ADDENDA (February 12, 2018)

(1) On frame coproducts

We are indebted to Dharmanand Baboolal for pointing to us the omission to adduce the openness of the coproduct injections, a very easy but fundamental and useful fact (for which we are unable to find a source; it might be folklore). It would fit in IV.5.

Openness of the coproduct injections. Recall the notation from IV.4: $\prod_i L_i = \{(x_i)_i \in \prod_i L_i \mid x_i = 1 \text{ but for finitely many } i\}$, $\mathsf{n} = \{(x_i)_i \mid \exists j, x_j = 0\}$, $x *_i u$ obtained from $u = (u_j)_j$ replacing u_i by x, the elements $\oplus_i a_i = {\downarrow}(a_i)_i \cup \mathsf{n}$ of $\bigoplus_i L_i$, and the injections $\iota_i : L_i \to \bigoplus_j L_j$ sending x to ${\downarrow}(x *_i \overline{1}) \cup \mathsf{n}$ ($\overline{1}_j = 1$ for all j).

In particular for two frames L, K we have $\mathsf{n} = \{(x, y) \in L \times K \mid x = 0 \text{ or } y = 0\}$, $a \oplus b = {\downarrow}(a, b) \cup \mathsf{n}$ and the coproduct injections $\iota_L = (a \mapsto a \oplus 1) : L \to L \oplus K$ and $\iota_K = (b \mapsto 1 \oplus b) : K \to L \oplus K$.

The following proof is very easy. Only, in the general case it is slightly obscured by the notation necessary for dealing with general index sets. Therefore we present, first, a proof for two frames and then one for the general case. The reader will see that the latter is, but for the notation, identical with the former one (in fact it is just an exercise in formalism).

Theorem. *The coproduct injections $\iota_i : L_i \to \bigoplus_i L_i$ are open frame homomorphisms.*

Proof. (*For two frames*): By characterization $(**)$ in the proof of III.7.2 we need a mapping $\phi : L \oplus K \to L$ such that

$$x \wedge \phi(U) \le y \quad \text{iff} \quad (x \oplus 1) \cap U \subseteq (y \oplus 1). \tag{$*$}$$

The inclusion $(x \oplus 1) \cap U \subseteq (y \oplus 1)$ is the same as

$${\downarrow}(x, 1) \cap U \subseteq {\downarrow}(y, 1) \cup \mathsf{n}$$

which amounts to claiming that $(a, b) \in U$, $a \le x$ and $b \ne 0$ implies $a \le y$, and since U is a down-set we can rewrite it as

$$((a, b) \in U \text{ and } b \ne 0) \quad \Rightarrow \quad a \wedge x \le y.$$

If we set $\phi(U) = \bigvee\{a \mid \exists b \ne 0,\ (a, b) \in U\}$ we have by distributivity $x \wedge \phi(U) = \bigvee\{x \wedge a \mid \exists b \ne 0,\ (a, b) \in U\} \le y$ iff for all $b \ne 0$ such that $(a, b) \in U$, $a \wedge x \le y$. Thus $(*)$ holds.

(*The general case*): We need a ϕ such that

$$x \wedge \phi(U) \le y \quad \text{iff} \quad (x *_i \overline{1}) \cap U \subseteq y *_i \overline{1}.$$

The inclusion $(x *_i \overline{1}) \cap U \subseteq y *_i \overline{1}$ is the same as

$${\downarrow}(x *_i \overline{1}) \cap U \subseteq {\downarrow}(y *_i \overline{1}) \cup \mathsf{n}$$

which amounts to claiming that $(a_j)_j \in U$, $a_i \leq x$ and $(a_j)_j \notin \mathsf{n}$ implies $a_i \leq y$, and since U is a down-set we can rewrite it as

$$(a_j)_j \in U \smallsetminus \mathsf{n} \;\Rightarrow\; a_i \wedge x \leq y.$$

If we set $\phi(U) = \bigvee\{a_i \mid (a_j)_j \in U \smallsetminus \mathsf{n}\}$ we have by distributivity $x \wedge \phi(U) = \bigvee\{x \wedge a_i \mid (a_j)_j \in U \smallsetminus \mathsf{n}\}$ iff for all $(a_j)_j \in U \smallsetminus \mathsf{n}$, $a_i \wedge x \leq y$ which yields the required equivalence. $\square$

Note. The proof using Proposition III.7.2 right away is also easy:

Proof. (*For two frames*): By III.7.2 we need to show that each ι_i is a complete Heyting homomorphism, that is,

(1) $(\bigvee_j x_j) \oplus 1 = \bigvee_j (x_j \oplus 1)$, and

(2) $(x \to y) \oplus 1 = (x \oplus 1) \to (y \oplus 1)$.

(1) holds by IV.5.2. Regarding (2), we have

$$(x \to y) \oplus 1 = \bigvee_{c\colon\, c \wedge x \leq y} (c \oplus 1) \tag{A}$$

while

$$(x \oplus 1) \to (y \oplus 1) = \bigvee\{C \in L \oplus K \mid C \cap (x \oplus 1) \subseteq (y \oplus 1)\}. \tag{B}$$

Of course, (A)$\subseteq$(B) since $(c \oplus 1) \cap (x \oplus 1) = (c \wedge x) \oplus 1 \subseteq y \oplus 1$. Conversely, let $C \in L \oplus K$ such that $C \cap (x \oplus 1) \subseteq (y \oplus 1)$ and let $(a, b) \in C$ with $b \neq 0$. Then $(a \wedge x, b) \in C \cap (x \oplus 1)$ and therefore $a \wedge x \leq y$ and $a \oplus b \subseteq a \oplus 1$, the latter being part of the join in (A).

The general case is again a mere exercise in formalism. $\square$

(2) A shorter proof of Proposition III.6.5 (page 36):

Lemma. *Let S be a sublocale f L. If $a \in S$ then for every $b \in L$, $b \to a = \nu_S(b) \to a$.*

Proof. Trivially, $\nu_S(b) \to a \leq b \to a$. On the other hand, $b \leq (b \to a) \to a$, hence $\nu_S(b) \leq (b \to a) \to a$, and finally $b \to a \leq \nu_S(b) \to a$. $\square$

Proposition. *Let S be a sublocale. Then $S = \bigcap\{\mathfrak{c}(x) \vee \mathfrak{o}(y) \mid \nu_S(x) = \nu_S(y)\}$.*

Proof. Since $\nu_S(\nu_S(x)) = \nu_S(x)$ it suffices to show that if $\nu_S(x) = \nu_S(y)$ then $S \subseteq \mathfrak{c}(x) \vee \mathfrak{o}(y)$.

Hence let $a \in S$. We have by the lemma $a = (a \vee x) \wedge (x \to a) = (a \vee x) \wedge (\nu_S(x) \to a) = (a \vee x) \wedge (\nu_S(y) \to a) = (a \vee x) \wedge (y \to a) \in \mathfrak{c}(x) \vee \mathfrak{o}(y)$. $\square$

Another version of it:

Proposition. $S = \bigcap\{\mathfrak{c}(\nu_S(x)) \vee \mathfrak{o}(x) \mid x \in L\}$.

Proof. If $a \in S$ then for arbitrary x, $x \to a \in S$. Hence

$$a = (a \vee \nu_S(x)) \wedge (\nu_S(x) \to a) = (a \vee \nu_S(x)) \wedge (x \to a) \in \mathfrak{c}(\nu_S(x)) \vee \mathfrak{o}(x).$$

On the other hand, if $a \in \bigcap\{\mathfrak{c}(\nu_S(x)) \vee \mathfrak{o}(x) \mid x \in L\}$ then in particular $a \in \mathfrak{c}(\nu_S(a)) \vee \mathfrak{o}(a)$ and hence $a = x \wedge (a \to y)$ with $y \geq \nu_S(a)$. Since $a \leq a \to y$ we have $a \leq y$, hence $a \to y = 1$, so that $a = x \geq \nu_S(a)$, and $a \in S$. $\square$

(3) A shorter proof of Lemma VI.4.4.1 (page 106):

$(-) \wedge a$ has a right adjoint $r(x) = a^* \vee x$, and $(-) \vee a$ has a left adjoint $l(x) = a^* \wedge x$. $\square$