Topological structures on real-enriched categories

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This talk concerns extensions of Scott topology to real-enriched categories. A real-enriched category is a category with real numbers as enrichment. The definition is as follows. For each continuous t-norm & on the unit interval $[0,1]$, $\mathsf{V} = ([0,1], \&, 1)$ is a complete and symmetric monoidal closed category. A V-category is then a pair (X, α), where X is a set and $\alpha\colon X \times X \to [0,1]$ is a function such that

(i) $\alpha(x,x) = 1$ for all $x \in X$; and

(ii) $\alpha(y,z) \,\&\, \alpha(x,y) \leq \alpha(x,z)$ for all $x, y, z \in X$.

Such V-categories are called real-enriched categories. If we interpret the value $\alpha(x,y)$ as the truth degree that x precedes y, then (i) is reflexivity and (ii) is transitivity. So, real-enriched categories may be viewed as many-valued ordered sets. Since the quantale $([0,\infty]^{\mathrm{op}}, +, 0)$ is isomorphic to $([0,1], \cdot, 1)$, quasi-metric spaces are natural examples of real-enriched categories.

We focus on topological structures of real-enriched categories, including classical topological structures and V-approach structures. A V-approach structure on a set is a map $\delta\colon X \times 2^X \to [0,1]$ such that for all $x \in X$ and all $A, B \in 2^X$,

(A1) $\delta(x, \{x\}) = 1$;

(A2) $\delta(x, \varnothing) = 0$;

(A3) $\delta(x, A \cup B) = \delta(x, A) \vee \delta(x, B)$;

(A4) $\delta(x, A) \geq \Big(\inf\limits_{b \in B} \delta(b, A)\Big) \,\&\, \delta(x, B)$.

The value $\delta(x, A)$ is interpreted as the truth degree that x is close to A. So, a V-approach structure is a many-valued topology, a V-valued topology to be precise.

Scott topology is an important topology on partially ordered sets, it makes every directed subset converge to its supremum. In this talk we discuss extensions of Scott topology to the realm of real-enriched categories.

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