, 3)| s_\lambda(x_1, x_2, x_3)$$

$$3^3 = s_{(3)}(1,1,1) + 2s_{(2,1)}(1,1,1) + s_{(1,1,1)}(1,1,1) = 10 + 2 \times 8 + 1$$

The Littlewood-Richardson rule

- For $\mu, \nu \in \mathcal{P}_n$, we have the following $\mathfrak{gl}_n$ crystal isomorphism.
- Let $T \in B(\mu, n)$, $T' \in B(\nu, n)$, $T \otimes T' = w(T) \otimes w(T') = w(T)w(T')$ and

$$P(T \otimes T') = T \leftarrow w(T') \Rightarrow T \otimes T' \equiv P(T \otimes T')$$

- The recording tableau $Q(T \otimes T')$?
  - $T = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array}$, $T' = \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & 4 \\ \hline\end{array}$

$$T \leftarrow 2423 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 3 \\ \hline\end{array} \leftarrow 423 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 3 \\ \hline 4 & \\ \cline{1-1}\end{array} \leftarrow 23 = \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 2 & 3 & \\ \cline{1-2} 4 & & \\ \cline{1-1}\end{array} \leftarrow 3 = P(T \otimes T') = \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 2 & 3 & \\ \cline{1-2} 3 & 4 & \\ \cline{1-2}\end{array}$$

  - $Q(T \otimes T') = \begin{array}{|c|c|c|}\hline & & 1 \\ \hline & 1 & \\ \cline{1-2} 2 & 2 & \\ \cline{1-2}\end{array}$ is a Littlewood-Richardson tableau of shape λ/μ, $\mu = (2, 1)$ and content $\nu = (2, 2)$ with word 1122 a Yamanouchi word.
  - The set of Littlewood-Richardson tableaux of skew shape shape λ/μ and content ν is denoted by $L^{\lambda}_{\mu,\nu}$.
  - The Littlewood-Richardson coefficient $c^{\lambda}_{\mu,\nu} := |L^{\lambda}_{\mu,\nu}| \geq 0$.

- The map $T \otimes T' \longmapsto (P(T \otimes T'), Q(T \otimes T'))$ gives the following crystal isomorphism

$$B(\mu, n) \bigotimes B(\nu, n) \simeq \bigoplus_{\substack{\lambda \in \mathcal{P}_n \\ Q \in LR^{\lambda}_{\mu,\nu}}} B(Q) \simeq \bigoplus_{\substack{\lambda \in \mathcal{P}_n \\ |\lambda| = |\mu| + |\nu|}} B(\lambda, n)^{\oplus c^{\lambda}_{\mu,\nu}},$$

  where $B(Q) = B(\lambda, n) \times \{Q\}$ and $c^{\lambda}_{\mu,\nu} = |LR^{\lambda}_{\mu,\nu}|$
- $c^{(3,2,2)}_{(2,1),(2,2)} = 1$
- $s_\mu s_\nu = \sum_{\lambda \in \in \mathcal{P}_n} c^{\lambda}_{\mu,\nu} s_\lambda.$
- Tensor product decomposition of two $\mathfrak{gl}_n$ irreducible representations

$$V(\mu) \otimes V(\nu) = \bigoplus_{\substack{\lambda \in \mathcal{P}_n \\ |\lambda| = |\mu| + |\nu|}} V(\lambda)^{\oplus c^{\lambda}_{\mu,\nu}}$$

Lusztig-Schützenberger involution

- The Schützenberger–Lusztig involution $\xi : B(\lambda) \to B(\lambda)$ is the unique map of sets such that, for all $b \in B(\lambda)$, and $i \in I$,
  - $e_i \xi(b) = \xi f_{\theta(i)}(b)$
  - $f_i \xi(b) = \xi e_{\theta(i)}(b)$
  - $\mathrm{wt}(\xi(b)) = w_0 \mathrm{wt(b)}$

  where w_0 is the long element of the Weyl group W and θ is the I Dynkin diagram automorphism.
- Let $b \in B(\lambda)$ and $b = f_{j_r} \cdots f_{j_1}(u_\lambda)$, for $j_r, \dots, j_1 \in I$. Then

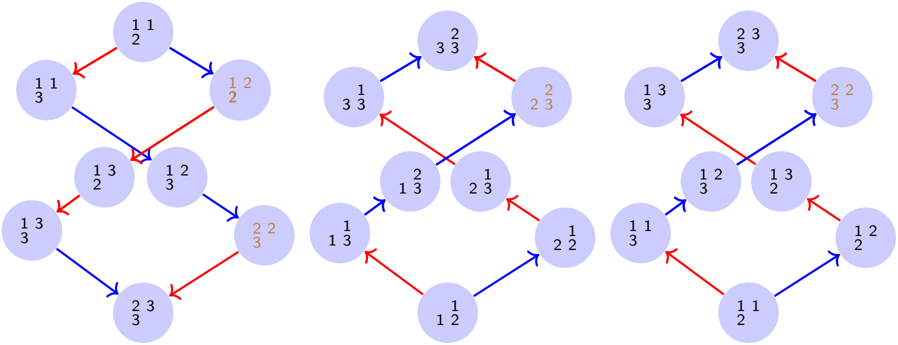
$$\xi(b) = e_{\theta(j_r)} \cdots e_{\theta(j_1)}(u_\lambda^{\mathrm{low}}), \quad \mathrm{wt}(\xi(b)) = w_0 \mathrm{wt}(b).$$

  In particular,
  - in type A_{n-1}, $\xi(b) = e_{n-j_r} \cdots e_{n-j_1}(u_\lambda^{\mathrm{low}})$, and $\mathrm{wt}(\xi(b)) = rev\, \mathrm{wt}(b)$, where rev is the reverse permutation (long element) of $\mathfrak{S}_n$,
    ξ reverses all arrows and colors, and weight. In particular, it interchanges the highest and lowest weight elements.

Lusztig–Schützenberger involution in type A

Schützenberger, 70' reversal/evacuation : $\begin{smallmatrix} 1 & 1 \\ 2 & \end{smallmatrix} \overset{\text{rotate}}{\rightarrow} \begin{smallmatrix} & 2 \\ 3 & 3 \end{smallmatrix} \overset{\text{rectification}}{\rightarrow} \begin{smallmatrix} 2 & 3 \\ 3 & \end{smallmatrix}$

$1 =$ blue $\quad 2 =$ red



$$B((2,1,0),3) \underset{\text{rotate}}{\rightarrow} B((2,2)/(1),3) \underset{\text{rectification}}{\rightarrow} B((2,1,0),3)$$

$$\xi\left(\begin{smallmatrix} 1 & 2 \\ 2 & \end{smallmatrix}\right) = evac\left(\begin{smallmatrix} 1 & 2 \\ 2 & \end{smallmatrix}\right) = \begin{smallmatrix} 2 & 2 \\ 3 & \end{smallmatrix}$$

Demazure crystals

- Let $G = GL_n(\mathbb{C})$ and B a Borel subgroup. Let the Lie algebras of G and B be $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ and $\mathfrak{b}$ a Borel subalgebra of $\mathfrak{g}$ respectively. Let $V(\lambda)$ be the irreducible G-module with highest weight λ.
- For $w \in W$, the Demazure module $V_w(\lambda) \subseteq V(\lambda)$ is the B-submodule defined

$$V_w(\lambda) = \mathcal{U}(\mathfrak{b}).V(\lambda)_{w\lambda},$$

  where $\mathcal{U}(\mathfrak{b})$ is the enveloping algebra of the Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$, and $V(\lambda)_{w\lambda}$ is the one dimensional weight space of $V(\lambda)$ with extremal weight $w\lambda$.
- The Kashiwara crystal $B(\lambda)$ is a combinatorial skeleton for the G-module $V(\lambda)$.
- Demazure characters are the characters of the B-submodules $V_w(\lambda)$.
- **Kashiwara and Littelmann have shown that Demazure characters can be obtained by summing the monomial weights over certain subsets $B_v = B_w(\lambda)$, $v \in W\lambda$, in the crystal $B(\lambda, n)$, called Demazure crystals.**
- $B_v = B_w(\lambda)$ **is the combinatorial skeleton of the Demazure module $V_w(\lambda)$, for $v \in W\lambda$.**
- How to detect a Demazure crystal $B_v = B_w(\lambda)$ in $B(\lambda, n)$?

Demazure keys: Dilatation of crystals

- Let m be a positive integer. There exists a unique embedding of crystals

$$\psi_m : B(\lambda) \hookrightarrow B(m\lambda)$$

such that for $b \in B(\lambda)$ and any path $b = f_{i_1} \cdots f_{i_l}(b_\lambda)$ in $B(\lambda)$, we have

$$\psi_m(b) = f_{i_1}^m \cdots f_{i_l}^m(b_{m\lambda}).$$

- $b_\lambda^{\otimes m}$ is of highest weight $m\lambda$ in $B(\lambda)^{\otimes m} \Rightarrow B(b_\lambda^{\otimes m})$ is a realization of $B(m\lambda)$ in $B(\lambda)^{\otimes m}$ with highest weight vertex $b_\lambda^{\otimes m}$.
- This gives a canonical embedding

$$\theta_m : \begin{cases} B(b_\lambda) \hookrightarrow B(b_\lambda^{\otimes m}) \subset B(b_\lambda)^{\otimes m} \\ b \longmapsto b_1 \otimes \cdots \otimes b_m \end{cases}$$

with important properties.

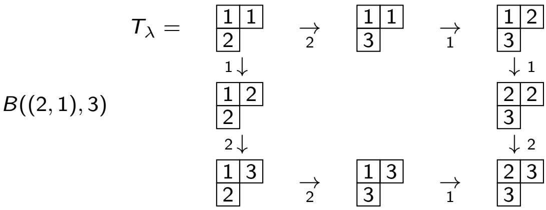
- For $\sigma \in W^\lambda$ the set of minimal coset representatives of W/W_λ, $\theta_m(b_{\sigma\lambda}) = b_{\sigma\lambda}^{\otimes m}$.
- When m has sufficiently many factors, there exist elements $\sigma_1, \ldots, \sigma_m$ in W^λ such that $\theta_m(b) = b_{\sigma_1\lambda} \otimes \cdots \otimes b_{\sigma_m\lambda}$ tensor product of keys.
  - the elements $b_{\sigma_1\lambda}$ and $b_{\sigma_m\lambda}$ in $\theta_m(b)$ do not depend on m,
  - up to repetition, the sequence $(\sigma_1\lambda, \ldots, \sigma_m\lambda)$ in $\theta_m(b)$ does not depend on the realization of the crystal $B(\lambda)$ and we have $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_m$.

Demazure keys:right and left

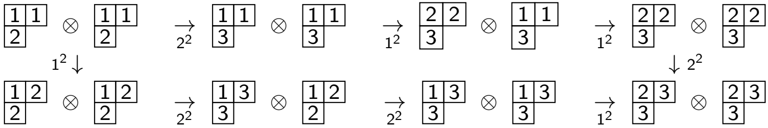
- For $b \in B(\lambda)$ and $\theta_m(b) = b_{\sigma_1\lambda} \otimes \cdots \otimes b_{\sigma_m\lambda}$:
  - The keys $K^+(b)$ and $K^-(b)$ of b are defined as follows:

    $$K^+(b) = b_{\sigma_1\lambda} \text{ and } K^-(b) = b_{\sigma_m\lambda}.$$

    In particular, $K^+(b_{\sigma\lambda}) = K^-(b_{\sigma\lambda}) = b_{\sigma\lambda}$ for any $\sigma \in W^\lambda$.
  - $K^-(b) \leq K^+(b)$ for any $b \in B(\lambda)$, and
  - $K^-(b) = K^+(b)$ if and only if b is in $O(\lambda) = \{b_{\sigma\lambda} : \sigma \in W^\lambda\}$ the set of keys in $B(\lambda)$.



$2-\textit{dilatation},\ B((2,1),3) \hookrightarrow B(T_\lambda^{\otimes 2}) \subseteq B((2,1))^{\otimes 2}$



$$K_+ \begin{array}{|c|c|}\hline 1&3\\\hline 2&\multicolumn{1}{c}{}\\\cline{1-1}\end{array} = \begin{array}{|c|c|}\hline 1&3\\\hline 3&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}, \quad K^- \begin{array}{|c|c|}\hline 1&3\\\hline 2&\multicolumn{1}{c}{}\\\cline{1-1}\end{array} = \begin{array}{|c|c|}\hline 1&2\\\hline 2&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}, \quad K_+ \begin{array}{|c|c|}\hline 1&2\\\hline 3&\multicolumn{1}{c}{}\\\cline{1-1}\end{array} = \begin{array}{|c|c|}\hline 2&2\\\hline 3&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}, \quad K^- \begin{array}{|c|c|}\hline 1&2\\\hline 3&\multicolumn{1}{c}{}\\\cline{1-1}\end{array} = \begin{array}{|c|c|}\hline 1&1\\\hline 3&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}$$

Lakshmibai-Seshadri (LS) paths

- The dilatated crystal $B(\lambda)^{\otimes m}$

$$
\begin{array}{ccccccc}
\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array}\otimes\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array} & \underset{2^2}{\rightarrow} & \begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array}\otimes\begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array} & \underset{1^2}{\rightarrow} & \begin{array}{|c|c|}\hline 2&2\\\hline 3\\\cline{1-1}\end{array}\otimes\begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array} & \underset{1^2}{\rightarrow} & \begin{array}{|c|c|}\hline 2&2\\\hline 3\\\cline{1-1}\end{array}\otimes\begin{array}{|c|c|}\hline 2&2\\\hline 3\\\cline{1-1}\end{array} \\
1^2\downarrow & & & & & & \downarrow 2^2 \\
\begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}\otimes\begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array} & \underset{2^2}{\rightarrow} & \begin{array}{|c|c|}\hline 1&3\\\hline 3\\\cline{1-1}\end{array}\otimes\begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array} & \underset{2^2}{\rightarrow} & \begin{array}{|c|c|}\hline 1&3\\\hline 3\\\cline{1-1}\end{array}\otimes\begin{array}{|c|c|}\hline 1&3\\\hline 3\\\cline{1-1}\end{array} & \underset{1^2}{\rightarrow} & \begin{array}{|c|c|}\hline 2&3\\\hline 3\\\cline{1-1}\end{array}\otimes\begin{array}{|c|c|}\hline 2&3\\\hline 3\\\cline{1-1}\end{array}
\end{array}
$$

- The corresponding crystal of LS paths

$$
\begin{array}{ccccccc}
(id;0,1) & \underset{2}{\rightarrow} & (s_2;0,1) & \underset{1}{\rightarrow} & (s_1s_2,s_2;0,1/2,1) & \underset{1}{\rightarrow} & (s_1s_2;0,1) \\
1\downarrow & & & & & & \downarrow 2 \\
(s_1;0,1) & \underset{2}{\rightarrow} & (s_2s_1,s_1;0,1/2,1) & \underset{2}{\rightarrow} & (s_2s_1;0,1) & \underset{1}{\rightarrow} & (s_1s_2s_1;0,1)
\end{array}
$$

  - Lakshmibai-Seshadri (LS) path of shape λ is a pair $(\nu; a)$ of sequences $\nu : \nu_0 > \cdots > \nu_s$ of elements in W/W_λ in strictly decreasing order and $a : a_0 = 0 < a_1 < \cdots < a_s < a_{s+1} = 1$ of rational numbers in strictly increasing order, satisfying certain integrability conditions. We may regard π as a piecewise linear function such that

$$\pi(t) = \sum_{k=1}^{i-1}(a_k - a_{k-1})\nu_k\lambda + (t - a_{i-1})\nu_i\lambda, \quad a_{i-1} \leq t \leq a_i$$

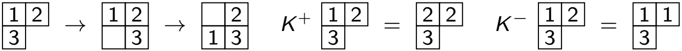
  - For $\pi = (id; 0, 1)$, one has $\pi_\lambda(t) = \lambda t$, $t \in [0, 1]$ and $\theta_m(K(\lambda))$ gives π_λ.
  - $i(\pi) = \nu_0$ is the initial direction (right key) and $e(\pi) = \nu_r$, the final direction (left key) of the path.

The crystal of LS paths

- We denote by $B^{LS}(\lambda)$ the set of all LS paths of shape λ.
- $\pi = (id; 0, 1)$ identified with $\pi_\lambda(t) = \lambda t,\ t \in [0, 1]$ is in $B^{LS}(\lambda)$.
- $B^{LS}(\lambda)$ has crystal structure isomorphic to $B(\lambda)$ with highest weight element π_λ given by $\theta_m(K(\lambda))$.
- There is a unique crystal isomorphism between $B(\lambda)$ and $B^{LS}(\lambda)$ that sends $K(\lambda)$ to π_λ.

Lascoux's keys: jeu de taquin

- An effective way to compute the right and the left key of a SSYT.



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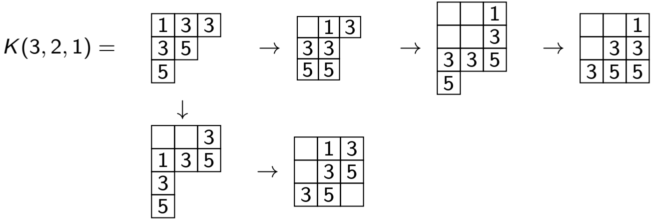


Lascoux's keys or keys à la Lascoux

$$T = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 3 & 6 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array} \;\to\; \begin{array}{|c|c|c|}\hline & 2 & 4 \\ \hline 1 & 3 \\ \cline{1-2} 5 & 6 \\ \cline{1-2}\end{array} \;\to\; \begin{array}{|c|c|c|}\hline & & 2 \\ \hline & & 4 \\ \hline 1 & 3 & 6 \\ \hline 5 \\ \cline{1-1}\end{array} \;\to\; \begin{array}{|c|c|c|}\hline & & 2 \\ \hline & 3 & 4 \\ \hline 1 & 5 & 6 \\ \hline\end{array}$$

$$\downarrow$$

$$\begin{array}{|c|c|c|}\hline & & 4 \\ \hline 1 & 2 & 6 \\ \hline 3 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array} \;\to\; \begin{array}{|c|c|c|}\hline & 2 & 4 \\ \hline & 3 & 6 \\ \hline 1 & 5 & \\ \hline\end{array}$$

$$K^+T = \begin{array}{|c|c|c|}\hline 2 & 4 & 4 \\ \hline 4 & 6 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array} \qquad K^-T = \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 3 & 5 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array}$$

Demazure crystal and its opposite

- Demazure crystal
  - If $v = \sigma\lambda$ where $\sigma = s_{i_\ell} \cdots s_{i_1} \in W$ is a reduced word, we define the Demazure crystal B_v (also denoted $B_\sigma(\lambda)$ to be

$$B_v = \{f_{i_\ell}^{k_\ell} \cdots f_{i_1}^{k_1}(K(\lambda)) \mid (k_\ell, \ldots, k_1) \in \mathbb{Z}_{\geq 0}^\ell\} \setminus \{0\}.$$

    $B_e(\lambda) = B_\lambda = \{K(\lambda)\}$, and if $\sigma = w_0$, $B_{w_0}(\lambda) = B(\lambda)$.
  - Decomposition into Demazure atoms. For $\rho \leq \sigma$ in W^λ, $B_\rho(\lambda) \subseteq B_\sigma(\lambda)$. The *Demazure crystal atom* $\overline{B}_v$ is defined to be

$$\overline{B}_v = B_v \setminus \bigsqcup_{\substack{u \in W\lambda \\ u < v}} B_u = B_v \setminus \bigsqcup_{K(u) < K(v)} B_u.$$

  - Decomposition into Demazure atoms

$$\overline{B}_v = \{b \in B(\lambda) : K^+(b) = K(v)\}$$
$$B_v = \bigsqcup_{\substack{v' \in W\lambda \\ v' \leq v}} \overline{B}_{v'} = \{b \in B(\lambda) : K^+(b) \leq K(v)\}.$$

Opposite Demazure cystal

- For $b \in B(\lambda)$, $\xi K_+(b) = K_-(\xi b)$, $\xi K(v) = K(w_0 v)$.
- For $u, v \in W\lambda$, $u \leq v$ in the induced Bruhat order $\Leftrightarrow K(u) \leq K(v) \Leftrightarrow \xi K(u) \geq \xi K(v)$.
- The *opposite Demazure crystal* $B^{w_0 v}$ is defined to be

$$\begin{aligned} B^{w_0 v} &:= \{e_{\theta(i_\ell)}^{k_\ell} \cdots e_{\theta(i_1)}^{k_1}(K(w_0\lambda)) \mid (k_\ell, \dots, k_1) \in \mathbb{Z}_{\geq 0}^{\ell}\} \setminus \{0\} \\ &= \xi(B_v) = \xi\{b \in B(\lambda) : K^+(b) \leq K(v)\} \\ &= \{b \in B(\lambda) : K^+(\xi b) \leq K(v)\} \\ &= \{b \in B(\lambda) : \xi K^-(b) \leq K(v)\} \\ &= \{b \in B(\lambda) : K^-(b) \geq \xi K(v)\} \\ &= \{b \in B(\lambda) : K^-(b) \geq K(w_0 v)\}. \end{aligned}$$

$$B^v = \xi(B_{w_0 v}) = \{b \in B(\lambda) : K^-(b) \geq K(v)\}.$$

- Opposite Demazure atom

$$\overline{B}^v = \{b \in B(\lambda) : K^-(b) = K(v)\} = \xi(\overline{B}_{w_0 v}).$$

- Decomposition into opposite Demazure atoms

$$B^{w_0 v} = \bigsqcup_{v' \in W\lambda,\, v' \leq v} \xi(\overline{B}_{v'}) = \bigsqcup_{v' \in W\lambda,\, v' \leq v} \overline{B}^{w_0 v'} = \bigsqcup_{\substack{v' \in W\lambda \\ v' \geq w_0 v}} \overline{B}^{v'}.$$

$$B^v = \bigsqcup_{\substack{v' \in W\lambda \\ v' \geq v}} \overline{B}^{v'}.$$

Relations between Demazure crystals

Let $u, v, x, y \in W\lambda$ and $b \in B(\lambda)$. Then

1. $B_x \subseteq B_y \Leftrightarrow B^x \supseteq B^y \Leftrightarrow x \leq y$.
2. $\overline{B}^u \cap \overline{B}_v \neq \emptyset \Leftrightarrow u \leq v$ which happens when

$$\overline{B}^u \cap \overline{B}_v = \{b \in B(\lambda) \mid K(u) = K^-(b) \leq K^+(b) = K(v)\} \supseteq \{K(u), K(v)\}.$$

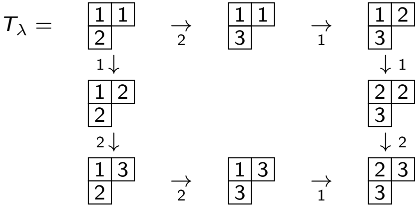
3. $B^u \cap B_v \neq \emptyset \Leftrightarrow u \leq v$ which happens when

$$B^u \cap B_v = \{b \in B(\lambda) \mid K(u) \leq K^-(b) \leq K^+(b) \leq K(v)\} \supseteq \{K(z) \mid z \in [u, v]\},$$

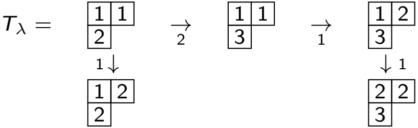
where $[u, v] \subseteq W\lambda$ is an interval in the induced Bruhat order.

Demazure character

- $B((2,1,0),3)$



- Demazure crystal $B_{s_1 s_2}((2,1,0)) = \{b \in B((2,1,0),3) : K^+(b) \le K(s_1 s_2(2,1,0))\}$



$$K^+ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} = \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$$

- Demazure character and opposite Demazure character

$$\kappa_{s_1 s_2 \lambda}(x_1, x_2, x_3) = \sum_{b \in B_{s_1 s_2}(\lambda)} x^{wt(b)} = \sum_{\substack{b \in B(\lambda) \\ K^+(b) \le K(s_1 s_2 \lambda)}} x^{wt(b)}$$

$$\kappa^{s_1 s_2 \lambda}(x_1, x_2, x_3) = \sum_{b \in B^{s_1 s_2}(\lambda)} x^{wt(b)} = \sum_{\substack{b \in B(\lambda) \\ K^-(b) \ge K(s_1 s_2 \lambda)}} x^{wt(b)}.$$

Schubert varieties and Demazure crystals

- Let G be a simply-connected semisimple algebraic group over $\mathbb{C}$, $\mathfrak{g}$ its Lie algebra, and $T \subseteq G$ a maximal torus. We also fix $T \subseteq B \subseteq G$, B a Borel subgroup of G (a subgroup of G containing a maximal torus). Let B^- be the corresponding opposite Borel subgroup, that is, it is the unique Borel subgroup of G with the property $B \cap B^- = T$.
- For simplicity, let $G = GL_n(\mathbb{C})$, B be the subgroup of upper triangular matrices and B^- the subgroup of lower triangular matrices.
- Bruhat decomposition and (full) flag variety.
  The Bruhat decomposition which describes the $B \times B$, respectively $B^- \times B$ orbits in G which are parameterized by W

$$G = \bigsqcup_{w \in W} BwB = \bigsqcup_{w \in W} B^- wB.$$

- Let $G/B = \{gB : g \in G\}$ be the (full) flag variety in G.
-

$$G/B = \bigsqcup_{w \in W} BwB/B = \bigsqcup_{w \in W} B^- wB/B.$$

- The Schubert cell C_w is $C_w = BwB/B = B\dot{w}$, and
- The opposite Schubert cell C^w is $C^w = w_0 C_{w_0 w} = B^- wB/B = B^- \dot{w}$.
- The Schubert variety X_w, respectively the opposite Schubert variety X^w, in G/B are

$$X_w = \bigsqcup_{v \leq w} C_v, \ \ X^w = \bigsqcup_{u \geq w} C^u = w_0 X_{w_0 w} \subseteq G/B,$$

  where $\leq$ denotes the (strong) Bruhat order on W.

Relations among Schubert varieties

- For $w, w' \in W$,

$$X_w \subseteq X_{w'} \Leftrightarrow w \leq w' \Leftrightarrow X^w \supseteq X^{w'}.$$

- The Richardson variety X_α^β in G/B corresponding to the pair (α, β), $\alpha, \beta \in W$, is the (set theoretic) intersection

$$X_\alpha^\beta := X_\alpha \cap X^\beta = \bigsqcup_{\beta \leq v' \leq u' \leq \alpha} C_{u'} \cap C^{v'} \neq \emptyset \Leftrightarrow \beta \leq \alpha.$$

Borel-Weil theorem, Demazure modules and Schubert varieties

- Let $\mathfrak{g}$ be the Lie algebra of G. Let $V(\lambda)$ be the irreducible highest weight G-module over $\mathbb{C}$ with highest weight λ, and let $B(\lambda)$ its combinatorial skeleton.
- Let L_λ be a line bundle on the flag variety G/B.
- By the Borel-Weyl theorem the space $H^0(G/B, L_\lambda)$ of global sections is a G-module isomorphic to $V(\lambda)^*$ the dual of $V(\lambda)$,

$$V(\lambda)^* \simeq H^0(G/B, L_\lambda).$$

- Kashiwara constructed a specific $\mathbb{C}$-basis of $H^0(X_w, L_\lambda)$ via the quantized enveloping algebra associated to $\mathfrak{g}$, specialized at $q = 1$. This $\mathbb{C}$-basis, $\{G_\lambda^{up}(b) : b \in B(\lambda)\}$ the upper global basis (specialized at $q = 1$) is compatible with Schubert varieties and opposite Schubert varieties:

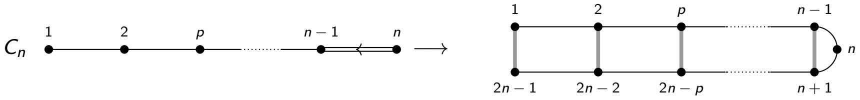
$$H^0(X_w, L_\lambda) \simeq V_w(\lambda)^* \quad \mathbb{C}\text{-basis } \{G_\lambda^{up}(b) : b \in B_w(\lambda)\}$$

$$H^0(X^w, L_\lambda) \simeq V^w(\lambda)^* \quad \mathbb{C}\text{-basis } \{G_\lambda^{up}(b) : b \in B^w(\lambda)\}.$$

- Associated to the combinatorial path model given by the LS paths of shape λ, Littelmann constructed also a basis for $H^0(G/B, L_\lambda)$ compatible with Schubert varieties and opposite Schubert varieties and their intersections.

Cartan type C_n

Folding A_{2n-1}:



- In type C_n, $\alpha_n = 2e_n$ and $\alpha_n^\vee = e_n$ and the Weyl group $W = B_n$ the hyperocthaedral group, and for $v \in \mathbb{Z}^n$, $w_0 v = -v$. The Dynkin automorphism $\theta = id$.
- $B_n \simeq B_n^A := \langle \tilde{s}_i := s_i^A s_{2n-i}^A, \tilde{s}_n := s_n^A, 1 \leq i < n \rangle$ as a subgroup of $\mathfrak{S}_{2n}$.
- B_n is the free group generated by $r_1, \ldots, r_{n-1}, r_n$ subject to the relations

$$r_i^2 = 1,\ 1 \leq i \leq n, \tag{1}$$

$$(r_i r_j)^2 = 1,\ 1 \leq i < j \leq n,\ |i - j| > 1, \tag{2}$$

$$(r_i r_{i+1})^3 = 1, 1 \leq i \leq n - 2, \tag{3}$$

$$(r_{n-1} r_n)^4 = 1. \tag{4}$$

- Kashiwara crystal Let V be an Euclidean space with inner product $\langle \cdot, \cdot \rangle$. Fix a root system Φ with simple roots $\{\alpha_i \mid i \in I\}$ where I is an indexing set and a weight lattice $\Lambda \supseteq$ -span$\{\alpha_i \mid i \in I\}$. A *Kashiwara crystal* of type Φ is a nonempty set $\mathfrak{B}$ together with maps :

$$e_i, f_i : \mathfrak{B} \to \mathfrak{B} \sqcup \{0\} \quad \varepsilon_i, \varphi_i : \mathfrak{B} \to \sqcup\{-\infty\} \quad \text{wt} : \mathfrak{B} \to \Lambda$$

where $i \in I$ and $0 \notin \mathfrak{B}$ is an auxiliary element, satisfying the following conditions:

(a) if $a, b \in \mathfrak{B}$ then $e_i(a) = b \Leftrightarrow f_i(b) = a$. In this case, we also have $\text{wt}(b) = \text{wt}(a) + \alpha_i$, $\varepsilon_i(b) = \varepsilon_i(a) - 1$ and $\varphi_i(b) = \varphi_i(a) + 1$;

(b) for all $a \in \mathfrak{B}$, we have

$$\varphi_i(a) = \langle \text{wt}(a), \alpha_i^\vee \rangle + \varepsilon_i(a) \text{ with } \alpha_i^\vee = \frac{2\alpha_i}{\langle \alpha_i, \alpha_i \rangle}.$$

1. $\varphi_i(a) = \max\{k \in_{\geq 0} \mid f_i^k(a) \neq 0\}$ and $\varepsilon_i(a) = \max\{k \in_{\geq 0} \mid e_i^k(a) \neq 0\}$.

Kashiwara-Nakashima tableaux

- Let $B^C(\lambda)$ be the irreducible C_n-crystal with highest weight a partition λ of at most n parts.
  We realize $B^C(\lambda)$ as the crystal of symplectic Kashiwara–Nakashima tableaux of shape λ on the alphabet

$$\{1 < \cdots < n < \bar{n} < \cdots < \bar{1}\}.$$