The topological μ-calculus

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Modal logic enjoys well-known topological semantics dating back to Tarski and McKinsey, where the modal $\diamond$ may be interpreted as topological closure or, alternately, as the Cantor derivative [8]. These semantics readily extend to the language of the μ-calculus [7]. In this presentation we will provide a general introduction to the topological μ-calculus and survey the state of the art and open questions.

Since topological operators such as the closure and interior operators are already idempotent, this version of the μ-calculus behaves quite differently from its relational variant. In the closure semantics, a certain polyadic operator known as the *tangled closure* [3] is expressively complete due to results of Dawar and Otto [2]. This remains true for semantics based on the Cantor derivative over spaces satisfying a regularity condition known as T_d. Goldblatt and Hodkinson [5] studied the μ-calculus in this setting, providing completeness results for various classes including that of metric spaces. However, many of the techniques used break down when dropping the T_d assumption, which among other things allows one to embed the topological μ-calculus into the relational one and thus draw on known results.

In more recent work, A. Baltag, N. Bezhanishvili and the speaker [1] have shown that completeness results for arbitrary spaces may be obtained directly via *final submodel* methods [4]. The question remains whether the expressive completeness of the tangled fragments holds in the general topological setting as well. Preliminary results by Baltag et al. [1] and Q. Gougeon [6] suggest a negative answer, although the latter also proposes a compelling candidate for an expressively complete *hybrid tangle.*

Finally, one may ask which classes of spaces are μ-calculus definable without being modally definable, with no examples being known for a surprisingly long time. Gougon's thesis also provides the first such examples, leading to the introduction of *imperfect spaces.* Such classes of spaces may or may not be representative of all μ-calculus definable classes, but there are reasons to conjecture that they are, especially if the question of expressive completeness of tangle-like fragments is settled in the affirmative.

References

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