A general Fubini theorem for the Riesz paradigm

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We prove a general theorem on symmetric monoidal adjunctions, and as a corollary we establish a Fubini theorem for integrals on arbitrary convergence spaces that generalizes (and entails) the classical Fubini theorem for Radon measures on compact Hausdorff spaces. For a symmetric monoidal adjunction $F \dashv G : \mathcal{L} \to \mathcal{X}$ with $\mathcal{L}$ and $\mathcal{X}$ closed, we formulate a Fubini theorem as the statement that an associated *monad of natural distributions* $\mathbb{D} = \mathcal{L}([-, R], R)$ on $\mathcal{X}$ is *commutative* [2]. Under a mild completeness hypothesis on $\mathcal{L}$, we show that if each cotensor $[X, R]$ of the unit object R in $\mathcal{L}$ is reflexive (where $X \in \mathcal{X}$), then $\mathbb{D}$ is indeed commutative. This result is applicable to the example of $\mathcal{X} = \{\text{convergence spaces}\}$ and $\mathcal{L} = \{\text{real vector spaces in } \mathcal{X}\}$, since Butzmann [1] showed that each vector space of continuous functions $[X, \mathbb{R}]$ is indeed reflexive, so we deduce that the continuous linear functionals $[X, \mathbb{R}] \to \mathbb{R}$ satisfy a Fubini theorem.

Our result also entails a Fubini theorem for *vector-valued* integration, since by analogy with [3], the monad $\mathbb{D}$ in the setting of convergence spaces defines a theory of vector-valued integration, with $\mathbb{D}$-algebras being convergence vector spaces equipped with operations of integration with respect to natural distributions. We define a closed reflective subcategory $\widetilde{\mathcal{L}}$ of $\mathcal{L}$ which contains all reflexive $\mathcal{L}$-objects and embeds fully into $\mathcal{X}^{\mathbb{D}}$. In the setting of convergence, all $\widetilde{\mathcal{L}}$-objects (such as Cauchy-complete locally convex topological vector spaces) thus support vector-valued integration.

REFERENCES

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