# Game comonads: logical and homotopical aspects

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Game comonads were introduced in [1, 5] to provide a categorical approach to finite model theory and descriptive complexity, which typically focus on logic fragments with a finite amount of *logical resources*, such as finite-variable logics or logics with bounded quantifier rank, and the corresponding combinatorial parameters of (relational) structures. In many cases, equivalence in these logic fragments can be captured in a syntax-free way by means of model-comparison games, such as Ehrenfeucht–Fraïssé and pebble games.

The key insight underlying game comonads is that collections of plays in these games can be naturally organised into endofunctors on categories of relational structures that carry the structure of *comonads*, and their *coalgebras* encode both the corresponding logic fragments and the combinatorial parameters.

This idea has proved to be very robust and the whole pattern, covering a range of resource-bounded logics, has been axiomatised at a general categorical level in [2, 3]. These results have led to new connections between two areas within logic in computer science that have largely been disjoint: finite model theory and descriptive complexity, and semantics and categorical structures of computation.

In this talk, I will give an overview of the main ideas underpinning game comonads, with an emphasis on their:

(i) Logical aspects, including a categorical view on homomorphism counting results [6, 7] and equi-resource homomorphism preservation theorems [4];

(ii) Homotopical aspects, including the use of *upgrading arguments* in preservation theorems, akin to small object arguments in abstract homotopy, and a homotopical view of modal logic and the Łoś–Tarski preservation theorem [8].

## References

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