Regularity theory and free boundary problems: from PDE to interfaces

Satellite Conference of the
European Congress of Mathematics ECM2024

Book of abstracts

Coimbra, 22-26 July 2024

Under the scope of:

9TH EUROPEAN CONGRESS OF MATHEMATICS ECM Sevilla2024 July 15th to 19th

EUROPEAN MATHEMATICAL SOCIETY

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Mini-course

Gradient regularity estimates

Giuseppe Mingione
University of Parma

Abstract

Recent years have witnessed a blossoming of regularity results for solutions to nonlinear elliptic and parabolic, potentially degenerate equations, especially as far as gradient estimates are concerned. Here I will try to survey a certain number of those, dividing them in two classes.

1. Uniformly elliptic problems. In this case I will give a reasonable list of pointwise nonlinear potential estimates allowing to reduce the gradient regularity of general nonlinear quasilinear equations to that of the plain Poisson equation.
2. Nonuniformly elliptic problems. In this case I will outline novel methods to get gradient estimates bridging uniformly and nonuniformly elliptic theories and finally coming to illustrate the solution found by Cristiana De Filippis and myself to the longstading open problem of the validity of Schauder estimates for nonuniformly elliptic equations and functionals.

Plenary talks

Higher free boundary regularity for semilinear obstacle-type problems

Mark Allen
Brigham Young University

Abstract

We consider obstacle problems with a semilinear right hand side. We show how to utilize the Hodograph transform to obtain higher regularity of the free boundary once C^1 regularity of the free boundary is known. Typically, the hodograph transform is used for the Bernoulli free boundary problem whereas the Legendre transform is used for the obstacle free boundary problem. The semilinear right hand sides we consider do not accommodate the Legendre transform. Consequently, we utilize the Hodograph transform in conjunction with Schauder estimates, and Sobolev estimates to bootstrap C^1 to C^∞ regularity for the free boundary.

A priori estimates for singularities of the Lagrangian mean curvature flow

Arunima Bhattacharya
University of North Carolina at Chapel Hill

Abstract

In this talk, we will discuss interior Hessian estimates for shrinkers, expanders, translators, and rotators of the Lagrangian mean curvature flow, and further extend this result to a broader class of Lagrangian mean curvature type equations. We assume the Lagrangian phase to be hypercritical, which results in the convexity of the potential of the initial Lagrangian submanifold. Convex solutions of the second boundary value problem for certain such equations were constructed by Brendle-Warren 2010, Huang 2015, and Wang-Huang-Bao 2023. We will also briefly introduce the fourth-order Hamiltonian stationary equation and mention some recent results on the regularity of solutions of certain fourth-order PDEs, which are critical points of variational integrals of the Hessian of a scalar function. Examples include volume functionals on Lagrangian submanifolds. This talk is partially based on joint work with Jeremy Wall.

Quantified Legendreness

Cristiana De Filippis
University of Parma

Abstract

Convex, even polynominals are one of the most prominent model examples of autonomous, nonuniformly elliptic variational integrals with power-type degree of nonuniformity, cf. Marcellini ARMA '89. In this talk I will describe a novel quantification of nonuniform ellipticity via convex duality that allows proving essentially optimal regularity results for possibly degenerate vectorial problems. From recent joint work with Lukas Koch (MPI Leipzig) and Jan Kristensen (Oxford).

Some results on Anisotropic PDEs

Eurica Henriques
Centre of Mathematics CMAT
University of Trás-os-Montes e Alto Douro

Abstract

In this presentation, I start by revisit several previous works concerning anisotropic differential equations, namely the Hölder continuity of the local weak solutions to

$$u_t - \operatorname{div}\left(u^{\gamma(x,t)} Du\right) = 0 \ , \quad \gamma(x,t) > 0 \ ;$$

$$u_t - \sum_{i=1}^{N} (u^{m_i})_{x_i x_i} = 0 \ , \quad m_i > 0 \ ;$$

written for nonnegative bounded functions u defined in $\Omega_T = \Omega \times (0, T]$, being Ω a bounded domain in $\mathbf{R}^N$ and $0 < T < \infty$.

From these works a recent collaboration, with S. Ciani and I. Skrypnik, emerged. I'll briefly discuss the ongoing projects related to the study of these and other anisotropic PDEs, pointing out some of the difficulties one needs to address and overcome.

Inhomogeneous one-phase Stefan problem

David Jesus

July 22, 2024

Abstract

The Stefan problem consists of a parabolic free boundary problem which models problems of melting or solidification, where the caloric energy changes discontinuously across the melting temperature. It constitutes probably the most important problem in this family and has attracted much attention from experts in the field. The two-phase inhomogeneous equation can be written as

$$\begin{cases} \partial_t u - \Delta u = f, & \textit{in } \{(x,t) \in \Omega \times [0,T] : \; u > 0\}, \\ \partial_t u - \Delta u = f, & \textit{in } \{(x,t) \in \Omega \times [0,T] : \; u \leq 0\}^0, \end{cases}$$

with the following free boundary condition

$$-\frac{\partial_t u^-}{|\nabla u^-|} = \frac{\partial_t u^+}{|\nabla u^+|} = |\nabla u^+| - |\nabla u^-|, \quad \textit{on } \partial\{(x,t) \in \Omega \times [0,T] : \; u > 0\} \cap \Omega.$$

Recently, De Silva, Forcillo and Savin managed to develop a more flexible argument, adapting the idea of De Silva in the elliptic case to allow for a non-stationary free boundary, and developed an improvement of flatness based on a Harnack inequality. However, only the homogeneous one-phase case was considered, that is, they assumed $u \geq 0$ and $f = 0$.

In a recent paper with Fausto Ferrari (University of Bologna), Nicolò Forcillo (Michigan State University) and Davide Giovagnoli (University of Bologna), we managed to extend the above result to the inhomogeneous one-phase case. A special care has to be given when the source term is very negative near the free boundary as this can cause the water to freeze and violate the free boundary condition. To avoid this situation, a new nondegeneracy assumption was utilized to get a parabolic Hopf-type lemma for negative source terms.

The Burkholder functional and some related variational problems

Jan Kristensen
Oxford University

Abstract

The area formula of Gronwall and Bieberbach can be viewed as a precise way to express that the 2D Jacobian is a null Lagrangian. In this talk I discuss a quasiconvexity inequality for the Burkholder functional in the context of planar quasiconformal maps. This inequality can be viewed as an extention of the area formula to an Lp context. It also motivates the introduction of a new convexity notion that we call principal quasiconvexity. We give additional examples of such functionals, relevant in planar nonlinear elasticity, and show how it in combination with Stoilow factorization allows to establish existence of minimizers in related variational problems. The talk is based on joint work with Kari Astala (Helsinki), Daniel Faraco (Madrid), Andre Guerra (ETH), and Aleksis Koski (Aalto).

Harmonic Moving Frames, Renormalised Energy, and Interplay between Loewner and Willmore Energies

Alexis Michelat*

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Abstract

In this talk, we will explain how one can rewrite the Loewner energy—a conformally invariant energy of planar curves—as the Bethuel–Brezis–Hélein renormalised energy of singular harmonic maps arising as limiting objects in the Ginzburg–Landau model (joint work with Yilin Wang). If time allows, we will describe a procedure to link the Willmore energy—or renormalised area—of minimal surfaces in 3-dimensional hyperbolic space with the Loewner energy of the simple curve that they asymptotically bound at infinity (work in progress with Paul Laurain).

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The Lawson-Osserman conjecture for the minimal surface system

Connor Mooney
University of California Irvine

Abstract

In their seminal work on the minimal surface system, Lawson and Osserman conjectured that Lipschitz graphs that are critical points of the area functional with respect to outer variations are also critical with respect to domain variations. We will discuss the proof of this conjecture for two-dimensional graphs of arbitrary codimension. This is joint work with J. Hirsch and R. Tione.

Leapfrogging vortex rings for Euler equations

Monica Musso
University of Bath

Abstract

We consider the Euler equations for incompressible fluids in 3-dimension. A classical question that goes back to Helmholtz is to describe the evolution of vorticities with a high concentration around a curve. The work of Da Rios in 1906 states that such a curve must evolve by the so-called "binormal curvature flow". Existence of true solutions whose vorticity is concentrated near a given curve that evolves by this law is a long-standing open question that has only been answered for the special case of a circle travelling with constant speed along its axis, the thin vortex-rings, and of a helical filament, associated to a translating-rotating helix. In this talk I will consider the case of two vortex rings interacting between each other, the so-called leapfrogging. The results are in collaboration with J. Davila, M. del Pino and J. Wei.

Local regularity results for parabolic systems with general growth

Jihoon Ok
Sogang University

Abstract

We discuss on regularity theory for parabolic systems of the form

$$u_t - \mathrm{div} A(Du) = 0 \quad \text{in} \quad \Omega_T = \Omega \times (0, T],$$

where $u : \Omega_T \to \mathbb{R}^N$, $u = u(x,t)$, is a vector valued function and the nonlinearity $A : \mathbb{R}^{nN} \to \mathbb{R}^{nN}$ satisfies a general Orlicz growth condition characterized by exponents p and q, subject to the inequality $\frac{2n}{n+2} < p < q$. It is noteworthy that when if $p < 2 < q$, the degeneracy of the system remains indeterminate.

This talk focuses on presenting developments in the realm of regularity results concerning the spatial gradient of solutions of the above system, which include the higher higher integrability, Hölder continuity when $A(\xi)$ satisfies the Uhlenbeck structure, i.e., $A(\xi) = \frac{\varphi'(|\xi|)}{|\xi|}\xi$, and partial Hölder continuity. These results are joint works with Giovanni Scilla and Bianca Stroffolini from University of Naples Federico II, and Peter Hästö from University of Helsinki.

Kinetic integral equations

Giampiero Palatucci
University of Parma

Abstract

I will present some recent results for weak solutions to a wide class of kinetic integral equations, where the diffusion term in velocity is an integro-differential operator having nonnegative kernel is of fractional order s in $(0, 1)$ with merely measurable coefficients. In particular, I will focus on Harnack-type inequalities for nonnegative weak solutions that does not require the usual a priori boundedness. The talk is based on a series of papers by Anceschi, Kassmann, Piccinini, Weidner and myself.

Variational and Quasi-variational solutions to thick flows

José Francisco Rodrigues
CMAFcIO/Ciências/ULisboa, Portugal

Abstract

We formulate the flow of thick fluids as evolution variational and quasi-variational inequalities, with a variable threshold on the absolute value of the deformation rate tensor, which may be a priori given (variational case) or may depend on the solution itself (quasi-variational case). We show the existence of strong and weak solutions in different cases to a problem which may be considered a free boundary problem for the Navier-Stokes equations. The results are based on new extensions of the continuous dependence of weak solutions to the variational inequalities for the Navier-Stokes equations with constraints on the derivatives, and on their respective generalised Lagrange multipliers. This is a joint work with Lisa Santos (UMinho)

Regularity of free interfaces in transmission problems arising from the jump of conductivity

María Soría-Carro
Rutgers University

Abstract

Transmission problems are a class of boundary value problems that describe diffusion processes driven by discontinuous laws across interfaces. The mathematical theory started in the 1950s with the pioneering work of M. Picone in elasticity, and since then, it has been an active research area with numerous developments.

In this talk, we will introduce a parabolic *free* transmission problem motivated by the jump of conductivity in composite materials that undergo a phase transition. Our main goal is to establish strong regularity properties of the free boundary (i.e., the transition surface), following the classical strategy:

(i) *Flat free boundaries are* $C^{1,\alpha}$;

(ii) $C^{1,\alpha}$ *implies smooth.*

We will discuss some of the main ideas and techniques we used to prove (i) and (ii), which are largely inspired by the seminal works of D. Kinderlehrer, L. Nirenberg, and J. Spruck (1978), L. Caffarelli (1989), and D. De Silva (2011).

This is a joint work with Dennis Kriventsov (Rutgers University).

ABP maximum principle for Lp-viscosity solutions of parabolic equations with singular terms

Andrzej Święch
Georgia Institue of Technology, GaTech

Abstract

We will present several results on Aleksandrov-Bakelman-Pucci and Bony type maximum principles for L^p-viscosity solutions of fully nonlinear, uniformly parabolic equations with singular drift terms. In particular, we will discuss how to obtain the ABP maximum principle in a version with contact sets for viscosity solutions of such equations and what integrability of the gradient coefficient is required. Implications of the new maximum principle results for the theory of L^p-viscosity solutions will also be discussed. This is a joint work with S. Koike.

Fully nonlinear dead-core systems

Rafayel Teymurazyan
KAUST and University of Coimbra

Abstract

We will present recent advances on fully nonlinear dead-core systems coupled with strong absorption terms. The main challenge in dealing with those systems is the lack of comparison principle and of the classical Perron's method. Nevertheless, we discover a chain reaction, exploiting the properties of an equation along the system and obtaining higher sharp regularity across the free boundary. Additionally, we prove geometric measure estimates and obtain coincidence of the free boundaries. We also derive Liouville type theorems for entire solutions. These results are new, even for linear systems.

Hidden convexity in the nonlinear Schrödinger equations, equations for barotropic fluids and other related models

Dmitry Vorotnikov
University of Coimbra

Abstract

We study the systems of nonlinear PDEs that can be written in the form

$$\partial_t v = L(\mathbf{F}(v))$$

with some symmetry assumptions that formally yield conservation of a convex entropy $\mathbf{K} : \mathbb{R}^n \to \mathbb{R}_+$. Here L is a linear operator and $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}_+^{N\times N}$ is such that $\frac{1}{2}\mathrm{Tr}\mathbf{F} = \mathbf{K}$. These problems can be viewed as generalized conservation laws. The examples include the equations of motion of compressible barotropic fluids, the NLS and NLKG equations, various models with quadratic nonlinearity such as Euler's equations of inviscid fluids, the ideal MHD and many others. We discuss the dual (ballistic) formulation of such problems in anisotropic matrix-valued Orlicz spaces generated by the Legendre transform of $\mathbf{K}$ and in the spaces of matrix-valued measures. We study solvability and consistency of the dual formulation. As an application, we prove a "Dafermos' principle", i.e., that no (properly defined) subsolution of such problems can dissipate the entropy faster than the strong solution provided that the latter exists.

Contributed talks

Optimal Regularity for the Variable Coefficients Parabolic Signorini Problem

Vedansh Arya
University of Jyväskylä

Abstract

In this talk, we discuss the optimal regularity of the variable coefficient parabolic Signorini problem with $W_p^{1,1}$ coefficients and L^p inhomogeneity, where $p > n+2$ with n being the space dimension. Relying on a parabolic Carleman estimate and an epiperimetric inequality, we demonstrate the optimal regularity of the solutions as well as the regularity of the regular free boundary. This work is based on a joint effort with Wenhui Shi.

Sharp bounds for higher Steklov-Dirichlet Eigenvalue

Anisa Chorwadwala
IISER Pune

Abstract

We consider mixed Steklov-Dirichlet eigenvalue problem on smooth bounded domains in Riemannian manifolds. Under certain symmetry assumptions on multiconnected domains in a Euclidean space with a spherical hole, we obtain isoperimetric inequalities for k-th Steklov-Dirichlet eigenvalues for k between 2 and $n+1$. We extend Theorem 3.1 of GPPS from Euclidean domains to domains in space forms, that is, we obtain sharp lower and upper bounds of the first Steklov-Dirichlet eigenvalue on bounded star-shaped domains in the unit n-sphere and in the hyperbolic space.

GPPS Nunzia Gavitone, Gloria Paoli, Gianpaolo Piscitelli, and Rossano Sannipoli. An isoperimetric inequality for the first Steklov–dirichlet laplacian eigenvalue of convex sets with a spherical hole. Pacific Journal of Mathematics, 320 (2) 241– 259, 2023.

Optimal error estimates for a Discontinuous Galerkin method on curved boundaries

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The ever-increasing complexity of real-word applications has raised important challenges in the quest for accurate and efficient numerical methods for solving partial differential equations. In particular, we are interested in solving boundary value problems in curved boundary domains considering the Discontinuous Galerkin (DG) method. The question that arises concerns the reduction of the order of convergence of numerical methods when considering the approximation of the domain by a polygonal mesh. In [2] we developed a strategy called DG-ROD (Reconstruction for Off-site Data) method, which is based on a polynomial reconstruction of the boundary condition imposed on the computational domain [3].
In this talk, we present a study on the existence and uniqueness of the solution and derive error estimates for a boundary-value problem with homogeneous Dirichlet boundary conditions [1, 4]. We prove that the DG–ROD solution exhibits an optimal $\mathcal{O}(h^{N+1})$ convergence rate in the L^2-norm when N-degree piecewise polynomials are used, under certain regularity conditions on the solution.

Keywords: Arbitrary curved boundaries; Discontinuous Galerkin method; Reconstruction for off-site data method; Error estimate

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CONTINUITY AND HARNACK INEQUALITIES FOR ELLIPTIC FUNCTIONALS WITH GENERALIZED ORLICZ GROWTH

M.O. Savchenko[1,2]

joint project with I.I. Skrypnik[2], **Y.A. Yevgenieva**[2]

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To explain the point of view of this research consider the energy integrals $\int\limits_{\Omega} \varPhi_i(x, |\nabla u|)dx$,

$\varPhi_1(x,v) = v^p + a_1(x)v^q, \quad a_1(x) \geqslant 0, \qquad \underset{B_r(x_0)}{\operatorname{osc}} a_1(x) \leqslant A\, r^{q-p}, \quad A>0, \quad v>0.$

$\varPhi_2(x,v) = v^p\big(1 + a_2(x)\log(1+v)\big), \qquad a_2(x) \geqslant 0, \qquad \underset{B_r(x_0)}{\operatorname{osc}} a_2(x) \leqslant \dfrac{A}{\log \frac{1}{r}}, \quad A>0, \quad v>0.$

In particular, these conditions imply

$$\sup_{B_r(x_0)} \varPhi_i\big(x, \frac{v}{r}\big) \leqslant \gamma(K) \inf_{B_r(x_0)} \varPhi_i\big(x, \frac{v}{r}\big), \quad r \leqslant v \leqslant K, \quad i = 1, 2. \tag{1}$$

It is well known (see [1]) that the minimizers of the corresponding integrals of the calculus of variations satisfy Harnack's type inequality, or more generally (see [2]), Harnack's type inequality is valid under the conditions

$$\underset{B_r(x_0)}{\operatorname{osc}} a_1(x) \leqslant A\,\big[\log\frac{1}{r}\big]^L r^{q-p}, \quad \underset{B_r(x_0)}{\operatorname{osc}} a_2(x) \leqslant A\frac{\big[\log\log\frac{1}{r}\big]^L}{\log\frac{1}{r}},$$

if $L>0$ is sufficiently small. These conditions yield

$$\sup_{B_r(x_0)} \varPhi_1\big(x, \frac{v}{r}\big) \leqslant \gamma(K) \inf_{B_r(x_0)} \varPhi_1\big(x, \frac{v}{r}\big), \quad r \leqslant v \leqslant K\,\lambda(r), \quad \lambda(r) = \big[\log\frac{1}{r}\big]^{-\frac{L}{q-p}}, \tag{2}$$

and

$$\sup_{B_r(x_0)} \varPhi_2\big(x, \frac{v}{r}\big) \leqslant \gamma(K)\,\Lambda(r) \inf_{B_r(x_0)} \varPhi_2\big(x, \frac{v}{r}\big), \quad r \leqslant v \leqslant K, \quad \Lambda(r) = \big[\log\log\frac{1}{r}\big]^L. \tag{3}$$

To take into account the non-uniformly elliptic case, we set

$$a(x) = \big|\log\big|\log\frac{1}{|x-x_0|}\big|\big|^{L_1}, \quad x_0 \in \Omega,$$

and let $\varPhi_1(v) = v^p + v^q$, $\varPhi_2(v) = v^p\big(1+\log(1+v)\big)$, then

$$\gamma^{-1}\, a(x)\varPhi_i(v) \leqslant \varPhi_i(x,v) \leqslant \gamma\,\varPhi_i(v), \quad L_1 < 0, \quad i = 1, 2,$$

$$\gamma^{-1}\,\varPhi_i(v) \leqslant \varPhi_i(x,v) \leqslant \gamma\, a(x)\,\varPhi_i(v), \quad L_1 > 0, \quad i = 1, 2$$

provided that $B_r(x_0) \subset B_R(x_0) \subset \Omega$ and R is sufficiently small and the bounded local solutions of the corresponding elliptic equations satisfy Harnack's type inequality [3] if

$$\frac{1}{a(x)} \in L^t(\Omega) \quad \text{and} \quad a(x) \in L^s(\Omega) \tag{4}$$

with some t, $s>1$, i.e. if L_1 is sufficiently small. Our aim is to combine logarithmic, non-logarithmic and non-uniformly elliptic conditions (1)–(4). Obviously, conditions (1)–(4) imply for i=1,2

$$\Big(r^{-n} \int\limits_{B_r(x_0)} [\varPhi_i\big(x, \frac{v}{r}\big)]^s\, dx\Big)^{\frac{1}{s}} \Big(r^{-n} \int\limits_{B_r(x_0)} [\varPhi_i\big(x, \frac{v}{r}\big)]^{-t}\, dx\Big)^{\frac{1}{t}} \leqslant \gamma(K)\Lambda(x_0, r), \quad r \leqslant v \leqslant K\,\lambda(r), \tag{5}$$

with some s, t and the precise choice of $\lambda(r)$ and $\Lambda(x_0, r)$.

Another interesting example is the energy integral $\int\limits_{\Omega} \Phi_3(x, |\nabla u|)\, dx$,

$$\Phi_3(x,v) = v^{p(x)}, \quad \underset{B_r(x_0)}{\operatorname{osc}} p(x) \leqslant \frac{\bar{\mu}(r)}{\log \frac{1}{r}}, \quad \lim_{r\to 0} \bar{\mu}(r) = \infty, \quad \lim_{r\to 0} \frac{\bar{\mu}(r)}{\log \frac{1}{r}} = 0, \quad v > 0.$$

We also consider the integrals of this type. And, of course, it would be interesting to unify our approach. More precisely, we will prove continuity and Harnack's inequality for functions belonging to the corresponding non uniformly elliptic De Giorgi classes $DG_\Phi(B_R(x_0))$.

We write $W^{1,\Phi}(B_R(x_0))$ for the class of functions $u \in W^{1,1}(B_R(x_0))$ with $\int\limits_{B_R(x_0)} \Phi(x, |\nabla u|)dx < \infty$ and we say that a measurable function $u : B_R(x_0) \to \mathbb{R}$ belongs to the elliptic class $DG^{\pm}_\Phi(B_R(x_0))$ if $u \in W^{1,\Phi}(B_R(x_0)) \cap L^\infty(B_R(x_0))$ and there exist numbers $c > 0$, $q > 1$ such that for any ball $B_r(x_0) \subset B_R(x_0)$, any $k \in \mathbb{R}$ and any $\sigma \in (0,1)$ the following inequalities hold:

$$\int\limits_{A^{\pm}_{k,r}} \Phi\big(x, |\nabla u|\big)\zeta^q(x)dx \leqslant c \int\limits_{A^{\pm}_{k,r}} \Phi\bigg(x, \frac{(u-k)_\pm}{\sigma r}\bigg)dx, \tag{6}$$

here $(u-k)_\pm := \max\{\pm(u-k), 0\}$, $A^{\pm}_{k,r} := B_r(x_0) \cap \{(u-k)_\pm > 0\}$, $\zeta(x) \in C_0^\infty(B_r(x_0)), 0 \leqslant \zeta(x) \leqslant 1$, $\zeta(x) = 1$ in $B_{(1-\sigma)r}(x_0)$ and $|\nabla\zeta(x)| \leqslant \dfrac{1}{\sigma r}$. We also say that $u \in DG^{\pm}_\Phi(\Omega)$ if $u \in DG^{\pm}_\Phi(B_R(x_0))$ for any $B_{8R}(x_0) \subset \Omega$. We set also $DG_\Phi(B_R(x_0)) = DG^{-}_\Phi(B_R(x_0)) \cap DG^{+}_\Phi(B_R(x_0))$ and $DG_\Phi(\Omega) = DG^{-}_\Phi(\Omega) \cap DG^{+}_\Phi(\Omega)$.

Further we suppose that $\Phi(x,v) : B_R(x_0) \times \mathbb{R}_+ \to \mathbb{R}_+$ is a non-negative function satisfying the following properties: for any $x \in B_R(x_0)$ the function $v \to \Phi(x,v)$ is increasing and $\lim\limits_{v\to 0} \Phi(x,v) = 0$, $\lim\limits_{v\to+\infty} \Phi(x,v) = +\infty$. We also assume that

(Φ_0) There exists $c_0 > 0$ such that for any $x \in B_R(x_0)$ there holds

$$\Phi(x,1) \geqslant c_0.$$

(Φ) There exist $1 < p < q$ such that for $x \in B_R(x_0)$ and for $w \geqslant v > 0$ there holds

$$\Big(\frac{w}{v}\Big)^p \leqslant \frac{\Phi(x,w)}{\Phi(x,v)} \leqslant \Big(\frac{w}{v}\Big)^q.$$

($\Phi^{\lambda}_{\Lambda,x_0}$) There exist continuous, non-decreasing function $0 < \lambda(r) \leqslant 1$ and continuous, non-increasing function $\Lambda_\lambda(x_0, r) \geqslant 1$ on the interval $(0, R)$ such that for any $B_r(x_0) \subset B_R(x_0)$, for any $K > 0$ there holds

$$\sup_{r\leqslant v\leqslant K\,\lambda(r)} \Lambda_\Phi\big(x_0, r, \frac{v}{r}\big) \leqslant c_1(K)\Lambda_\lambda(x_0, r),$$

$$\frac{1}{tp} + \frac{1}{sp} < \frac{1}{n}, \quad t \in \big(\max(1, \frac{1}{p-1}), \infty\big], \quad s \in (1, \infty],$$

here $c_1(K)$ is some fixed positive number depending on K, s and t and

$$\Lambda_\Phi\big(x_0, r, \frac{v}{r}\big) := \Lambda_{-,\Phi}\big(x_0, r, \frac{v}{r}\big)\ \Lambda_{+,\Phi}\big(x_0, r, \frac{v}{r}\big),$$

$$\Lambda_{-,\Phi}\big(x_0, r, \frac{v}{r}\big) := \bigg(r^{-n} \int\limits_{B_r(x_0)} \big[\Phi(x, \frac{v}{r})\big]^{-t}\, dx\bigg)^{\frac{1}{t}}, \ \Lambda_{+,\Phi}\big(x_0, r, \frac{v}{r}\big) := \bigg(r^{-n} \int\limits_{B_r(x_0)} \big[\Phi(x, \frac{v}{r})\big]^{s}\, dx\bigg)^{\frac{1}{s}}.$$

We will also write (Φ^{λ}_{Λ}) if condition ($\Phi^{\lambda}_{\Lambda,x_0}$) holds for any $B_R(x_0) \subset B_{8R}(x_0) \subset \Omega$ and set

$$\Lambda_\lambda(r) := \sup_{x_0\in\Omega, B_{8R}(x_0)\subset\Omega} \Lambda_\lambda(x_0, r).$$

Sometimes we will also need the following technical assumption

(λ) There exist positive constants c_2 and c_3 such that

$$\lambda(\rho) \leqslant \left(\frac{\rho}{r}\right)^{c_2} \lambda(r), \quad \Lambda_\lambda(x_0, r) \leqslant \left(\frac{\rho}{r}\right)^{c_3} \Lambda_\lambda(x_0, \rho), \quad 0 < r \leqslant \rho.$$

Our first result is the interior continuity of the functions belonging to the corresponding De Giorgi classes.

Theorem 1. *[4] Let $u \in DG_\Phi(B_R(x_0))$ and let conditions (Φ_0), (Φ), $(\Phi^\lambda_{\Lambda,x_0})$ be fulfilled. There exist numbers C_1, $\beta_1 > 0$ depending only on the data such that if*

$$\int\limits_0 \exp\big(C_1 \big[\Lambda_\lambda(x_0, r)\big]^{\beta_1}\big) \frac{dr}{\lambda(r)} < +\infty, \quad \int\limits_0 \lambda(r) \exp\big(-C_1 \big[\Lambda_\lambda(x_0, r)\big]^{\beta_1}\big) \frac{dr}{r} = +\infty, \tag{7}$$

then $u(x)$ is continuous at point x_0.

If additionally, $u \in DG_\Phi(\Omega)$, condition (Φ^λ_Λ) holds and

$$\int\limits_0 \exp\big(C_1 \big[\Lambda_\lambda(r)\big]^{\beta_1}\big) \frac{dr}{\lambda(r)} < +\infty, \quad \int\limits_0 \lambda(r) \exp\big(-C_1 \big[\Lambda_\lambda(r)\big]^{\beta_1}\big) \frac{dr}{r} = +\infty, \tag{8}$$

then $u(x) \in C(\Omega)$.

Next result is the Harnack inequality. We will distinguish several cases, first we will assume that $\Lambda_\lambda(r) \leqslant$ *const*, $0 < r \leqslant R$. Note that the case $\lim\limits_{r\to 0} \Lambda_1(r) = \infty$ is possible.

Theorem 2. *[4] Let $u \in DG^-_\Phi(\Omega)$, $u \geqslant 0$, let conditions (Φ_0), (Φ), (Φ^λ_1), (λ) be fulfilled. Then there exist numbers $C_2 > 0$, $\theta \in (0, 1)$ depending only on the data such that*

$$\Big(\rho^{-n} \int\limits_{B_\rho(x_0)} u^\theta \, dx\Big)^{\frac{1}{\theta}} \leqslant \frac{C_2}{\lambda(\rho)} \Big\{ \min_{B_{\frac{\rho}{2}}(x_0)} u + \rho \Big\}, \quad 0 < \rho \leqslant R, \tag{9}$$

provided that $B_{8R}(x_0) \subset \Omega$.

In addition, if $u \in DG_\Phi(\Omega)$ and condition (Φ^1_Λ) holds , then there exist numbers C_3, $\beta_2 > 0$ depending only on the data such that

$$\max_{B_{\frac{\rho}{2}}(x_0)} u \leqslant C_3 \frac{[\Lambda_1(\rho)]^{\beta_2}}{\lambda(\rho)} \Big\{ \min_{B_{\frac{\rho}{2}}(x_0)} u + \rho \Big\}, \quad 0 < \rho \leqslant R, \tag{10}$$

provided that $B_{8R}(x_0) \subset \Omega$.

We formulate our next theorem under the assumption $\lambda(r) \equiv 1$, moreover, its formulation requires more complicated conditions on the function $\Lambda_1(r)$, so we will prove it only in the model case, namely, we will assume that $\Lambda_1(r) = [\log\log \frac{1}{r}]^L, L > 0$.

Theorem 3. *[4] Let $u \in DG_\Phi(\Omega) \cap C(\Omega)$, $u \geqslant 0$ and let conditions (Φ_0), (Φ), (Φ^1_Λ) be fulfilled. Let $\Lambda_1(\rho) = [\log\log \frac{1}{\rho}]^L$, $\rho \in (0, 1)$, $L > 0$. Then there exists number $C_4 > 0$, depending only on the data and L such that*

$$u(x_0) \leqslant C_4 \log \frac{1}{\rho} \Big\{ \min_{B_{\frac{\rho}{2}}(x_0)} u + \rho \Big\}, \quad 0 < \rho \leqslant R, \tag{11}$$

provided that $B_{8R}(x_0) \subset \Omega$ and L is small enough.

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