A pointfree account of Carathéodory's Extension Theorem

Tomáš Jakl [a]

Workshop on Algebra, Logic and Topology in Coimbra
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Classical Carathéodory's Extension Theorem

Theorem

A measure $m\colon \mathcal{B} \to [0,1]$ *on a Boolean algebra* $\mathcal{B} \subseteq \mathcal{P}(X)$ *uniquely extends to a countably additive measure on* $\sigma(\mathcal{B})$.

Minimal σ*-algebra contaning B*

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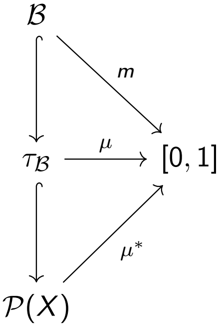
Proof.



1. Extend m to a **countably additive** function

$$\mu(U) = \sup\{m(B) \mid B \in \mathcal{B},\ B \subseteq U\}$$

2. Extend μ to an **outer measure**

$$\mu*(M) = \inf\{\mu(U) \mid U \in \tau_{\mathcal{B}},\ M \subseteq U\}$$

3. μ^* is a measure on measurable subsets $\mathcal{H} \subseteq \mathcal{P}(X)$. Restrict μ^* to $\sigma(\mathcal{B}) \subseteq \mathcal{H}$.

□

Extension theorem by Igor Kříž and Aleš Pultr

Abstract σ-algebra is a Boolean algebra which has *countable* joins.

Abstract finitely (resp. countably) additive measure $m\colon B \to [0,1]$ satisfies

1. $m(0_B) = 0, \quad m(1_B) = 1,$
2. $m(a \vee b) + m(a \wedge b) = m(a) + m(b)$
3. (resp. $\sum_{i=0}^{\infty} m(a_i) = m(\bigvee_{i=0}^{\infty} a_i)$ if a_i's are pairwise disjoint)

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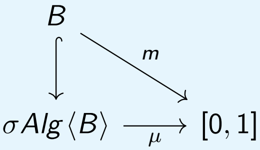
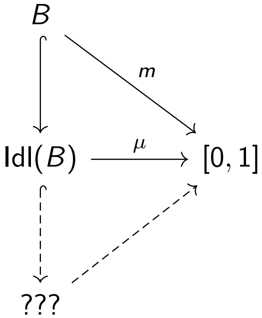
Theorem (Kříž, Pultr 2010)

Every finitely additive $m\colon B \to [0,1]$ *uniquely extends to a countably additive measure* $\mu\colon \sigma Alg\langle B\rangle \to [0,1]$ *such that*



Enlarges the space. On the other hand, useful for integration over infinite-dimensional spaces!

What instead of $\mathcal{P}(X)$?



Finitely additive $m\colon B \to [0,1]$ extends to a **valuation** $\mu\colon \mathrm{Idl}(B) \to [0,1]$,

$$\mu(I) = \sup\{m(a) : a \in I\}$$

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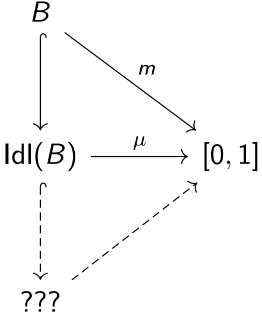

Finitely additive $m\colon B \to [0,1]$ extends to a **valuation** $\mu\colon \mathrm{Idl}(B) \to [0,1]$, i.e.

1. μ is a finitely additive measure
2. For a directed $A \subseteq^{\uparrow} \mathrm{Idl}(B)$:

$$\sup_{I \in A} \mu(I) = \mu\left(\bigvee\nolimits^{\uparrow} A\right)$$

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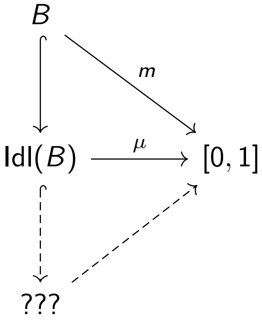

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We need a complete Boolean algebra which

- embeds $\mathrm{Idl}(B)$, and
- has the same (frame-theoretic) points as B has.

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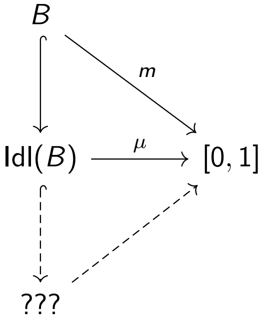

$\mathrm{Idl}(B)$ is a frame!

$a \wedge \bigvee_i b_i = \bigvee_i (a \wedge b_i)$

e.g. $\mathcal{O}(X, \tau) = \tau$

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Frame Theory intermezzo: Sublocales

A subspace $M \subseteq X$ introduces a frame congruence $\sim_M$ on $\mathcal{O}(X)$:

$$U \sim_M V \quad \text{iff} \quad U \cap M = V \cap M$$

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Congruences are equivalently represented as **sublocales** $S \subseteq L$

1. $\forall A \subseteq S, \quad \bigwedge A \in S$
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The mapping "congruences $\mapsto$ sublocales":

$$\sim \subseteq L \times L \quad \longmapsto \quad \{\text{largest elements of } \sim\text{-equivalence classes}\}$$

Every subspace of X introduces a sublocale of $\mathcal{O}(X)$
but not vice versa!

The complete lattice (coframe) of sublocales

$$\mathcal{S}(L) \;=\; \{S \subseteq L \mid S \text{ is a sublocale}\}, \quad \text{ordered by } \subseteq .$$

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Open and closed sublocales ($a \in L$):

$$\mathfrak{o}(a) = \{a \to x \mid x \in L\} \quad \text{and} \quad \mathfrak{c}(a) = \uparrow a$$

They are complemented in $\mathcal{S}(L)$.

$\bigvee_i \mathfrak{o}(a_i) = \mathfrak{o}(\bigvee_i a_i)$, $\quad \mathfrak{c}(a) \vee \mathfrak{c}(b) = \mathfrak{c}(a \wedge b)$, $\quad \ldots \quad$ (as expected)

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Join-sublattice $\mathcal{S}_{\mathfrak{c}}(L) \subseteq \mathcal{S}(L)$

$$\mathcal{S}_{\mathfrak{c}}(L) \;=\; \left\{ \begin{array}{c} \text{the set of sublocales obtained as} \\ \textbf{joins of closed sublocales} \end{array} \right\}$$

Always a frame!

Theorem (Picado, Pultr, Tozzi 2016)

If L is subfit then $\mathcal{S}_{\mathfrak{c}}(L)$ is a complete Boolean algebra and

$$a \in L \longmapsto \mathfrak{o}(a) \in \mathcal{S}_{\mathfrak{c}}(L)$$

is an injective frame homomorphisms $L \hookrightarrow \mathcal{S}_{\mathfrak{c}}(L)$.

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Moreover

- If X is a T_1 space, then $\mathcal{S}_{\mathfrak{c}}(\mathcal{O}(X)) \cong \mathcal{P}(X)$.
- In case of $X = \mathsf{spec}(B)$, we have $\mathcal{O}(X) \cong \mathsf{Idl}(B)$ and so

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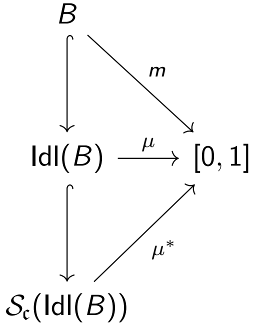
- $\Longrightarrow$ instead of $\mathcal{P}(X)$ take $\mathcal{S}_{\mathfrak{c}}(\mathrm{Idl}(B))$

Putting it together



Valuation $\mu\colon \mathsf{Idl}(B) \to [0,1]$ extends to an **outer measure** $\mu^*\colon \mathcal{S}_{\mathfrak{c}}(\mathsf{Idl}(B)) \to [0,1]$,

$$\mu^*(x) = \inf\{\mu(i) \mid i \in \mathsf{Idl}(B),\ x \leq i\}$$

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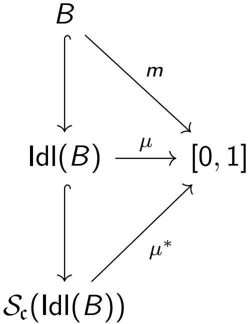

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1. μ^* is monotone
2. $\mu^*(x \vee y) + \mu^*(x \wedge y) \leq \mu^*(a) + \mu^*(b)$
3. For a directed $(x_i)_{i=0}^{\infty} \subseteq^{\uparrow} \mathcal{S}_{\mathfrak{c}}(\mathrm{Idl}(B))$:

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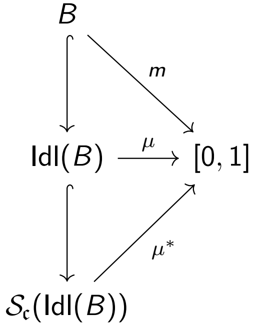

Valuation $\mu \colon \mathsf{Idl}(B) \to [0,1]$ extends to an **outer measure** $\mu^* \colon \mathcal{S}_{\mathfrak{c}}(\mathsf{Idl}(B)) \to [0,1]$, i.e.

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Furthermore

$$\mathcal{H} = \{x \in \mathcal{S}_{\mathfrak{c}}(\mathsf{Idl}(B)) \mid \mu^*(x) + \mu^*(\neg x) \leq 1\}$$

is a σ-algebra (containing $\sigma_{\mathcal{S}}(B)$) and so $\mu^* \restriction_{\mathcal{H}}$ is a measure.

Pointfree Carathéodory's Extension Theorem

Theorem

A finitely additive measure $m\colon B \to [0,1]$ uniquely extends to a countably additive measure on $\sigma_{\mathcal{S}}(B) \subseteq \mathcal{S}_{\mathfrak{c}}(\mathsf{Idl}(B))$.

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Corollary

There are bijective correspondences between

- *finitely additive measures* $B \to [0,1]$
- *regular countably additive measures* $\sigma_{\mathcal{S}}(B) \to [0,1]$
- *regular valuations* $\sigma_{\mathcal{S}}(\mathsf{Idl}(B)) \to [0,1]$

Comparison with the classical result

For a Boolean algebra $\mathcal{B} \subseteq \mathcal{P}(X)$, it might happen that

$$\bigcup_i B_i \in \mathcal{B} \quad \text{for some infinite} \quad \{B_i\}_i \subseteq \mathcal{B}.$$

However, in the Stone space $\operatorname{spec}(\mathcal{B})$ (i.e. in the "sobrification")

$$\bigcup_i [\![B_i]\!] \neq [\![\bigcup_i B_i]\!] = \left(\overline{\bigcup_i [\![B_i]\!]}\right)^\circ$$

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$\Longrightarrow$ We don't need the **extra assumption** for $m \colon \mathcal{B} \to [0,1]$:

For any pairwise disjoint $\{B_i\}_{i=0}^{\infty} \subseteq \mathcal{B}$ such that $\bigcup_i B_i \in \mathcal{B}$

$$m(\bigcup_i B_i) = \sum_{i=0}^{\infty} m(B_i)$$

The continuous map $U\colon (X, \mathcal{P}(X)) \to (\mathsf{spec}(\mathcal{B}), \mathcal{P}(\mathsf{spec}(\mathcal{B})))$

$$U\colon x \longmapsto \{B \in \mathcal{B} \mid x \in B\}$$

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introduces a frame homomorphism $h\colon \mathcal{P}(\mathrm{spec}(\mathcal{B})) \to \mathcal{P}(X)$

$$h\colon M \mapsto \{x \mid U(x) \in M\}$$

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Which restricts to $\sigma_{\mathcal{S}}(\mathcal{B}) \twoheadrightarrow \sigma(\mathcal{B})$

$\sigma_{\mathcal{S}}(\mathcal{B}) \subseteq \mathcal{S}_{\mathfrak{c}}(\mathsf{Idl}(\mathcal{B})) \cong \mathcal{P}(\mathsf{spec}(\mathcal{B}))$

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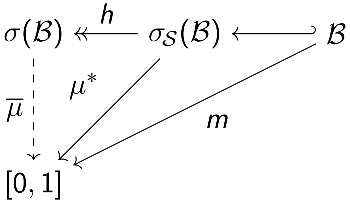

Define $\overline{\mu}(M) = \mu^*(U[M])$

If the "extra assumption" holds for m, we obtain the Carathéodory's measure!

Canonical extensions

For a Boolean algebra B, we have

$$B \hookrightarrow B^\delta$$

Characterised as

1. B is join–meet and meet–join dense in B^δ
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2. the embedding is compact

Recall

- B^δ is a complete Boolean algebra,
- for the Stone dual X of B we have $B^\delta \cong (\mathcal{P}(X), \subseteq)$, and
- B^δ can be constructed entirely choice-free.

Consequently

- $B^\delta \cong \mathcal{P}(X) \cong \mathcal{S}_{\mathfrak{c}}(\mathsf{Idl}(B))$

Theorem (Ball, Pultr 2017)

Assume that L is subfit, $L \hookrightarrow M$, and for any $x < y$ in M there is $a < b$ in L such that

$$x \wedge b \leq a \quad \text{and} \quad y \vee a \geq b.$$

If M is a Boolean frame then $\mathcal{S}_{\mathfrak{c}}(L) \cong M$.

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Proof that $B^\delta \cong \mathcal{S}_{\mathfrak{c}}(\mathsf{Idl}(B))$ algebraically:

For $x < y$ pick a join of B's $i \in B^\delta$ such that

$$x \leq i \quad \text{and} \quad y \not\leq i$$

and pick a meet of B's $f \in B^\delta$ such that

$$f \leq y \quad \text{and} \quad f \not\leq i$$

Then, $a = i \vee \neg f$ and $b = 1$ satisfy the conditions. □

Generalisation to distributive lattices?

We know $D^\delta \cong \mathrm{Up}(X, \leq)$ for the Priestly space $(X, \tau, \leq)$ of D.

Is there a frame-theoretic construction for D^δ?

However

- $\mathrm{Idl}(D)$ need not be subfit
- $\mathrm{Idl}(D) \not\hookrightarrow \mathcal{S}_\mathfrak{c}(\mathrm{Idl}(D))$

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What instead of $\mathcal{S}_{\mathfrak{c}}(-)$? Something like $\mathcal{S}_{\mathfrak{o}}(L)$? ... is it a frame?

Extension theorem by Alex Simpson (2011)

Different approach

$$\mathcal{S}^\sigma(L) = \{S \subseteq L \mid S \text{ is a } \sigma\text{-sublocale of } L\}$$

Theorem

If L is a **fit** *σ-frame, then a valuation $\mu\colon L \to [0,1]$ uniquely extends to a valuation $\mu^*\colon \mathcal{S}^\sigma(L) \to [0,1]$ such that*

$$\begin{array}{ccc} L & & \\ \big\downarrow & \overset{\mu}{\searrow} & \\ \mathcal{S}^\sigma(L) & \xrightarrow[\mu^*]{} & [0,1] \end{array}$$

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Although $\sigma(B) \subseteq \mathcal{S}^\sigma(\mathrm{Idl}(B))$, $\mathcal{S}^\sigma(L)$ is a coframe, not a σ-algebra!
$\Longrightarrow$ We can't talk about points, it doesn't specialise to point-set setting.

On the other hand, it "resolves" Banach-Tarski paradox!

Concluding remarks

- Kříž–Pultr's solution factors through ours

$$
\begin{array}{ccccc}
B & & & & \\
\big\downarrow & \searrow & & \overset{m}{\searrow} & \\
\sigma\mathsf{Alg}\langle B\rangle & \xrightarrow{\exists!} & \sigma_{\mathcal{S}}(B) & \xrightarrow{\mu^*} & [0,1]
\end{array}
$$

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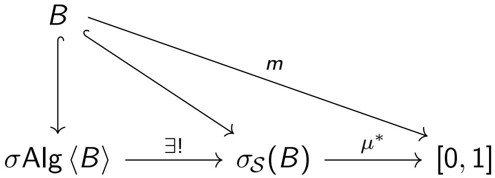

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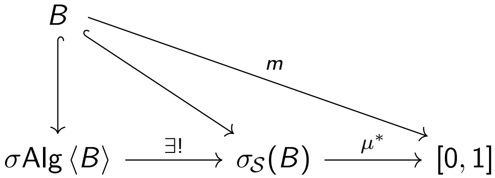

- It would be nice to construct D^δ frame-theoretically.

- The same reasoning as in the classical case applies.

- Common in Kříž–Pultr + TJ: We can study measure theory in a point-free fashion and only add points at the end, if needed.

Thank you!

and ...

Happy Birthday Aleši!

Aleš is influential in so many areas of mathematics:

1. Algebraic topology
2. Category theory
3. Duality theory
4. Fuzzy logic/sets
5. General algebra
6. Graph theory
7. Mathematical analysis
8. Pointfree topology
9. ...

Happy Birthday Aleši!

The most common words in Aleš's 185 titles:

(papers and book chapters combined)

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